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Optimal Fuzzy Control with Application to Discounted Cost Production Inventory Planning Problem

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Abstract— **A fuzzy optimal control model was formulated minimizing the objective function with discounted cost for the length of infinite horizon. We developed an equation of optimality in case of fuzzy optimal control problem. We revisited a special fuzzy control model with quadratic objective functional form for linear Liu's fuzzy control system. As an application, we investigated the infinite horizon production inventory planning problem with nonzero discount rate. We employed fuzzy optimal control to model inventory production planning problem with fuzzy variables and solved.**

Index Terms—Fuzzy optimal control, Linear quadratic problem, Production planning, Fuzzy control system.

I. INTRODUCTION

Applications of optimal control problems involve the control of dynamic systems that evolve over time either continuous-time systems or discrete-time systems. In order to deal with such systems, stochastic optimal control theory has been investigated and widely applied to physical, biological, finance, economics, production and inventory, marketing, maintenance and replacement, and the consumption of natural resources. Whenever the system is characterized with white noise and is represented by a controlled stochastic process then it is called stochastic control system which are described by Ito's stochastic differential equations. A traditional approach to solving optimal control problems consists of formulation of optimality conditions directly, use of the calculus of variations and Pontryagin's maximum principle (Bellman [2]), and then solving the resulting equations to obtain the solution to the problems.

Uncertainty is an important commodity in the modeling business and it is inherit in most dynamic systems. It is generally agreed that an important point in the evolution of the modern concept of uncertainty was introduced by Zadeh [13] via membership function. The use of fuzzy sets and fuzzy logic, in the social sciences appears to have been limited to mainly to psychology, with very little application to the analysis of economic problems and economic data. In the area of econometrics, however, there have been surprisingly few applications of fuzzy set/logic techniques. The way fuzziness has been applied widely in the modeling of control systems in different ways since the seminal contributions of Zadeh [13, 14] and his colleagues. The Mamdani [8] introduced fuzzy control system to control a steam engine and boiler combination by a set of control rules. The Takagi and Sugeno
[10] and Sugeno and Kang [11] approach to dealing with fuzzy control system which were introduced for generation of fuzzy rules from a given input-output data set.

The most successful application area of fuzzy systems has undoubtedly been the area of fuzzy control. Fuzzy control theory has been applied to industrial production. Our presentation of fuzzy control includes the connection between fuzzy controllers and production planning problem, the importance of which has increasingly been recognized. The idea of applying fuzzy sets to control problems was presented for the first time more explicitly in Chang and Zadeh [4]. The actual research on fuzzy controllers was initiated by Mamdani and Assilian [7]; Mamdani [9]. Mamdani's work influenced other researchers to explore the applicability of fuzzy controllers to various control problems. Recently, fuzzy process and fuzzy calculus are introduced by Liu [6] as fuzzy counterpart of stochastic process and Brownian Motion who expressed fuzzy di□erential equation with Liu process in order to deal with the evolution of fuzziness with time. The control problem of production planning with discount rate has been studied by many authors like Fleming, Sethi and Soner [5], Sethi et. al. [12], and Baten and Kamil [1] in a stochastic manufacturing systems. Bensoussan et. al. [3] employed stochastic optimal control theory to model production planning problem for the continuous-time case and Zhu [15] for fuzzy optimal control with a finite horizon. But this paper seems to be a new literature establishing an alternative fuzzy optimal control with an application to fuzzy discounted cost production inventory planning problem for an infinite horizon.

The objective of this paper is to give an application to fuzzy production planning problem by using the developed equation of optimality. In order to this in section 2, we formulated the fuzzy optimal control problem with fuzzy process and we developed an equation of optimality in the context of fuzzy optimal control problem for an infinite horizon. In section 3, we proved that the value function of fuzzy linear quadratic control problem satisfied associated

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partial differential equation applying the equation of optimality. We also obtained the optimal feedback control in terms of state variable for linear fuzzy quadratic problem with discounted rate for an infinite horizon. The section 4 devoted with the result of an application to fuzzy production inventory planning problem.

II. FUZZY OPTIMAL CONTROL PROBLEM

We employed fuzzy optimal control to model inventory production planning problem with fuzzy variables and solved.

An optimization model is formulated on fuzzy optimal control problem to minimize the expected discounted cost control for Liu's fuzzy control system.

We assume that the dynamics of the state equation is governed by a special fuzzy differential equation based on Liu's fuzzy control system according to

$$dx_t = g\left(t, x_t, y_t\right) d_t + h\left(t, x_t, y_t\right) dL_t \tag{2.1}$$

where L_t is a standard Liu process at time *t*, x_t is the state variable at time *t*, y_t is a control at time *t* and $t \geq 0$.

The purpose of a fuzzy optimal control model is to choose the best cost control y_t such that x_t is optimized, that is,

$$J(y) = E\left[\int_0^{\infty} e^{-\rho t} F\left(t, x_t, y_t\right) dt\right] \tag{2.2}$$

over $y \in \mathrm{A}$ where A is denoted as the class of all admissible controls of fuzzy Liu processes, *E* represents the expected value operator of fuzzy variable and $\rho > 0$ constant non-negative discount rate.

