Multidimensional Matrix Mathematics: Multidimensional Null and Identity Matrices and Multidimensional Matrix Outer and Inner Products, Part 3 of 6

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***Abstract*—This is the first series of research papers to define multidimensional matrix mathematics, which includes multidimensional matrix algebra and multidimensional matrix calculus. These are new branches of math created by the author with numerous applications in engineering, math, natural science, social science, and other fields. Cartesian and general tensors can be represented as multidimensional matrices or vice versa. Some Cartesian and general tensor operations can be performed as multidimensional matrix operations or vice versa. However, many aspects of multidimensional matrix math and tensor analysis are not interchangeable. Part 3 of 6 defines the multidimensional null matrix and multidimensional identity matrix. Also, the multidimensional matrix algebra operations for outer product and inner product are defined.**

***Index Terms*—multidimensional matrix math, multidimensional matrix algebra, multidimensional matrix calculus, matrix math, matrix algebra, matrix calculus, tensor analysis**

I. Introduction

Part 3 of 6 defines the multidimensional null matrix and multidimensional identity matrix. Also, part 3 of 6 defines the multidimensional matrix algebra operations for outer product and inner product.

II. Multidimensional Null Matrix

All elements of a multidimensional null matrix, which is represented by **NULL,** are zeros. The multidimensional null matrix can have any number of dimensions and any number of elements in each dimension. The dimensions of a multidimensional null matrix can be indicated through subscripted indices as in $\mathbf{NULL}_{s*t*u*\,...\,*z}$.

Addition of a multidimensional matrix **A** to a multidimensional null matrix with the same number of elements in each dimension results in the same multidimensional matrix **A.** That is, **A** + **NULL** = **A** where **A** and **NULL** have the same number of elements in each dimension.

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In the following example, a 4-D matrix **A** with dimensions of 3 * 2 * 1 * 3 is added to a 4-D null matrix $\mathbf{NULL}_{3*2*1*3}$ with the result being the same as the initial 4-D matrix **A** with dimensions of 3 * 2 * 1 * 3:

$$\left[\left[\begin{matrix}2&8\\4&10\\6&12\end{matrix}\right],\left[\begin{matrix}14&20\\16&22\\18&24\end{matrix}\right],\left[\begin{matrix}26&32\\28&34\\30&36\end{matrix}\right]\right]$$
$$+\left[\left[\begin{matrix}0&0\\0&0\\0&0\end{matrix}\right],\left[\begin{matrix}0&0\\0&0\\0&0\end{matrix}\right],\left[\begin{matrix}0&0\\0&0\\0&0\end{matrix}\right]\right]$$
$$=\left[\left[\begin{matrix}2&8\\4&10\\6&12\end{matrix}\right],\left[\begin{matrix}14&20\\16&22\\18&24\end{matrix}\right],\left[\begin{matrix}26&32\\28&34\\30&36\end{matrix}\right]\right]$$

Multidimensional matrix **A** subtracted by a multidimensional null matrix with the same number of elements in each dimension results in the same multidimensional matrix **A.** That is, **A** – **NULL** = **A** where **A** and **NULL** have the same number of elements in each dimension.

Multiplication of a multidimensional matrix **A** by a multidimensional null matrix that meets the conformability requirements for multidimensional matrix multiplication results in another multidimensional null matrix. That is, **A** * $\mathbf{NULL}_{b*c*u*v*\,...\,*z}$ = $\mathbf{NULL}_{a*c*u*v*\,...\,*z}$ where **A** has dimensions of $a * b * u * v * ... * z$. Also, $\mathbf{NULL}_{a*b*u*v*\,...\,*z}$ * **A** = $\mathbf{NULL}_{a*c*u*v*\,...\,*z}$ where **A** has dimensions of $b * c * u * v * ... * z$.

