Fixed Points Results via Iterates of Four Maps in TVS-valued Cone Metric Spaces

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***Abstract*—: We obtain common fixed points of four mappings satisfying a contractive type condition by demonstrating the convergence of their sequence of iterates in TVS valued cone metric spaces. Our results generalize some well-known recent results in the literature.**

***Index Terms*— contractive type mapping; non-normal cones; fixed point ; cone metric space.**

I. INTRODUCTION AND PRELIMINARIES

Fixed point theorems are very important tools for providing evidence of the existence and uniqueness of solutions to various mathematical models (i.e., differential, integral and partial differential equations) representing phenomena happening in different fields, such as steady state temperature distribution, chemical equations, economic theories, financial analysis and biomedical research. The literature of the last four decades flourishes with results which discover fixed points of self and nonself nonlinear operators in a metric space. For most of them, their reference result is the Banach contraction theorem, which states that if X is a complete metric space and T a single valued contractive self mapping on X, then T has a unique fixed point in X. This theorem looks simple but plays a fundamental role in fixed point theory and has become even more important because being based on iteration, it can be easily implemented on a computer. Common fixed point theorems deals with the guarantee that a family $\{T_i \,:\, i \in \Omega\}$ of self mappings on a set X has one or more common fixed points, i.e., the system $x = T_i x \ (i \in \Omega)$ of functional equations has one or more simultaneous solutions. Recently Beg, Azam and Arshad [4], studied common fixed points of a pair of maps on topological vector space(TVS) valued cone metric spaces which is bigger than that of introduced by Huang and Zhang [5]. In this paper we obtain common fixed points of a pair of mappings satisfying Banach contractive condition without the assumption of normality in TVS-valued cone metric spaces. Our results improve and generalize some significant recent results(e.g., see [1,5,8,10]).

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Let (E, τ) be always a topological vector space (TVS) and P a subset of E. Then, P is called a cone whenever
(i) P is closed, non-empty and $P \neq \{0\}$,
(ii) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b,
(iii) $P \cap (-P) = \{0\}$.

For a given cone $P \subseteq E,$ we can define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P.$ $x < y$ will stand for $x \leq y$ and $x \neq y,$ while $x << y$ will stand for $y - x \in \operatorname{int} P,$ where $\operatorname{int} P$ denotes the interior of $P.$

Definition 1: Let X *be a non-empty set. Suppose the mapping* $d : X \times X \to E$ *satisfies*

(d_1) $0 \leq d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = 0$ *if and only if* $x = y,$
$(\mathbf{d}_2)$ $d(x, y) = d(y, x)$ *for all* $x, y \in X,$
(d_3) $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in X.$

Then d *is called a TVS-valued cone metric on* X *and* (X, d) *is called a TVS-valued cone metric space.*

If E is a real Banach space then (X, d) is called cone metric space [3].

Definition 2: Let (X, d) *be a TVS-valued cone metric space,* $x \in X$ *and* $\{x_n\}_{n \geq 1}$ *a sequence in* $X.$ *Then*

(i) $\{x_n\}_{n \geq 1}$ *converges to* x *whenever for every* $c \in E$ *with* $0 << c$ *there is a natural number* N *such that* $d(x_n, x) << c$ *for all* $n \geq N.$

We denote this by $\lim_{n \to \infty} x_n = x$ *or* $x_n \to x.$

(ii) $\{x_n\}_{n \geq 1}$ *is a Cauchy sequence whenever for every* $c \in E$ *with* $0 << c$ *there is a natural number* N *such that* $d(x_n, x_m) << c$ *for all* $n, m \geq N.$

(iii) (X, d) *is a complete cone metric space if every Cauchy sequence is convergent.*

A pair (f,T) of self-mappings on a TVS-valued cone metric space X is said to be compatible if, for any sequence $x_n \in X$ with $fx_n \to t, Tx_n \to t,$ and for arbitrary $c \in \operatorname{int} P,$ there exists a natural number n_0 such that

$$d(fTx_n;Tfx_n) << c; \text{ for all } n > n_0.$$

The pair (f,T) is said to be weakly compatible if they commute at their coincidence point (i.e., $fTx = Tfx$ whenever $fx = Tx$). A self-mapping T on a TVS-valued cone metric X is called continuous at a point $x_0 \in X$ if, for every sequence $x_n \in X$, $x_n \to x_0$ implies $Tx_n \to Tx_0$

