Bivariate Degradation Modeling Based on Gamma Process

Jinglun Zhou, Zhengqiang Pan, *Member, IAENG* and Quan Sun

***Abstract*—Many highly reliable products have two or more performance characteristics (PCs). The PCs may be independent or dependent each other. If they are dependent, it is very important to find the joint distribution function of the PCs. In this paper, suppose that a product has two PCs and the PCs degradation can be governed by Gamma process. And the dependence of the PCs can be described by copula function. In order for estimating the product's reliability as accurate as possible, the parameters of the two PCs and copula function can be estimated as a whole. The model in such a situation is very complicated and analytically intractable, hence very cumbersome from a computational viewpoint. So the Bayesian MCMC method is developed to this problem that allows the maximum likelihood estimator (MLE) of the parameters to be evaluated in an efficient manner. As a nice application of the proposed model, an illustrative example about fatigue cracks is presented.**

***Index Terms*—Bivariate degradation model; Gamma process; Bayesian MCMC; Copula function.**

I. INTRODUCTION

In recent years, the situation and the increasing demand for high quality and reliability products have motivated the extensive growth of research activities in the degradation area. Many models for modeling and analyzing degradation paths are proposed including constant-stress degradation test and accelerated degradation test. But most of the previous research which focuses on degradation area considers only one PC. In practice, modern products usually have complex structure and more functions. This means that modern highly reliable product may have multiple degradation measures. For example, a rubidium discharge lamp is the key component of rubidium frequency standard which performance degradation can be described by the rubidium consumption or the decreasing of lamp's intensity. Another example is a lighting system consists of many LED lamps for different purposes of lighting. The design and the characteristic of the LED system demands may generate more than one degradation mechanism that lead to failure [1]. Therefore, multivariate or at least bivariate degradation model is needed to estimate the reliability of modern products. This analysis is needed not only for design and technical purposes but also as important information for the management and decision makers.

There are only few works dealing with estimation system reliability of bivariate or multivariate degradation data for industrial purpose [2-4]. Moreover, some works about multiple failure modes including failure time data are discussed [5-7]. However, these works use either independence assumption of the PCs, multivariate normal distribution, or modeling with covariates and modification to single failure classifications. In fact, these assumptions may be insufficient because in reality common usage history and complexity of the modern product may require relaxation of these assumptions.

Sari [8] answered the question that how to quantify the reliability of a product/system which has two or more degraded PCs with each PC leading to a failure mechanism of the system in his Ph.D thesis. He modeled the degradation data with generalized linear model (GLM) and described the dependence of these PCs by copula function.

But in Sari's work, he estimated the parameters of the PCs separately firstly, and then infer the copula parameter. If the PCs are dependent, maybe the parameters of different PC will influence each other. So we think we should deal with these parameters including copula parameter as a whole. Furthermore, Gamma process has many good properties so that it can be taken to describe the degradation of products extensively [9-11]. In this paper, we suppose that a product has two PCs and the PCs can be governed by Gamma process. And the dependence of the PCs can be described by copula function. We try to estimate the parameters of the two PCs and copula function as a whole. The model in such a situation is very complicated and analytically intractable, so we apply Bayesian MCMC method for our problem to evaluate the parameters efficiently. As a nice application of the proposed model, an illustrative example about fatigue cracks is presented.

The rest of the paper is organized as follows. In Section II, the bivariate degradation model based on Gamma process is introduced. In Section III, inference method for the model parameters is presented. A numerical example about fatigue cracks is given in Section IV.