In the following example, a 3-D matrix with dimensions of 3 * 2 * 3 is multiplied by a 3-D null matrix $\mathbf{NULL}_{2*3*3}$ with the resulting 3-D null matrix $\mathbf{NULL}_{3*3*3}$ as shown:

$$\left[\begin{matrix}\left[\begin{matrix}2&8\\4&10\\6&12\end{matrix}\right]\\\left[\begin{matrix}14&20\\16&22\\18&24\end{matrix}\right]\\\left[\begin{matrix}26&32\\28&34\\30&36\end{matrix}\right]\end{matrix}\right]*\left[\begin{matrix}\left[\begin{matrix}0&0&0\\0&0&0\end{matrix}\right]\\\left[\begin{matrix}0&0&0\\0&0&0\end{matrix}\right]\\\left[\begin{matrix}0&0&0\\0&0&0\end{matrix}\right]\end{matrix}\right]=\left[\begin{matrix}\left[\begin{matrix}0&0&0\\0&0&0\\0&0&0\end{matrix}\right]\\\left[\begin{matrix}0&0&0\\0&0&0\\0&0&0\end{matrix}\right]\\\left[\begin{matrix}0&0&0\\0&0&0\\0&0&0\end{matrix}\right]\end{matrix}\right]$$

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III. Multidimensional Identity Matrix or Multidimensional Unit Matrix

The multidimensional identity matrix is also known as a multidimensional unit matrix. When a multidimensional matrix is multiplied by its multidimensional identity matrix, the multidimensional matrix product is identical to the original multidimensional matrix.

A. Description of Multidimensional Identity Matrices

A multidimensional identity matrix must have two or more dimensions. A multidimensional identity matrix is an *identity multiplier* for two of these dimensions. This means that if multiplying these two dimensions of a multidimensional matrix **A** and a multidimensional identity matrix **UNIT** that has the same number of dimensions, then the multidimensional matrix product is equal to the original multidimensional matrix **A.** When a multidimensional matrix is being multiplied in two dimensions by its multidimensional identity matrix, the multidimensional identity matrix must be an identity multiplier in these same two dimensions. A multidimensional identity matrix must have an equal number of elements in each of the two dimensions for which it is an identity multiplier.

B. Notation for Multidimensional Identity Matrices

A multidimensional identity matrix is indicated by **UNIT** in bold uppercase letters. Individual elements within a multidimensional identity matrix are indicated by $unit_{ijk\,.\,.\,.\,q}$. Note that if **I** were used to refer to multidimensional identity matrices as it is often used for identity matrices in classical matrix algebra, then the lowercase letter *i* would refer to individual elements within the multidimensional identity matrix and this would conflict with the subscripted index *i*.

The variable $N_{da}(\mathbf{A})$ refers to the number of elements in the first dimension being multiplied in multidimensional matrix **A,** the variable $N_{db}(\mathbf{A})$ refers to the number of elements in the second dimension being multiplied in multidimensional matrix **A,** the variable $N_{da}(\mathbf{UNIT})$ refers to the number of elements in the first dimension being multiplied in the multidimensional identity matrix, and the variable $N_{db}(\mathbf{UNIT})$ refers to the number of elements in the second dimension being multiplied in the multidimensional identity matrix.

The index $index_d(\mathbf{M})$ refers to the position in the *d*th dimension in multidimensional matrix **M.** Therefore, $index_{da}(\mathbf{A})$ refers to the position in the first dimension being multiplied in multidimensional matrix **A,** $index_{db}(\mathbf{A})$ refers to the position in the second dimension being multiplied in multidimensional matrix **A,** $index_{da}(\mathbf{UNIT})$ refers to the position in the first dimension being multiplied in the multidimensional identity matrix, and $index_{db}(\mathbf{UNIT})$ refers to the position in the second dimension being multiplied in the multidimensional identity matrix.

In subscripted parentheses with the notation for a multidimensional identity matrix, one can indicate the number of elements in each dimension of the multidimensional identity matrix and the two dimensions for which it is an identity multiplier. That is, in the notation $\mathbf{UNIT}_{(s\,*\,t\,*\,u\,*\,.\,.\,.\,*\,z,\;da,\;db)}$, the variables $s * t * u * . . . * z$ indicate the number of elements in each dimension of the multidimensional identity matrix, the variable *da* indicates the first dimension for which it is an identity multiplier, and the variable *db* indicates the second dimension for which it is an identity multiplier.

C. Elements of a Multidimensional Identity Matrix

In a multidimensional identity matrix, the elements are ones in positions where the indices of the two dimensions being multiplied are equal. That is, elements are ones in positions where $index_{da}(\mathbf{UNIT}) = index_{db}(\mathbf{UNIT})$. In a multidimensional identity matrix, the elements are zeros in positions where the indices of the two dimensions being multiplied are not equal. That is, elements are zeros in positions where $index_{da}(\mathbf{UNIT}) \neq index_{db}(\mathbf{UNIT})$.