II. MAIN RESULTS

Theorem 3: Let (X,d) be a complete TVS-valued cone metric space and the mappings $S,T,f,g : X \to X$ satisfy:

$$d(Sx,Ty) \le \lambda\, d(fx,gy)$$

for all $x,y \in X$ where $0 \le \lambda < 1$. If

$$SX \subseteq gX, TX \subseteq fX,$$

f is continuous, (S,f) is compatible and (T,g) is weakly compatible, then $S,T,$ f and g have a unique common fixed point.

Proof: Let x_0 be an arbitrary point in X, Choose a point x_1 in X such that $y_1 = gx_1 = Sx_0$. This can be done since $SX \subseteq gX$. Similarly, choose a point x_2 in X such that $y_2 = fx_2 = Tx_1$. Continuing this process and having chosen x_n in X. We obtain x_{n+1} in X such that

$$\begin{aligned} y_{2k+1} &= gx_{2k+1} = Sx_{2k} \\ y_{2k+2} &= fx_{2k+2} = Tx_{2k+1}, k = 0,1,2,\ldots. \end{aligned}$$

Then,

$$\begin{aligned} d(gx_{2k+1},fx_{2k+2}) &= d(Sx_{2k},Tx_{2k+1}) \\ &\le \lambda d(fx_{2k},gx_{2k+1}). \end{aligned}$$

Similarly,

$$\begin{aligned} d(fx_{2k+2},gx_{2k+3}) &= d(Sx_{2k+2},Tx_{2k+1}) \\ &\le \lambda(fx_{2k+2},gx_{2k+1}) \\ &\le \lambda^2 d(fx_{2k},gx_{2k+1}). \end{aligned}$$

Now by induction, we obtain for each $k = 0,1,2,...,$

$$d(fx_{2k+2},gx_{2k+3}) \le \lambda^{2k+2} d(fx_0,gx_1).$$

For all $n,$ we have

$$\begin{aligned} d(y_{n+1},y_{n+2}) &\le \lambda d(y_n,y_{n+1}) \\ &\le \ldots \le \lambda^{n+1} d(y_0,\, y_1). \end{aligned}$$

It follows that for $m > n,$

$$\begin{aligned} d(y_m,\; y_n) &\le d(y_n,y_{n+1}) + d(y_{n+1},y_{n+2}) + ... \\ &\quad + d(y_{m-1},y_m) \\ &\le \left[\lambda^n + \lambda^{n+1} + ... + \lambda^{m-1}\right] d(y_0,y_1) \\ &\le \left[\frac{\lambda^n}{1-\lambda}\right] d(y_0,y_1). \end{aligned}$$

Let $\mathbf{0} << c$ be given, choose a symmetric neighborhood V of 0 such that $c + V \subseteq intP$ Also, choose a natural number N_1 such that $\left[\frac{\lambda^n}{1-\lambda}\right] d(y_0,y_1) \in V,$ for all $n \ge N_1$. Then, $\frac{\lambda^n}{1-\lambda} d(y_1,y_0) << c,$ for all $n \ge N_1$. Thus,

$$d(y_m,y_n) \le \left[\frac{\lambda^n}{1-\lambda}\right] d(y_0,y_1) << c,$$

for all $m > n.$ Therefore, $\{y_n\}_{n\ge 1}$ is a Cauchy sequence. Since X is complete, there exists $z \in X$ such that $y_n \to z.$ For its subsequence we obtain,

$$gx_{2k+1} \to z,\, fx_{2k+2} \to z, Sx_{2k} \to z \text{ and } Tx_{2k+1} \to z.$$

Since f is continuous therefore

$$fgx_{2k+1} \to fz,\, ffx_{2k+2} \to fz,\, fSx_{2k} \to fz \text{ and}$$
$$fTx_{2k+1} \to fz.$$

As $fSx_{2k} \to fz$ and (S,f) is compatible, for arbitrary $c \in \operatorname{int} P,$ there exists a natural numbers n_1 and n_2 such that

$$d\left(Sfx_{2k},\, fSx_{2k}\right) << \frac{c}{2} \text{ for all } k > n_1$$

and

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$$d\left(fSx_{2k}, fz\right) << \frac{c}{2} \text{ for all } k > n_2.$$