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II. Biviviate Degradation Model Based on Gamma Process

A. Gamma Process

In order for the stochastic degradation process to be monotonic, we can best consider it as a Gamma process. A Gamma process is a stochastic process with independent, non-negative increments having a Gamma distribution with an identical scale parameter. The Gamma process with shape parameter $v > 0$ and scale parameter $u > 0$ is a continuous time stochastic process $\{X(t), t \geq 0\}$ with the following properties:

1) $X(0) = 0$ with probability one;

2) $X(t)$ has independent increments;

3) $X(t) - X(s) \sim Ga(v \cdot t - v \cdot s, u)$ for all $t > s \geq 0$.

where $Ga(x \mid v, u)$ is the probability density function of a random variable X which follows a Gamma distribution with shape parameter $v > 0$ and scale parameter $u > 0$ as follows:

$$Ga(x \mid v, u) = \frac{u^v}{\Gamma(v)} x^{v-1} \exp(-ux) I_{(0,\infty)}(x), \qquad (1)$$

where

$$I_{(0,\infty)}(x) = \begin{cases} 1 & x \in (0,\infty) \\ 0 & x \notin (0,\infty) \end{cases},$$

and $\Gamma(v) = \int_0^\infty x^{v-1} e^{-x} dx$ is Gamma function.

From c), we know that the degradation increment governed by Gamma process is a linear function of t. Where it is not linear, however, it is often found that a monotonic transformation of the time scale can make it linear. Whitmore and Schenkelberg [12] discussed this situation about Wiener diffusion process in their work. Denote the transformation by $\tau = \tau(t)$,

where t denotes the clock or calendar time and τ is the transformed time. We shall require the transformation to satisfy the initial condition $\tau(0) = 0$.

B. Copula Function

In this paper, we use copula function to describe the dependence of the PCs. So here we introduce copula function firstly. Copula functions offer a far more flexible method for combining marginal distributions into multivariate distributions and offer an enormous improvement in capturing the real correlation pattern. More details can be found in Nelson [13].

Suppose that we have two marginal CDFs $G^{(1)}(x^{(1)})$ and $G^{(2)}(x^{(2)})$, where $X^{(1)}$ and $X^{(2)}$ are the random variables. According to Sklar's Theorem, if $H(x^{(1)}, x^{(2)})$ is a joint distribution with margins $G^{(1)}(x^{(1)})$ and $G^{(2)}(x^{(2)})$, then there exists a copula C such that for all $(x^{(1)}, x^{(2)})$ in the defined range

$$H(x^{(1)}, x^{(2)}) = C\left(G^{(1)}(x^{(1)}), G^{(2)}(x^{(2)})\right).$$

There are some 1-parameter copulas that can be used. For example: Elliptical family (i.e. Gaussian, t) and Archimedean family (i.e. Frank, Gumbel, Clayton, *et al*). One of the popular Archimedean copulas is Frank copula. It is a symmetric copula (for bivariate data) given by

$$C_\alpha(u,v) = -\frac{1}{\alpha} \ln\left(1 + \frac{(\exp(-\alpha u)-1)(\exp(-\alpha v)-1)}{\exp(-\alpha)-1}\right),$$

and its generator is:

$$\varphi_\alpha(t) = -\ln\left(\frac{\exp(-\alpha t)-1}{\exp(-\alpha)-1}\right),$$

where $\alpha \in (-\infty, \infty) \setminus \{0\}$.

The relationship of between Kendall's tau τ and the Frank copula parameter α is given by

$$\frac{[1 - D_1(\alpha)]}{\alpha} = \frac{1-\tau}{4}, \qquad (2)$$

where

$$D_1(\alpha) = \frac{1}{\alpha} \int_0^\alpha \frac{t}{\exp(t)-1} dt,$$

is a Debye function of the first kind.

In this paper, the Frank copula is used to describe the dependence of the PCs.

C. Model for Bivariate Degradation Data

Suppose that a product has two PCs and the PCs can be governed by Gamma process. During the degradation test experiment, N items are tested and M measurements for all the items are observed up to the termination time T, which results in degradation measurements $\mathbf{x}_{i,j} = \left(x_{i,j}^{(1)}, x_{i,j}^{(2)}\right)$ of the i th item at corresponding time t_j, that means the measurement times of the two PCs and N items are the same (balanced data). In general, the bivariate degradation data for this model can be presented in the form,

$$\mathbf{X}_{(2n \times m)} = \begin{pmatrix} \mathbf{X}^{(1)} \\ \mathbf{X}^{(2)} \end{pmatrix} = \begin{pmatrix} x_{1,1}^{(1)} & \cdots & x_{1,m}^{(1)} \\ \vdots & & \vdots \\ x_{n,1}^{(1)} & \cdots & x_{n,m}^{(1)} \\ x_{1,1}^{(2)} & \cdots & x_{1,m}^{(2)} \\ \vdots & & \vdots \\ x_{n,1}^{(2)} & \cdots & x_{n,m}^{(2)} \end{pmatrix}.$$