For example, consider a 3-D identity matrix that has dimensions of 2 * 2 * 2 and that is an identity multiplier for the second dimension and third dimension. If the second dimension and third dimension of 3-D matrix **A** with dimensions of 2 * 2 * 2 is multiplied by multidimensional identity matrix $\mathbf{UNIT}_{(2\,*\,2\,*\,2,\;2,\;3)}$, then the multidimensional matrix product is equal to **A.** The elements $unit_{ijk}$ of the multidimensional identity matrix $\mathbf{UNIT}_{(2\,*\,2\,*\,2,\;2,\;3)}$ are ones where $j = k$ and the elements of the multidimensional identity matrix are zeros where $j \neq k$. That is, $unit_{111} = 1$, $unit_{211} = 1$, $unit_{122} = 1$, and $unit_{222} = 1$ whereas $unit_{121} = 0$, $unit_{221} = 0$, $unit_{112} = 0$, and $unit_{212} = 0$.

$$\mathbf{UNIT}_{(2\,*\,2\,*\,2,\;2,\;3)} = \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \end{bmatrix}$$

D. Conformability Requirements for Multiplication of a Multidimensional Matrix by Its Multidimensional Identity Matrix

In multidimensional matrix multiplication, a multidimensional matrix and its multidimensional identity matrix must meet conformability requirements. Therefore, the number of elements in the first dimension being multiplied in multidimensional matrix **A,** the number of elements in the second dimension being multiplied in multidimensional matrix **A,** the number of elements in the first dimension being multiplied in the multidimensional identity matrix, and the number of elements in the second dimension being multiplied in the multidimensional identity matrix must be equal. That is, $N_{da}(\mathbf{A}) = N_{db}(\mathbf{A}) = N_{da}(\mathbf{UNIT}) = N_{db}(\mathbf{UNIT})$. Also, the number of elements in each dimension not being multiplied in the first multidimensional matrix must equal the number of elements in the same dimension not being multiplied in its multidimensional identity matrix.

For example, if the second dimension and fourth dimension of multidimensional matrix **A** and multidimensional matrix **UNIT** are being multiplied, then $N_2(\mathbf{A}) = N_4(\mathbf{A}) = N_2(\mathbf{UNIT}) = N_4(\mathbf{UNIT})$, $N_1(\mathbf{A}) = N_1(\mathbf{UNIT})$, $N_3(\mathbf{A}) = N_3(\mathbf{UNIT})$, and $N_d(\mathbf{A}) = N_d(\mathbf{UNIT})$ for d >= 5.

E. Multiplication of a Multidimensional Matrix by Its Multidimensional Identity Matrix

Multidimensional matrix **A** multiplied by its multidimensional identity matrix **UNIT** results in the same multidimensional matrix **A** where **A** and **UNIT** meet the conformability requirements for multidimensional matrix multiplication. That is, **UNIT** * **A** = **A** and **A** * **UNIT** = **A** where **A** and **UNIT** meet the conformability requirements for multidimensional matrix multiplication.

In the following example, the first dimension and second dimension of a 4-D matrix **A** with dimensions of 3 * 3 * 2 * 2

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and its 4-D identity matrix, $\mathbf{UNIT}_{(3 * 3 * 2 * 2,\ 1,\ 2)}$, are multiplied with the result being the original 4-D matrix **A** with dimensions of 3 * 3 * 2 * 2. The multidimensional identity matrix has dimensions of 3 * 3 * 2 * 2 and is an identity multiplier for the first dimension and second dimension.

$$a_{ijkl} = \sum_{x=1}^{n} a_{ixkl} * unit_{xjkl}$$

where $n = \mathrm{N}_{db}(\mathbf{A}) = \mathrm{N}_{da}(\mathbf{UNIT})$.

$$\begin{bmatrix} \begin{bmatrix}1&7&13\\3&9&15\\5&11&17\end{bmatrix}, & \begin{bmatrix}0&6&12\\2&8&14\\4&10&16\end{bmatrix} \\ \begin{bmatrix}19&25&31\\21&27&33\\23&29&35\end{bmatrix}, & \begin{bmatrix}18&24&30\\20&26&32\\22&28&34\end{bmatrix} \end{bmatrix} *_{(1,\,2)}$$

$$\begin{bmatrix} \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, & \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \\ \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, & \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \end{bmatrix} =$$

$$\begin{bmatrix} \begin{bmatrix}1&7&13\\3&9&15\\5&11&17\end{bmatrix}, & \begin{bmatrix}0&6&12\\2&8&14\\4&10&16\end{bmatrix} \\ \begin{bmatrix}19&25&31\\21&27&33\\23&29&35\end{bmatrix}, & \begin{bmatrix}18&24&30\\20&26&32\\22&28&34\end{bmatrix} \end{bmatrix}$$