It implies that

$$\left(Sfx_{2k}, fz\right) \le \left(Sfx_{2k}, fSx_{2k}\right) + \left(fSx_{2k}, fz\right)$$
$$<< \frac{c}{2} + \frac{c}{2} = c \text{ for all } k > \max\{n_1, n_2\},$$

that is $Sfx_{2k} \to fz$. Choose natural numbers $N_1, N_2, N_3,$ and N_4 such that

$$d\left(fz, Sfx_{2k}\right) << \frac{(1-\lambda)c}{4} \text{ for all } k \ge N_1,$$

$$d\left(ffx_{2k}, fz\right) << \frac{(1-\lambda)c}{4} \text{ for all } k \ge N_2,$$

$$d\left(z, gx_{2k+1}\right) << \frac{(1-\lambda)c}{4} \text{ for all } k \ge N_3$$

and

$$d\left(Tx_{2k+1}, z\right) << \frac{(1-\lambda)c}{4} \text{ for all } k \ge N_4.$$

For $k \ge \max\{N_1, N_2, N_3, N_4\}$, by performing a simple calculation we can have

$$d\left(fz, z\right) << \frac{c}{m}, \text{for all } m \ge 1.$$

So, $\frac{c}{m} - d\left(fz, z\right) \in P,$ for all $m \ge 1.$ Since $\frac{c}{m} \to \mathbf{0}$ (as $m \to \infty$) and P is closed, $-d(gu, Su) \in P.$ But $d\left(fz, z\right) \in P.$ Therefore, $d\left(fz, z\right) = \mathbf{0}$. Hence $fz = z.$ To assert $Sz = z,$ choose natural numbers $N_4, N_5,$ such that

$$d\left(z, gx_{2k+1}\right) << \frac{c}{2\lambda} \text{ for all } k \ge N_5$$

and

$$d\left(Tx_{2k+1}, z\right) << \frac{c}{2} \text{ for all } k \ge N_6.$$

Now , for $k \ge \max\{N_5, N_6\}$ we have

$$\begin{aligned} d(Sz,z) &\le d(Sz, Tx_{2k+1}) + d(Tx_{2k+1}, z) \\ &\le \lambda d(fz, gx_{2k+1}) + d(Tx_{2k+1}, z) \\ &\le \lambda d(z, gx_{2k+1}) + d(Tx_{2k+1}, z) \\ &\le \lambda \frac{c}{2\lambda} + \frac{c}{2} = c. \end{aligned}$$

By a similar argument (as in the proof of $fz = z$), we have $Sz = z.$ Now we shall show $Tz = gz.$ As $SX \subseteq gX,$ there exists $u \in X$ such that $z = Sz = gu.$ Since,

$$\begin{aligned} d(Tu, gu) &\le d(Tu, Sz) \le \lambda d(gu, fz) \\ &\le \lambda d(z, z) = 0, \end{aligned}$$

therefore $z = Tu = gu$ and weakly compatibility of T and g implies that

$$gz = gTu = Tgu = Tz.$$

To see $Tz = z,$ consider,

$$d(Tz,z) = d(Tz,Sz) \le \lambda d(gz,fz) = \lambda d(Tz,z).$$

It yields $Tz = z,$ which further implies $gz = z.$ Hence,

$$z = gz = fz = Sz = Tz.$$

Example 4: Let $X = [0,1]$ *and* E *be the set of all real valued functions on* X *which also have continuous derivatives on* X*. Then* E *is a vector space over* $\mathbb{R}$ *under the following operations:*

$$(x+y)(t) = x(t) + y(t), \ (\alpha x)(t) = \alpha x(t),$$

for all $x, y \in E, \alpha \in \mathbb{R}.$ *Let* τ *be the strongest vector (locally convex) topology on* E*. Then* (E, τ) *is a topological vector space which is not normable and is not even metrizable. Define* $d : X \times X \to E$ *as follows:*

$$(d(x,y))(t) = |x-y|e^t,$$

$$P = \{(x \in E : x(t) \ge 0 \text{ for all } t \in X\}.$$

Then (X, d) *is a TVS-valued cone metric space. Let* $S, T, f, g : X \to X$ *be such that*

$$Sx = \frac{x}{x+10}, Tx = \frac{x}{x+16}, \ fx = \frac{x}{5} \text{ and } gx = \frac{x}{8}.$$