For $k = 1,2$, let

$$\Delta x_{i,j}^{(k)} = x_{i,j+1}^{(k)} - x_{i,j}^{(k)}, \; \Delta \tau_j^{(k)} = \tau_{j+1}^{(k)} - \tau_j^{(k)}.$$

By the independent increment property of the Gamma process, we have independent but not identical random variables

$$\Delta x_{i,j}^{(k)} \sim G\left(v^{(k)} \cdot \Delta \tau_j^{(k)}, u^{(k)}\right), \qquad (3)$$

So the probability density function (PDF) of $\Delta x_{i,j}^{(k)}$ is

$$g\left(\Delta x_{i,j}^{(k)}\right) = \frac{\left(u^{(k)}\right)^{v^{(k)} \cdot \Delta \tau_j^{(k)}}}{\Gamma\left(v^{(k)} \cdot \Delta \tau_j^{(k)}\right)} \left(\Delta x_{i,j}^{(k)}\right)^{v^{(k)} \cdot \Delta \tau_j^{(k)} - 1} \cdot \exp\left(-u^{(k)} \cdot \Delta x_{i,j}^{(k)}\right), \qquad (4)$$

where $i = 1, \cdots, N$, $j = 1, \cdots, M-1$.

Assume that in the case of different pre-determined measurement times of the PCs, the dependence can be

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ignored, that is, for $\forall i$, we think $\Delta x_{i,j}^{(1)}$ and $\Delta x_{i,k}^{(2)}$ are independent, satisfying $j \neq k$. Now, consider the joint distribution of $\Delta x_{i,j}^{(1)}$ and $\Delta x_{i,j}^{(2)}$.

$$\begin{aligned} H(\Delta x_{i,j}^{(1)}, \Delta x_{i,j}^{(2)}) &= C\left(G^{(1)}(\Delta x_{i,j}^{(1)}), G^{(2)}(\Delta x_{i,j}^{(2)})\right) \\ &= P\left(X^{(1)} \leq \Delta x_{i,j}^{(1)}, X^{(2)} \leq \Delta x_{i,j}^{(2)}\right) \end{aligned}.$$

Set $u(i,j) = G^{(1)}\left(\Delta x_{i,j}^{(1)}\right)$ and $v(i,j) = G^{(2)}\left(\Delta x_{i,j}^{(2)}\right)$, and $c\left(u(i,j), v(i,j)\right)$ is the PDF of $C\left(u(i,j), v(i,j)\right)$. We can get the log-likelihood function of this model as follows

$$L = \sum_{i=1}^{N} \sum_{j=1}^{M-1} \ln c\left(u(i,j), v(i,j)\right). \tag{5}$$

We can obtain the parameters from (5). The method of parameter estimation will be introduced in Section III.

Let $\omega^{(1)}$ and $\omega^{(2)}$ be the threshold value of the two PCs. With the non-decreasing property of the degradation function, the product reliability can be written as

$$\begin{aligned} R(t) &= P\left(X_t^{(1)} < \omega^{(1)}, X_t^{(2)} < \omega^{(2)}\right) \\ &= C\left(G^{(1)}(\omega^{(1)} \mid t), G^{(2)}(\omega^{(2)} \mid t)\right) \end{aligned}. \tag{6}$$

If the two PCs are independent, the product reliability can be expressed simply as

$$R(t) = R^{(1)}(t) \cdot R^{(2)}(t). \tag{7}$$

where

$$R^{(k)}(t) = P\left(X_t^{(k)} < \omega^{(k)}\right) = G\left(\omega^{(k)} \mid t\right). \tag{8}$$

III. Parameter Estimation Method

In this section, we will introduce the parameter estimation method for (5). The model is very complicated from computational view, so it is difficult to infer the parameters by analytical approach. In our work, we will apply Bayesian MCMC method to estimate the parameters.