In the following example, the third dimension and fourth dimension of a 4-D matrix with dimensions of 2 * 1 * 3 * 3 and its 4-D identity matrix, $\mathbf{UNIT}_{(2 * 1 * 3 * 3,\ 3,\ 4)}$, are multiplied with the result being the original 4-D matrix with dimensions of 2 * 1 * 3 * 3. The multidimensional identity matrix has dimensions of 2 * 1 * 3 * 3 and is an identity multiplier for the third dimension and fourth dimension.

$$a_{ijkl} = \sum_{x=1}^{n} unit_{ixkl} * a_{xjkl}$$

where $n = \mathrm{N}_{db}(\mathbf{UNIT}) = \mathrm{N}_{da}(\mathbf{A})$.

$$\begin{bmatrix} \begin{bmatrix}1\\1\end{bmatrix}, & \begin{bmatrix}0\\0\end{bmatrix}, & \begin{bmatrix}0\\0\end{bmatrix} \\ \begin{bmatrix}0\\0\end{bmatrix}, & \begin{bmatrix}1\\1\end{bmatrix}, & \begin{bmatrix}0\\0\end{bmatrix} \\ \begin{bmatrix}0\\0\end{bmatrix}, & \begin{bmatrix}0\\0\end{bmatrix}, & \begin{bmatrix}1\\1\end{bmatrix} \end{bmatrix} *_{(3,\,4)} \begin{bmatrix} \begin{bmatrix}1\\2\end{bmatrix}, & \begin{bmatrix}7\\8\end{bmatrix}, & \begin{bmatrix}13\\14\end{bmatrix} \\ \begin{bmatrix}3\\4\end{bmatrix}, & \begin{bmatrix}9\\10\end{bmatrix}, & \begin{bmatrix}15\\16\end{bmatrix} \\ \begin{bmatrix}5\\6\end{bmatrix}, & \begin{bmatrix}11\\12\end{bmatrix}, & \begin{bmatrix}17\\18\end{bmatrix} \end{bmatrix} =$$

$$\begin{bmatrix} \begin{bmatrix}1\\2\end{bmatrix}, & \begin{bmatrix}7\\8\end{bmatrix}, & \begin{bmatrix}13\\14\end{bmatrix} \\ \begin{bmatrix}3\\4\end{bmatrix}, & \begin{bmatrix}9\\10\end{bmatrix}, & \begin{bmatrix}15\\16\end{bmatrix} \\ \begin{bmatrix}5\\6\end{bmatrix}, & \begin{bmatrix}11\\12\end{bmatrix}, & \begin{bmatrix}17\\18\end{bmatrix} \end{bmatrix}$$

IV. Multidimensional Matrix Outer Product

The multidimensional matrix outer product corresponds to the outer product of tensors. The notation ⊗ is used to indicate a multidimensional matrix outer product of two multidimensional matrices.

The multidimensional matrix outer product $\mathbf{A} \otimes \mathbf{B} = \mathbf{C}$ of multidimensional matrix **A** with *r* dimensions and multidimensional matrix **B** with *s* dimensions is a multidimensional matrix **C** with *r* + *s* dimensions like an outer product of tensors.

The indices of multidimensional matrix **A** are *ia, ja, ka, . . ., qa*. The indices of multidimensional matrix **B** are *ib, jb, kb, . . ., qb*. The indices of multidimensional matrix **C** are *ic, jc, kc, . . ., qc*. In this research paper, when two letters or more letters are used for each index, to avoid confusing the second or third letter of an index as a separate index, the indices for an element are separated by commas like in $a_{ia,\, ja,\, ka,\, \ldots,\, qa}$.

