Fixed Points Results via Iterates of Four Maps in TVS-valued Cone Metric Spaces

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Abstract—We obtain common fixed points of four mappings satisfying a contractive type condition. Our results generalize some well-known recent results in the literature.

Index Terms—contractive type mapping; non-normal cones; fixed point; cone metric space.

1. INTRODUCTION AND PRELIMINARIES

Fixed point theorems are important tools for providing evidence of the existence and uniqueness of solutions to various mathematical models (i.e., differential, integral and partial differential equations) representing phenomena happening in different fields, such as steady state temperature distribution, chemical equations, economic theories, financial analysis and biomedical research. The literature of the last four decades flourishes with results which discover fixed points of self and nonself nonlinear operators in a metric space. For most of them, their reference result is the Banach contraction theorem, which states that if \( X \) is a complete metric space and \( T \) a single valued contractive self mapping on \( X \), then \( T \) has a unique fixed point in \( X \). This theorem looks simple but plays a fundamental role in fixed point theory and has become even more important because being based on iteration, it can be easily implemented on a computer. Common fixed point theorems deals with the guarantee that a family \( \{T_i : i \in \Omega\} \) of self mappings on a set \( X \) has one or more common fixed points, i.e., the system \( x = T_i x \ (i \in \Omega) \) of functional equations has one or more simultaneous solutions. Recently Beg, Azam and Arshad [4], studied common fixed points of a pair of maps on topological vector space (TVS) valued cone metric spaces which is bigger than that of introduced by Huang and Zhang [5]. In this paper we common fixed points of a pair of mappings satisfying Banach contractive condition without the assumption of normality in TVS-valued cone metric spaces. Our results improve and generalize some significant recent results (e.g., see [1,5,8,10]).

Let \( (E, \tau) \) be always a topological vector space (TVS) and \( P \) a subset of \( E \). Then, \( P \) is called a cone whenever

(i) \( P \) is closed, non-empty and \( P \neq \{0\} \),

(ii) \( ax + by \in P \) for all \( x, y \in P \) and non-negative real numbers \( a, b \),

(iii) \( P \cap (-P) = \{0\} \).

For a given cone \( P \subseteq E \), we can define a partial ordering \( \leq \) with respect to \( P \) by \( x \leq y \) if and only if \( y - x \in P \). \( x < y \) will stand for \( x \leq y \) and \( x \neq y \), while \( x << y \) will stand for \( y - x \in \text{int } P \), where \( \text{int } P \) denotes the interior of \( P \).

Definition 1: Let \( X \) be a non-empty set. Suppose the mapping \( d : X \times X \to E \) satisfies

\[
\begin{align*}
(d_1) \quad & 0 \leq d(x, y) \quad \text{for all } x, y \in X \\
& d(x, y) = 0 \quad \text{if and only if } x = y, \\
(d_2) \quad & d(x, y) = d(y, x) \quad \text{for all } x, y \in X, \\
(d_3) \quad & d(x, y) \leq d(x, z) + d(z, y) \quad \text{for all } x, y, z \in X.
\end{align*}
\]

Then \( d \) is called a TVS-valued cone metric on \( X \) and \( (X, d) \) is called a TVS-valued cone metric space.

If \( E \) is a real Banach space then \( (X, d) \) is called cone metric space [3].

Definition 2: Let \( (X, d) \) be a TVS-valued cone metric space, \( x \in X \) and \( \{x_n\}_{n \geq 1} \) a sequence in \( X \). Then

(i) \( \{x_n\}_{n \geq 1} \) converges to \( x \) whenever for every \( c \in E \) with \( 0 << c \) there is a natural number \( N \) such that \( d(x_n, x) << c \) for all \( n \geq N \).

We denote this by \( \lim_{n \to \infty} x_n = x \) or \( x_n \to x \).

(ii) \( \{x_n\}_{n \geq 1} \) is a Cauchy sequence whenever for every \( c \in E \) with \( 0 << c \) there is a natural number \( N \) such that \( d(x_n, x_m) << c \) for all \( n, m \geq N \).