In most practical applications where the Bayesian approach is used, it is difficult to compute analytically the posterior distribution. The MCMC method can be used to generate a sample from the posterior distribution large enough based on a Markov Chain so that any desired feature of the posterior distribution can be accurately summarized. Now, the two most popular MCMC algorithms are the Gibbs sampling and the Metropolis-Hastings algorithm.

Gibbs sampling is an algorithm to generate a sequence of samples from the joint probability distribution of two or more random variables. It is a special case of single-component Metropolis-Hastings algorithm using as proposal density $q\left(\theta' \mid \theta^{(t)}\right)$ the full conditional posterior distribution $f\left(\theta_j \mid \theta_{\setminus j}, y\right)$, where

$$\theta_{\setminus j} = \left(\theta_1, \cdots, \theta_{j-1}, \theta_{j+1}, \cdots, \theta_d\right)^T.$$

Such proposal distributions result in acceptance probability $\alpha = 1$.

The algorithm can be summarized by the following steps [14]:

Step 1: Set initial values $\theta^{(0)}$;

Step 2: For $t = 1, \cdots, T$ repeat the following steps:

Set $\theta = \theta^{(t-1)}$;

For $j = 1, \cdots, d$, update θ_j from $\theta_j \sim f\left(\theta_j \mid \theta_{\setminus j}, y\right)$;

Set $\theta^{(t)} = \theta$ and save it as the generated set of values at $t+1$ iteration of the algorithm.

In this paper, we use WinBUGS to implement the Gibbs sampling. And we can estimate the parameters efficiently.

IV. Illustrative Example

In this Section, we implement the proposed models in Section II. The numerical data is taken from Meeker and Escobar [15] in Table C.14 which is reported in Lu and Meeker (1993) and read from Figure 4.52 in Bogdanoff and Kozin (1985). In the original data, 21 samples are tested for fatigue crack development and the measurements are taken at the same measurement times. The time unit is in million cycles. To demonstrate the bivariate degradation model, we choose 20 samples and the data will be treated as if half of it will be supposed that there is a product with two possible fatigue crack positions and this assumption is valid for every item tested. The data will be used is the data measured only until 0.09 million cycles. The item is considered failed if one of the two cracks size exceeds 1.6 in. Table 1 lists the crack data. Fig. 1 and Fig. 2 show the cumulative degradation and the degradation increments of the fatigue cracks size respectively.

Firstly, we consider modeling the data by a standard Gamma process (P1). The PDF of the degradation increment $\Delta x_{i,j}^{(k)}$ is denoted as (4). But from Fig.2, we can see that the degradation increment is not a constant each 0.01 million cycles, it continues to increase along with cycles, that is, the cumulative degradation is not a linear function of cycles. So, we also consider using Gamma process with a time scale transformation (P2) to model the data. We choose the transformation function as $\tau(t) = t^{\gamma}$.

Take Frank copula to describe the dependence of the two cracks and suppose that the copula parameter does not depend on cycles. The parameters can be estimated in terms of (5) by Bayesian MCMC method. Let

$$\boldsymbol{\theta}_{P1} = (v^{(1)}, u^{(1)}, v^{(1)}, u^{(2)}, \alpha),$$

and

$$\boldsymbol{\theta}_{P2} = (v^{(1)}, u^{(1)}, \gamma^{(1)}, v^{(1)}, u^{(2)}, \gamma^{(2)}, \alpha),$$

which are the parameters of the model with P1 and P2 respectively.