The elements of multidimensional matrix **C** are the products of the elements of multidimensional matrix **A** and multidimensional matrix **B** where the indices of each element $c_{ic,\, jc,\, kc,\, \ldots,\, qc}$ in **C** are determined by a concatenation of the indices for elements being multiplied together in **A** and **B** and with the indices of the element $a_{ia,\, ja,\, ka,\, \ldots,\, qa}$ in **A** being first and the indices of the element $b_{ib,\, jb,\, kb,\, \ldots,\, qb}$ in **B** being second like an outer product of tensors. That is, $c_{ic,\, jc,\, kc,\, \ldots,\, qc} = a_{ia,\, ja,\, ka,\, \ldots,\, qa} * b_{ib,\, jb,\, kb,\, \ldots,\, qb} = c_{ia,\, ja,\, ka,\, \ldots,\, qa,\; ib,\, jb,\, kb,\, \ldots,\, qb}$.

For example, consider the outer product of 2-D matrix **A** with dimensions of 2 * 2 and 3-D matrix **B** with dimensions of 2 * 3 * 2. The resulting 5-D matrix **C** has dimensions of 2 * 2 * 2 * 3 * 2 and each element in **C** is determined by $c_{ijklm} = a_{ij} * b_{klm}$.

$$\begin{bmatrix}1&2\\3&4\end{bmatrix} \otimes \begin{bmatrix} \begin{bmatrix}5&6&7\\8&9&10\end{bmatrix} \\ \begin{bmatrix}11&12&13\\14&15&16\end{bmatrix} \end{bmatrix} =$$

$$\begin{bmatrix} \begin{bmatrix} \begin{bmatrix}5&10\\15&20\end{bmatrix}, & \begin{bmatrix}6&12\\18&24\end{bmatrix}, & \begin{bmatrix}7&14\\21&28\end{bmatrix} \\ \begin{bmatrix}8&16\\24&32\end{bmatrix}, & \begin{bmatrix}9&18\\27&36\end{bmatrix}, & \begin{bmatrix}10&20\\30&40\end{bmatrix} \end{bmatrix} \\ \begin{bmatrix} \begin{bmatrix}11&22\\33&44\end{bmatrix}, & \begin{bmatrix}12&24\\36&48\end{bmatrix}, & \begin{bmatrix}13&26\\39&52\end{bmatrix} \\ \begin{bmatrix}14&28\\42&56\end{bmatrix}, & \begin{bmatrix}15&30\\45&60\end{bmatrix}, & \begin{bmatrix}16&32\\48&64\end{bmatrix} \end{bmatrix} \end{bmatrix}$$

This example corresponds to the outer product of a second order tensor and a third order tensor.

V. Multidimensional Matrix Inner Product

The multidimensional matrix inner product corresponds to the inner product of tensors. An inner product of two tensors is a contraction of the outer product with respect to two indices for different components of the tensors. Similarly, an inner product of two multidimensional matrices is a contraction of the outer product with respect to two indices for different components of the multidimensional matrices.

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For example, following is a multidimensional matrix **A** with indices i and j and a multidimensional matrix **B** with indices k, l, and m.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} \begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix} \\ \begin{bmatrix} 11 & 12 & 13 \\ 14 & 15 & 16 \end{bmatrix} \end{bmatrix}$$

Einstein summation convention is used in the ensuing description of the multidimensional matrix inner product to simply notation and be consistent with most descriptions of tensor inner product.

The inner product $a_{ij}b_{ilm}$ can be determined as follows:

$$a_{ij}b_{ilm} = a_{1j}b_{1lm} + a_{2j}b_{2lm} = c_{jlm}$$

This results in multidimensional matrix **C** with indices j, l, and m:

$$\mathbf{C} = \begin{bmatrix} \begin{bmatrix} 29 & 33 & 37 \\ 42 & 48 & 54 \end{bmatrix} \\ \begin{bmatrix} 53 & 57 & 61 \\ 74 & 84 & 90 \end{bmatrix} \end{bmatrix}$$

The inner product $a_{km}b_{klm}$ can be determined as follows:

$$a_{km}b_{klm} = a_{11}b_{1l1} + a_{12}b_{1l2} + a_{21}b_{2l1} + a_{22}b_{2l2} = d_l$$

This results in multidimensional matrix **D** with index l:

$$\mathbf{D} = \begin{bmatrix} 107 \\ 117 \\ 127 \end{bmatrix}$$

VI. Conclusion

Part 3 of 6 defined the multidimensional null matrix and multidimensional identity matrix. Also, part 3 of 6 defined the multidimensional matrix algebra operations for outer product and inner product.

Part 4 of 6 defines the multidimensional matrix algebra operations for transpose, determinant, and inverse. Also, part 4 of 6 defines multidimensional matrix symmetry and antisymmetry.

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