(iii) \( (X, d) \) is a complete cone metric space if every Cauchy sequence is convergent.
A pair \((f, T)\) of self-mappings on a TVS-valued cone metric space \(X\) is said to be compatible if, for any sequence \(x_n \in X\) with \(fx_n \to t, Tx_n \to t\), and for arbitrary \(c \in \text{int } P\), there exists a natural number \(n_0\) such that
\[
d(fTx_n ; Tfx_n) < c; \text{ for all } n > n_0.
\]
The pair \((f, T)\) is said to be weakly compatible if they commute at their coincidence point (i.e., \(fTx = Tfx\) whenever \(fx = Tx\)). A self-mapping \(T\) on a TVS-valued cone metric \(X\) is called continuous at a point \(x_0 \in X\) if, for every sequence \(x_n \in X\) such that \(x_n \to x_0\) implies \(Tx_n \to Tx_0\).

II. MAIN RESULTS

Theorem 3: Let \((X, d)\) be a complete TVS-valued cone metric space and the mappings \(S, T, f, g : X \to X\) satisfy:
\[
d(Sx, Ty) \leq \lambda d(fx, gy)
\]
for all \(x, y \in X\) where \(0 \leq \lambda < 1\). If \(SX \subseteq gX, TX \subseteq fX,\) \(f\) is continuous, \((S, f)\) is compatible and \((T, g)\) is weakly compatible, then \(S, T, f\) and \(g\) have a unique common fixed point.

Proof: Let \(x_0\) be an arbitrary point in \(X\), choose a point \(x_1\) in \(X\) such that \(y_1 = gx_1 = Sx_0\). This can be done since \(SX \subseteq gX\). Similarly, choose a point \(x_2\) in \(X\) such that \(y_2 = fx_2 = Tx_1\). Continuing this process and having chosen \(x_n\) in \(X\). We obtain \(x_{n+1}\) in \(X\) such that
\[
y_{2k+1} = gx_{2k+1} = Sx_{2k} \\
y_{2k+2} = fx_{2k+2} = Tx_{2k+1}, k = 0, 1, 2, \ldots .
\]
Then,
\[
d(gx_{2k+1}, fx_{2k+2}) = d(Sx_{2k}, Tx_{2k+1}) \\
\leq \lambda d(fx_{2k}, gx_{2k+1}).
\]

Similarly,
\[
d(fx_{2k+2}, gx_{2k+3}) = d(Sx_{2k+2}, Tx_{2k+3}) \\
\leq \lambda (fx_{2k+2}, gx_{2k+3}) \\
\leq \lambda^2 d(fx_{2k}, gx_{2k+1}).
\]
Now by induction, we obtain for each \(k = 0, 1, 2, \ldots\),
\[
d(fx_{2k+2}, gx_{2k+3}) \leq \lambda^{2k+2} d(fx_0, gx_1).
\]
For all \(n\), we have
\[
d(y_{n+1}, y_{n+2}) \leq \lambda d(y_n, y_{n+1}) \\
\leq \ldots \leq \lambda^{n+1} d(y_0, y_1).
\]
It follows that for \(m > n\),
\[
d(y_m, y_n) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \ldots + d(y_{m-1}, y_m) \\
\leq \left[\frac{\lambda^n + \lambda^{n+1} + \ldots + \lambda^{m-1}}{1 - \lambda}\right] d(y_0, y_1).
\]
Let \(0 << c\) be chosen, choose a symmetric neighborhood \(V\) of \(0\) such that \(c + V \subseteq \text{int } P\). Also, choose a natural number \(N_1\) such that \(\frac{x}{\lambda^n} d(y_0, y_1) \subseteq V\) for all \(n \geq N_1\). Then, \(\frac{x}{\lambda^n} d(y_1, y_0) << c\), for all \(n \geq N_1\). Thus,
\[
d(y_m, y_n) \leq \left[\frac{\lambda^n}{1 - \lambda}\right] d(y_0, y_1) << c,
\]
for all \(m > n\). Therefore, \(\{y_n\}_{n=1}^\infty\) is a Cauchy sequence. Since \(X\) is complete, there exists \(z \in X\) such that \(y_n \to z\). For its subsequence we obtain,
\[
gx_{2k+1} \to z, fx_{2k+2} \to z, Sx_{2k} \to z \text{ and } Tx_{2k+1} \to z.
\]
Since \(f\) is continuous therefore
\[
fgx_{2k+1} \to fz, ffx_{2k+2} \to fz, fSx_{2k} \to fz \text{ and } fTx_{2k+1} \to fz.
\]
As \(fSx_{2k} \to fz\) and \((S, f)\) is compatible, for arbitrary \(c \in \text{int } P\), there exists a natural numbers \(n_1\) and \(n_2\) such that
\[
d(Sfx_{2k}, fSx_{2k}) << \frac{c}{2} \text{ for all } k > n_1
\]
and
\[d(fSx_{2k}, fz) << \frac{c}{2} \text{ for all } k > n_2.\]