We use relatively non-informative priors for parameters $\boldsymbol{\theta}_{P1}$ and $\boldsymbol{\theta}_{P2}$ and estimate them by WinBUGS. During the simulation test, we find that the parameters are very stable after 50,000 iterations. So we generated 50,000 realizations of them from posterior. The last 40,000 were used in the estimation of mean, standard deviation, MCMC error and quantiles of the parameters. Table 2 lists the computational results. We also estimate the parameters without considering the dependence of the two fatigue cracks and list the results in Table 3. According to (2), we can compute the Kendall's tau easily for Frank copula. They are 0.55996 and 0.41046 for P1 and P2 respectively.

From Table 2 and Table 3, we know that some differences exist about the parameters if we consider the two cracks separately. It implies that the parameters of different PC may

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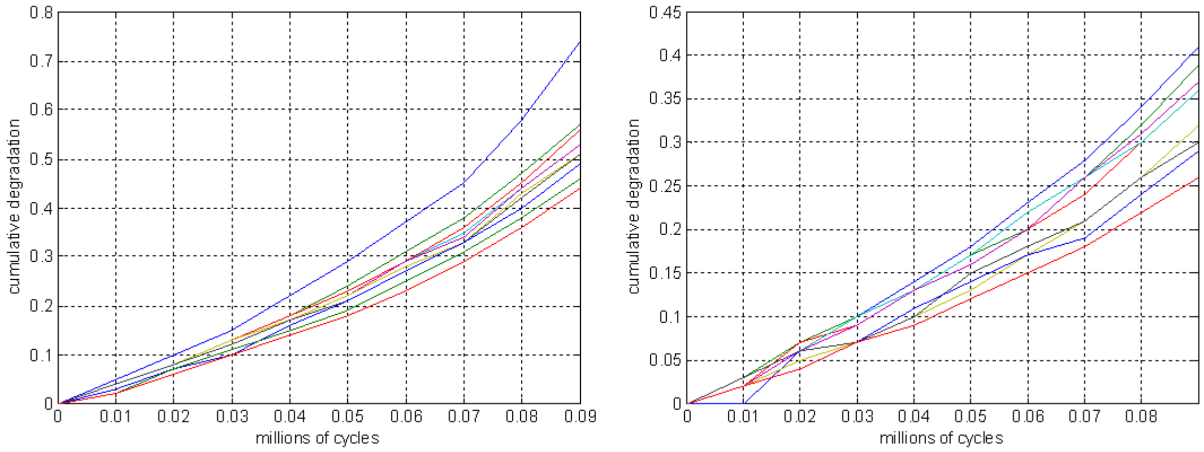
influence each other. So we think it is more rational to estimate them as a whole so that we can assess the product's reliability as accurate as possible.

We assess the reliability according to (6-8) considering the cracks to be dependent, independent and separate. Fig. 3 and Fig. 4 compare the product's marginal reliability of Crack A and Crack B. And Fig. 5 shows the product's reliability. We consider both independence and dependence for P1 and P2 in these figures.

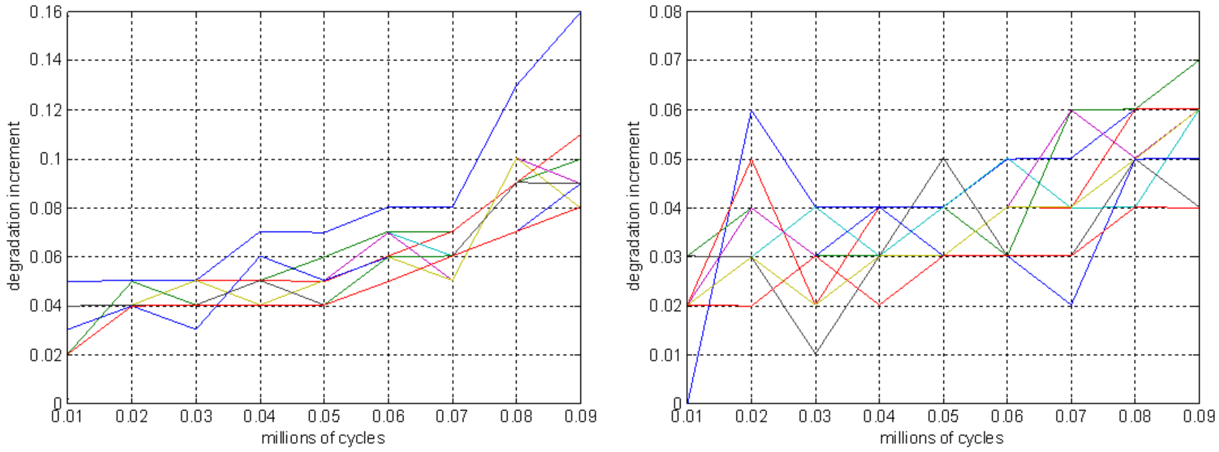
Table 1 Fatigue crack data (in) taken from Meeker and Escobar(1998)