For $x, y \in X,$ *we have*

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$$\begin{aligned}&\left|\frac{x}{x+10}-\frac{y}{y+16}\right|e^t\\&=\left|\frac{x(y+16)-y(x+10)}{(x+10)(y+16)}\right|e^t\\&\le\left|\frac{16x-10y}{160}\right|e^t\\&\le\frac{1}{2}\left|\frac{x}{5}-\frac{y}{8}\right|e^t,\end{aligned}$$

that is

$$d(Sx,Ty)(t)\le\frac{1}{2}d(fx,gy)(t) \text{ for all } t\in X.$$

It suffices to assume $\lambda=\frac{1}{2}$, *in order to satisfy all assumptions of the above theorem.*

If in Theorem 3, we choose $S=T$ and $f=g$, we obtain the following corollary.

Corollary 5: Let (X,d) *be a complete TVS-valued cone metric space and the mappings* $T\ f\ :\ X\to X$ *satisfy:*

$$d(Tx,Ty)\le\lambda\, d(fx,fy)$$

for all $x,y\in X$ *where* $0\le\lambda<1$. *If*

$$TX\subseteq fX,$$

f *is continuous,* (T,f) *is compatible then* T *and* f *have a unique common fixed point.*

In the following result continuity of f is not required whereas completeness of X is replaced with the completeness of TX or fX.

Theorem 6: Let (X,d) *be a TVS-valued cone metric space and the mappings* $T\ f\ :\ X\to X$ *satisfy:*

$$d(Tx,Ty)\le\lambda\, d(fx,fy)$$

for all $x,y\in X$ *where* $0\le\lambda<1$. *If*

$$TX\subseteq fX,$$

TX *or* fX *is complete,* (T,f) *is compatible then* T *and* f *have a unique common fixed point.*

Proof: As in the proof of Theorem 3, construct a Cauchy sequence $\{fx_n\}_{n\ge1}$ in fX such that

$$\begin{aligned}y_{2k+1}&=fx_{2k+1}=Tx_{2k}\\y_{2k+2}&=fx_{2k+2}=Tx_{2k+1}, k=0,1,2,\dots.\end{aligned}$$

By completeness of fX, there exists $u,v\in X$ such that $y_n\to v=fu$ (this holds also if TX is complete with $v\in TX$). Choose a natural number N such that

$$d(y_n,v)<<\frac{c}{2} \text{ for all } n\ge N.$$

Hence, for all $n\ge N$

$$\begin{aligned}d(fu,Tu)&\le d(fu,y_{2n+2})+d(y_{2n+2},Tu)\\&\le d(v,y_{2n+2})+d(Tx_{2n+1},Tu)\\&\le d(v,y_{2n+2})+\lambda d(fx_{2n+1},\ fu)\\&\le d(v,\ y_{2n+2})+d(y_{2n+1},v)\\&<<\frac{c}{2}+\frac{c}{2}=c.\end{aligned}$$

Thus,

$$d(fu,Tu)<<\frac{c}{m}, \text{for all } m\ge1.$$

So, $\frac{c}{m}-d(fu,Tu)\in P$, for all $m\ge1$. Since $\frac{c}{m}\to0\ (as\ m\to\infty)$ and P is closed, $-d(fu,Su)\in P$, but $P\cap(-P)=\{0\}$. Therefore, $d(fu,Tu)=0$. Hence

$$v=fu=Tu.$$

By weakly compatibility of (T,f), we have

$$Tv=Tfu=fTu=fv.$$

Then

$$d(Tv,v)=d(Tv,Tu)\le\lambda d(fv,fu)=\lambda d(Tv,v).$$

Thus v is a unique common fixed point of T and f.