It implies that
\[
(Sfx_{2k}, fz) \leq (Sfx_{2k}, fSx_{2k}) + (fSx_{2k}, fz) \\
<< \frac{c}{2} + \frac{c}{2} = c \text{ for all } k > \max\{n_1, n_2\},
\]

that is \(Sfx_{2k} \to fz\). Choose natural numbers \(N_1, N_2, N_3, \) and \(N_4\) such that
\[
d( fz, Sfx_{2k}) << \frac{(1-\lambda)c}{4} \text{ for all } k \geq N_1,\\
d( ffx_{2k}, fz) << \frac{(1-\lambda)c}{4} \text{ for all } k \geq N_2,\\
d( z, gx_{2k+1}) << \frac{(1-\lambda)c}{4} \text{ for all } k \geq N_3
\]

and
\[
d(Tx_{2k+1}, z) << \frac{(1-\lambda)c}{4} \text{ for all } k \geq N_4.
\]

For \(k \geq \max\{N_1, N_2, N_3, N_4\}\), by performing a simple calculation we can have
\[
d(fz, z) << \frac{c}{m} \text{ for all } m \geq 1.
\]

So, \(
\frac{c}{m} - d(fz, z) \in P, \text{ for all } m \geq 1. \) Since \(\frac{c}{m} \to 0\) as \(m \to \infty\) and \(P\) is closed,

\(-d(gu, Su) \in P. \) But \(d(fz, z) \in P. \) Therefore, \(d(fz, z) = 0.\) Hence \(fz = z. \) To assert \(Sz = z, \) choose natural numbers \(N_4, N_5, \) such that
\[
d(z, gx_{2k+1}) << \frac{c}{2\lambda} \text{ for all } k \geq N_5
\]

and
\[
d(Tx_{2k+1}, z) << \frac{c}{2} \text{ for all } k \geq N_6.
\]

Now, for \(k \geq \max\{N_5, N_6\}\) we have
\[
d(Sz, z) \leq d(Sz, Tx_{2k+1}) + d(Tx_{2k+1}, z) \\
\leq \lambda d(fz, gx_{2k+1}) + d(Tx_{2k+1}, z) \\
\leq \lambda d(z, gx_{2k+1}) + d(Tx_{2k+1}, z) \\
\leq \lambda \frac{c}{2\lambda} + \frac{c}{2} = c.
\]

By a similar argument (as in the proof of \(fz = z\)), we have \(Sz = z. \) Now we shall show \(Tz = gz. \) As
\[SX \subseteq gX, \text{ there exists } u \in X \text{ such that } z = Sz = gu. \] Since,
\[
d(Tu, gu) \leq d(Tu, Sz) \leq \lambda d(gu, fz) \\
\leq \lambda d(z, z) = 0,
\]

therefore \(z = Tu = gu\) and weakly compatibility of \(T\) and \(g\) implies
\[gz = gTu = Tgu = Tz.\]