Cracks	Item	Millions of Cycles									
		0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
Crack A	1	0.90	0.95	1.00	1.05	1.12	1.19	1.27	1.35	1.48	1.64
	2	0.90	0.94	0.98	1.03	1.08	1.14	1.21	1.28	1.37	1.47
	3	0.90	0.94	0.98	1.03	1.08	1.13	1.19	1.26	1.35	1.46
	4	0.90	0.94	0.98	1.03	1.07	1.12	1.19	1.25	1.34	1.43
	5	0.90	0.94	0.98	1.03	1.07	1.12	1.19	1.24	1.34	1.43
	6	0.90	0.94	0.98	1.03	1.07	1.12	1.18	1.23	1.33	1.41
	7	0.90	0.94	0.98	1.02	1.07	1.11	1.17	1.23	1.32	1.41
	8	0.90	0.93	0.97	1.00	1.06	1.11	1.17	1.23	1.30	1.39
	9	0.90	0.92	0.97	1.01	1.05	1.09	1.15	1.21	1.28	1.36
	10	0.90	0.92	0.96	1.00	1.04	1.08	1.13	1.19	1.26	1.34
Crack B	1	0.90	0.90	0.96	1.00	1.04	1.08	1.13	1.18	1.24	1.31
	2	0.90	0.93	0.97	1.00	1.03	1.07	1.10	1.16	1.22	1.29
	3	0.90	0.92	0.97	0.99	1.03	1.06	1.10	1.14	1.20	1.26
	4	0.90	0.93	0.96	1.00	1.03	1.07	1.12	1.16	1.20	1.26
	5	0.90	0.92	0.96	0.99	1.03	1.06	1.10	1.16	1.21	1.27
	6	0.90	0.92	0.95	0.97	1.00	1.03	1.07	1.11	1.16	1.22
	7	0.90	0.93	0.96	0.97	1.00	1.05	1.08	1.11	1.16	1.2
	8	0.90	0.92	0.94	0.97	1.01	1.04	1.07	1.09	1.14	1.19
	9	0.90	0.92	0.94	0.97	0.99	1.02	1.05	1.08	1.12	1.16
	10	0.90	0.92	0.94	0.97	0.99	1.02	1.05	1.08	1.12	1.16



(a) Crack A (b) Crack B

Fig.1 the cumulative degradation of fatigue cracks size



(a) Crack A (b) Crack B

Fig.2 the degradation increments of fatigue cracks size

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Table 2 Parameters estimation results considering the dependence

Process	Parameters	Results					
		mean	std	MC-error	2.5%	Median	97.5%
P1	$v^{(1)}$	695.9	102.5	5.603	515.0	688.3	909.1
	$u^{(1)}$	113.9	17.84	0.9582	82.33	112.6	151.2
	$v^{(2)}$	702.1	100.6	5.292	524.1	698.7	911.1
	$u^{(2)}$	184.0	27.98	1.452	133.8	183.1	242.4
	α	6.948	1.052	0.02876	5.05	6.906	9.153
P2	$v^{(1)}$	2797	480.3	28.87	1938	2753	3751
	$u^{(1)}$	217.8	31.02	1.585	159.1	216.3	280.2
	$\gamma^{(1)}$	1.313	0.04206	1.883e-3	1.232	1.313	1.396
	$v^{(2)}$	1809	343.2	20.27	1119	1804	2513
	$u^{(2)}$	267.1	41.36	2.075	189.9	265.6	353.0
	$\gamma^{(2)}$	1.240	0.04427	2.000e-3	1.152	1.240	1.326
	α	4.307	0.8826	0.02148	2.594	4.293	6.107