Corollary 7: Let (X,d) *be a TVS-valued cone metric space and the mappings* $T\ f\ :\ X\to X$ *satisfy:*

$$TX\subseteq fX \text{ and } d(T^nx,T^ny)\le\lambda d(fx,fy)$$

for all $x,y\in X$ *where* $0\le\lambda<1$. *Then* T *and* f *have a unique common fixed point, if* $Tf=fT$ *and one of the following conditions is satisfied:*

- X is complete and f is continuous

- fX is complete
- TX is complete.

Proof: By Corollary 5 and Theorem 6, we obtain $v \in X$ such that

$$fv = T^n v = v.$$

The result then follows from the fact that

$$\begin{aligned} d(Tv,v) &= d(TT^n v, T^n v) = d(T^n Tv, T^n v) \\ &\leq \lambda d(fTv, fv) = \lambda d(Tfv, fv) \\ &= \lambda d(Tv,v). \end{aligned}$$

Example 8: Let $X = C([1,3],\mathrm{R})$, (E,τ) *is the topological vector space of Example 4. Define* $d : X \times X \to E$ *as follows:*

$$d(x,y)(t) = \left(\sup_{s\in[1,3]} |x(s) - y(s)| \right) e^t$$

$$P = \{(x \in E : x(t)?0 \text{ for all } t \in X\}.$$

Then (X,d) *is a TVS-valued cone metric space. Define* $T : X \to X$ *by*

$$T(x(s)) = 4 + \int_1^s (x(u) + u^2) e^{u-1} du,$$

$$f(x(s)) = x(s).$$

For $x, y \in X$

$$\begin{aligned} d(Tx,Ty)(t) &= \left(\sup_{s\in[1,3]} |Tx(s) - Ty(s)| \right) e^t \\ &\leq \left(\int_1^3 \sup_{s\in[1,3]} |(x(u) - y(u))| e^2 du \right) e^t \\ &= 2e^2 d(x,y) e^t. \end{aligned}$$

Similarly,

$$d(T^n x, T^n y)(e^t) \leq e^{2n} \frac{2^n}{n!} d(x,y)(e^t).$$

Note that

$$e^{2n} \frac{2^n}{n!} = \frac{53}{100} \text{ if } n = 38.$$

Thus for $\lambda = \frac{53}{100}$, $n = 38$, *all conditions of Corollary 7 are satisfied and so* T *has a unique fixed point, which is the unique solution of the integral equation:*

$$x(s) = 4 + \int_1^s (x(u) + u^2) e^{u-1} du$$

or the differential equation:

$$x'(s) = (x + s^2) e^{s-1}, \quad s \in [1,3], \quad x(1) = 4.$$

REFERENCES

[1] M. Abbas and G. Jungck, Common fixed point results for non commuting mappings without continuity in cone metric spaces, J. Math. Anal. Appl., 341(2008), 416-420.
[2] M. Arshad, A. Azam and P. Vetro, Some common fixed point results in cone metric spaces, Fixed Point Theory Appl. 2009 (2009), 11 pages., Article ID 493965.
[3] A. Azam, M. Arshad and I. Beg, Common fixed points of two maps in cone metric spaces, Rend. Circ. Mat. Palermo, 57 (2008) 433--441.
[4] I. Beg, A. Azam and M. Arshad, Common fixed points for maps on topological vector space valued cone metric spaces, Internat. J. Math. and Math. Sciences, 2009(2009) 8 pages, Article ID 560264.
[5] L.-G. Huang and X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 332 (2007) 1468--1476.
[6] D. Klim, D. Wardowski, Dynamic processes and fixed points of set-valued nonlinear contractions in cone metric spaces, Nonlinear Analysis, 71 (2009) 5170-5175.
[7] D. Ilic and V. Rakocevic, Common fixed points for maps on cone metric space, J. Math. Anal. Appl., 341(2008), 876--882.
[8] S. Rezapour and R. Hamlbarani, Some notes on paper "Cone metric spaces and fixed point theorems of contractive mappings.", J. Math. Anal. Appl., 345 (2008) 719--724.
[9] S. Radenovic and B.E. Rhoades, Fixed point theorem for two non-self mappings in cone metric spaces, Comp. Math. Appl.,57 (2009) 1701--1707.
[10] Du Wei-Shih: A note on cone metric fxed point theory and its equivalence, Nonlinear Analysis, 72 (2010) 2259-2261.

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