To see \(Tz = z, \) consider,
\[
d(Tz, z) = d(Tz, Sz) \leq \lambda d(gz, fz) = \lambda d(Tz, z).
\]

It yields \(Tz = z, \) which further implies \(gz = z. \) Hence,
\[z = gz = fz = Sz = Tz.
\]

Example 4: Let \(X = [0, 1]\) and \(E\) be the set of all real valued functions on \(X\) which also have continuous derivatives on \(X.\) Then \(E\) is a vector space over \(\mathbb{R}\) under the following operations:
\[(x + y)(t) = x(t) + y(t), \quad (ax)(t) = ax(t),\]

for all \(x, y \in E, a \in \mathbb{R}. \) Let \(\tau\) be the strongest vector (locally convex) topology on \(E.\) Then \((E, \tau)\) is a topological vector space which is not normable and is not even metrizable. Define \(d : X \times X \to E\) as follows:
\[
(d(x, y))(t) = |x - y|e^t,
\]

\[P = \{(x \in E : x(t) \geq 0 \text{ for all } t \in X}\}.\]

Then \((X, d)\) is a TVS-valued cone metric space. Let \(S, T, f, g : X \to X\) be such that \(Sx = \frac{x}{x+10}, Tx = \frac{x}{x+16}, \) \(fx = \frac{x}{5}\) and \(gx = \frac{x}{8}.\)

For \(x, y \in X, \) we have
\[Sx = \frac{x}{x+10}, Tx = \frac{x}{x+16}, \quad fx = \frac{x}{5} \quad \text{and} \quad gx = \frac{x}{8}.\]
\[
\frac{x + 10 - \frac{y}{y + 16}}{e^t} = \frac{x(y + 16) - y(x + 10)}{(x + 10)(y + 16)} e^t \\
\leq \frac{16x - 10y}{160} e^t \\
\leq \frac{1}{2} \frac{x}{5} - \frac{y}{8} e^t,
\]

that is
\[
d(Sx, Ty)(t) \leq \frac{1}{2} d(fx, gy)(t) \text{ for all } t \in X.
\]

It suffices to assume \( \lambda = \frac{1}{2} \), in order to satisfy all assumptions of the above theorem.

If in Theorem 3, we choose \( S = T \) and \( f = g \), we obtain the following corollary.

**Corollary 5:** Let \((X, d)\) be a complete TVS-valued cone metric space and the mappings \( T, f : X \to X \) satisfy:

\[
d(Tx, Ty) \leq \lambda d(fx, fy)
\]

for all \( x, y \in X \) where \( 0 \leq \lambda < 1 \). If

\( TX \subseteq fX \), \( f \) is continuous, \((T, f)\) is compatible then \( T \) and \( f \) have a unique common fixed point.

In the following result continuity of \( f \) is not required whereas completeness of \( X \) is replaced with the completeness of \( TX \) or \( fX \).

**Theorem 6:** Let \((X, d)\) be a TVS-valued cone metric space and the mappings \( T, f : X \to X \) satisfy:

\[
d(Tx, Ty) \leq \lambda d(fx, fy)
\]

for all \( x, y \in X \) where \( 0 \leq \lambda < 1 \). If

\( TX \subseteq fX \),

\( TX \) or \( fX \) is complete, \((T, f)\) is compatible then \( T \) and \( f \) have a unique common fixed point.