Table 3 Parameters estimation results without considering the dependence

Process	Cracks	Parameters	Results					
			mean	std	MC-error	2.5%	Median	97.5%
P1	Crack A	v	729.1	108.1	2.773	533.1	724.5	955.7
		u	122.8	18.86	0.4834	88.39	122.0	162.4
	Crack B	v	711.7	105.3	2.608	522.2	706.9	934.5
		u	192.3	29.49	0.7285	139.0	190.9	254.3
P2	Crack A	v	2895	455.6	14.80	2023	2888	3820
		u	226.6	31.06	0.9659	166.5	226.2	290.2
		γ	1.317	0.03949	8.634e-4	1.242	1.316	1.396
	Crack B	v	2002	336.6	11.11	1386	1983	2712
		u	288.0	40.96	1.296	211.2	286.8	371.0
		γ	1.260	0.04096	8.466e-4	1.182	1.259	1.342

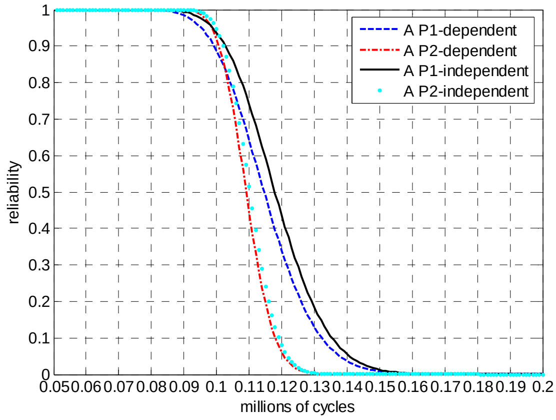

Fig. 3 product's marginal reliability of Crack A

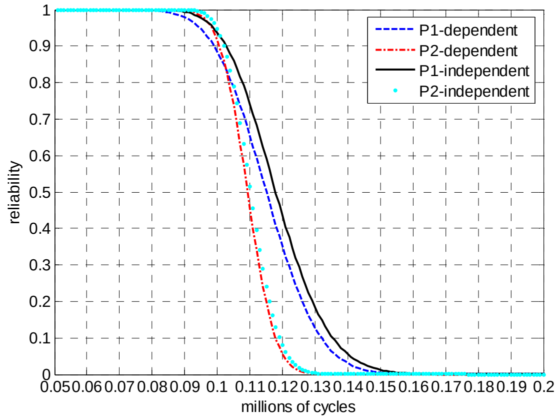

Fig. 5 product's reliability

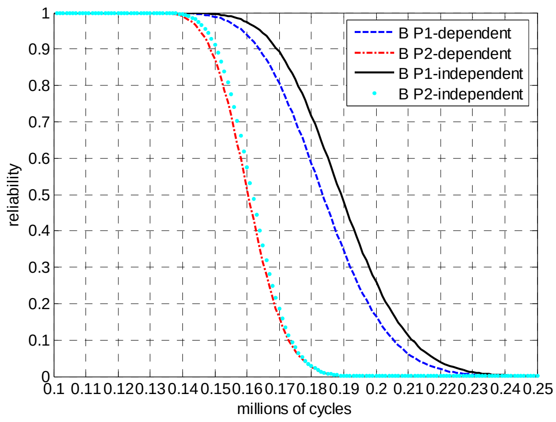

Fig. 4 product's marginal reliability of Crack B

In the following, we will draw a comparison between P1and P2 by Akaike information criterion (AIC). The AIC is defined by

$$AIC = -2 \times (\max log - likelihood) + 2m ,$$

where m is the number of unknown model parameters. The AIC is frequently used in engineering and statistics literature to give a guideline for a model selection. When there are several potential models available, the one with the smallest AIC among them can be selected as a good fitting model. Table 4 lists the AIC of P1 and P2 for the two cracks. From Table 4, we can see that P2 has smaller AIC for both of the two cracks. So we think that P2 can fit the data better and it is more rational to estimate the product's reliability by P2.