**Proof:** As in the proof of Theorem 3, construct a Cauchy sequence \( \{fx_n\}_{n \geq 1} \) in \( fX \) such that

\[
y_{2k+1} = fx_{2k+1} = Tx_{2k}
\]

\[
y_{2k+2} = fx_{2k+2} = Tx_{2k+1}, k = 0, 1, 2, \ldots
\]

By completeness of \( fX \), there exists \( u, v \in X \) such that \( y_n \to v = fu \) (this holds also if \( TX \) is complete with \( v \in TX \)). Choose a natural number \( N \) such that

\[
d(y_n, v) < c \frac{2}{m} \text{ for all } n \geq N.
\]

Hence, for all \( n \geq N \)

\[
d(fu, Tu) \leq d(fu, y_{2n+2}) + d(y_{2n+2}, Tu) \\
\leq d(v, y_{2n+2}) + d(Tx_{2n+1}, Tu) \\
\leq d(v, y_{2n+2}) + \lambda d(fx_{2n+1}, fu) \\
\leq d(v, y_{2n+2}) + d(y_{2n+1}, v) \\
<< \frac{c_2 + c}{c} = c.
\]

Thus,

\[
d(fu, Tu) << \frac{c}{m} \text{, for all } m \geq 1.
\]

So, \( \frac{c}{m} - d(fu, Tu) \in P \), for all \( m \geq 1 \). Since

\( \frac{c}{m} \to 0 \) (as \( m \to \infty \)) and \( P \) is closed,

\( -d(fu, Su) \in P \), but \( P \cap \{-P\} = \{0\} \). Therefore, \( d(fu, Tu) = 0 \). Hence

\( v = fu = Tu \).

By weakly compatibility of \((T, f)\), we have

\( Tv = Tf u = Tfu = fT u = fv \).

Then

\[
d(Tv, v) = d(Tv, Tu) \leq \lambda d(fv, fu) = \lambda d(Tv, v).
\]

Thus \( v \) is a unique common fixed point of \( T \) and \( f \).

**Corollary 7:** Let \((X, d)\) be a TVS-valued cone metric space and the mappings \( T, f : X \to X \) satisfy:

\( TX \subseteq fX \) and \( d(T^n x, T^n y) \leq \lambda d(fx, fy) \)

for all \( x, y \in X \) where \( 0 \leq \lambda < 1 \). Then \( T \) and \( f \) have a unique common fixed point, if \( Tf = fT \) and one of the following conditions is satisfied:

- \( X \) is complete and \( f \) is continuous
• \( TX \) is complete.

Proof: By Corollary 5 and Theorem 6, we obtain \( v \in X \) such that

\[ f v = T^n v = v. \]

The result then follows from the fact that

\[
d(T v, v) = d(T T^n v, T^n v) = d(T^n T v, T^n v) \leq \lambda d(f T v, f v) = \lambda d(T f v, f v).
\]

Example 8: Let \( X = C([1,3], \mathbb{R}) \), \((E, \tau)\) is the topological vector space of Example 4. Define \( d : X \times X \to E \) as follows:

\[
d(x, y)(t) = \left( \sup_{s \in [1,3]} |x(s) - y(s)| \right) e^t.
\]

Then \((X, d)\) is a TVS-valued cone metric space. Define \( T : X \to X \) by

\[
T(x(s)) = 4 + \int_1^s (x(u) + u^2)e^{u-1} du,
\]

\[ f(x(s)) = x(s). \]

For \( x, y \in X \)

\[
d(T x, Ty)(t) = \left( \sup_{s \in [1,3]} |T x(s) - Ty(s)| \right) e^t \leq \int_1^s \left( \sup_{u \in [1,3]} |x(u) - y(u)| e^{|u|} du \right) e^t = 2 e^2 d(x, y)e^t.
\]

Similarly,

\[
d(T^n x, T^n y)(e^t) \leq e^{2n} \frac{2n}{n!} d(x, y)(e^t).
\]

Note that

\[
e^{2n} \frac{2n}{n!} = \frac{53}{100} \quad \text{if} \quad n = 38.
\]

Thus for \( \lambda = \frac{53}{100}, \ n = 38, \) all conditions of Corollary 7 are satisfied and so \( T \) has a unique fixed point, which is the unique solution of the integral equation:

\[ x(s) = 4 + \int_1^s (x(u) + u^2)e^{u-1} du \]

or the differential equation:

\[ x'(s) = (x + s^2)e^{s-1}, \ s \in [1,3], \ x(1) = 4. \]

REFERENCES