Table 4 the AIC of P1 and P2 for the two cracks

Process	Cracks	AIC
P1	Crack A	190.45
	Crack B	213.24
P2	Crack A	177.36
	Crack B	210.73

In this section, we present an example to validate the bivariate degradation model based on Gamma process. From the example, we can see that Bayesian MCMC method is an effective method to estimate the parameters of such complicated model. Comparing the reliability in Fig. 3-5, we believe that the results estimated from Gamma process with a time scale transformation are more rational. So Gamma

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process with a time scale transformation is the better choice for this example. Moreover, the product is considered failed if one of the two cracks size exceeds the threshold value. That means the two cracks can be regarded as a series system, so the reliability of the product should be decided by the crack A. Fig. 3-5 shows that the consequences of reliability estimation are consonant with this conclusion.

REFERENCES

[1] J. K. Sari, M. J. Newby, A. C. Brombacher and L. C. Tang, “Bivariate constant stress degradation model: LED lighting system reliability estimation with two-stage modeling”, *Qual. Reliab. Engng. Int.,* vol 25, 2009, pp.1067-1084.

[2] V. Crk, “Reliability assessment from degradation data”, *RAMS*, 2000, pp.155-161.

[3] P. Wang, D. W. Coit, “Reliability prediction based on degradation modeling for systems with mulitple degradation measures”, *RAMS*, 2004, pp.302-307.

[4] D. Xu, W. B. Zhao. “Reliability prediction using multivariate degradation data”, *RAMS*, 2005, pp.337-341.

[5] W. Huang, R. G. Askin, “Reliability analysis of electronic devices with multiple competing failure modes involving performance aging degradation”, *Qual. Reliab. Engng. Int. ,* vol 19, 2003, pp.241-254.

[6] V. Bagdonavicius, A. Bikelis and V. Kazakevicius, “Statistical analysis of linear degradation and failure time data with multiple failure modes”, *Lifetime Data Anal.*, vol 10, 2004, pp.65-81.

[7] V. Bagdonavicius, A. Bikelis, V. Kazakevicius and M. Nikulin. “Analysis of joint multiple failure mode and linear degradation data with renewals”, *J. Stat. Plan. Infer.*, vol 137, 2007, pp.2191-2207.

[8] J. K. Sari, “Multivariate degradation modeling and its application to reliability testing”, *Ph.D Thesis*, National University of Singapore. 2007.

[9] C. Park, W.J.Padgett, “Accelerated degradation models for failure based on geometric Brownian motion and Gamma processes”, *Lifetime Data Anal.*, vol 25, 2005, pp.511-527.

[10] C. Park, W. J. Padgett. “Stochastic degradation models with several accelerating variables”, *IEEE Trans. on Reliab.*, vol 55, 2006, pp.379-390.

[11] S. T. Tseng, N. Balakrishnan, C. C. Tsai, “Optimal step-stress accelerated degradation test plan for Gamma degradation process”, *IEEE Trans. on Reliab.*, to be published.

[12] G. A. Whitmore, F. Schenkelberg, “Modeling accelerated degradation data using Wiener diffusion with a time scale transformation”. *Lifetime Data Anal.*, vol 3, 1997, pp.27-45.

[13] R. B. Nelson, *An Introduction to Copulas*, 2nd ed, New York: Springer Science + Business Media, Inc. 2006.

[14] I. Ntzoufras, *Bayesian Modeling Using WinBUGS*. New Jersey: John Wiley & Sons, Inc. 2009, pp.71-72.

[15] W. Q. Meeker, L. A. Escobar, *Statistical Methods for Reliability Data*, New Jersey: John Wiley & Sons, Inc. 1998, pp.639.

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