We define the value function whose value is the minimum value of the objective function of the fuzzy optimal control problem (2.2) for the fuzzy control system, that is,

$$V(x) = \inf_{y_t} E\left[\int_0^{\infty} e^{-\rho t} F\left(t, x_t, y_t\right) dt\right] = \inf_{y_t} J(y) \tag{2.3}$$

Theorem 2.1 *We assume that the value function V (t, x) is a twice differentiable function on* $[0,\infty)\times\mathbb{R}$. *If* y_t^* *is an optimal control, then* y_t^* *solves*

$$\inf_y \left\{ F(t,x,y) - \rho V(t,x) + \frac{\partial V(t,x)}{\partial t} + g(t,x,y)\frac{\partial V(t,x)}{\partial x} \right\} = 0 \tag{2.4}$$

which is called equation of optimality.

Proof: At time 0, the present value of the objective function is given by $J = \exp(-\rho t) V(t,x)$. Then we observed that $\frac{\partial J}{\partial t} = -\rho \exp(-\rho t) V(t,x) + \exp(-\rho t)\frac{\partial V(t,x)}{\partial t}$ and

$$\frac{\partial J}{\partial x} = \exp(-\rho t)\frac{\partial V(t,x)}{\partial x}$$

Using these relations in Bellman's (1957) equation of optimality, we obtain

$$\inf_y \left\{ \exp(-\rho t) F(t,x,y) - \rho \exp(-\rho t) V(t,x) + \exp(-\rho t)\frac{\partial V(t,x)}{\partial t} + \exp(-\rho t) g(t,x,y)\frac{\partial V(t,x)}{\partial t} \right\} = 0$$

Dividing both sides by $\exp(-\rho t)$, then we obtain the equation of optimality (2.4). The proof is completed.

III. FUZZY LINEAR QUADRATIC CONTROL PROBLEM

In this section we propose a special fuzzy control model with quadratic objective functional form for linear Liu's fuzzy control system. The objective of this linear quadratic control model is to find an optimal control y_t^* which is a function of *t*. In order to solve this problem for the explicit feedback controller, we follow equation of optimality (2.4).

In general, we state the following fuzzy linear quadratic control problem for an infinite horizon:

$$J(t,x;y) = E\left[\int_0^{\infty} e^{-\rho t}\left(Fx_t^2 + Gy_t^2 + Hx_t y_t + Lx_t + My_t + N\right) dt\right]$$

where *F, G, H, L, M* and *N* are all real numbers.

The value function of this linear quadratic control problem is formulated as follows

$$V(t,x) = \inf_{y_t} J(t,x;y) \tag{3.1}$$

Subject to

$$dx_t = y_t dt + \sigma x_t dL_t, \tag{3.2}$$

where *x* is the initial state and V (0,*x*) = 0.

Theorem 3.1 *We assume that* $G \neq 0$. *Let V (t, x) be the value function of linear quadratic control model (3.1) and (3.2). Then the value function V (t, x) satisfies the following partial differential equation*

$$-\frac{1}{4G}\left(\frac{\partial V(t,x)}{\partial x}\right)^2 + \left(F - \frac{H^2}{4G}\right)x^2 - \left(\frac{HM}{2G} + L + \frac{H}{2G}\frac{\partial V(t,x)}{\partial x}\right)x - \frac{M}{2G}\left(\frac{M}{2} + \frac{\partial V(t,x)}{\partial x}\right) + N - \rho V(t,x) + \frac{\partial V(t,x)}{\partial t} = 0 \tag{3.3}$$

and the optimal control satisfies

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$$y_t^* = -\frac{1}{2G}\left(\frac{\partial V(t,x)}{\partial x} + Hx + M\right).$$

Proof: It follows from the equation of optimality (2.4) that

$$\min_y U(y) = \min_y \Big\{ \left(Fx_t^2 + Gy_t^2 + Hx_t y_t + Lx_t + My_t + N\right) - \rho V(t,x) + \frac{\partial V(t,x)}{\partial t} + y_t \frac{\partial V(t,x)}{\partial t} \Big\} = 0 \tag{3.4}$$

where $U(y)$ denotes the term in the braces. Setting $\frac{\partial U(y)}{\partial y} = 0$ yields

$$2Gy + Hx + M + \frac{\partial V(t,x)}{\partial x} = 0 \ or$$

$$y_t^* = y = -\frac{1}{2G}\left(\frac{\partial V(t,x)}{\partial x} + Hx + M\right).$$

Substituting the last equality into the equation (3.4), we obtain

$$Fx^2 + \frac{1}{4G}\left(\frac{\partial V(t,x)}{\partial x} + Hx + M\right)^2 - \frac{Hx+M}{2G}\left(\frac{\partial V(t,x)}{\partial x} + Hx + M\right) + Lx + N - \rho V(t,x) + \frac{\partial V(t,x)}{\partial t} - \frac{1}{2G}\left(\frac{\partial V(t,x)}{\partial x} + Hx + M\right)\frac{\partial V(t,x)}{\partial x} = 0$$

which is just equation (3.6). The proof is complete.

Example 3.2 *We* assume that $G \neq 0$. Let V (t, x) be the value function of linear quadratic control model (3.1) and (3.2). We conjecture the value function $V(t,x) = Kx + \eta$, where $k > 0$, and $\eta > 0$. Then it follows from the equation of optimality (2.4) that

$$\min_y U(y) = \min_y \left\{ Fx_t^2 + Gy_t^2 + Hx_t y_t + Lx_t + My_t + N - \rho Kx - \rho\eta + Ky_t \right\} = 0 \tag{3.5}$$

where $U(y)$ denotes the term in the braces. Setting $\frac{\partial U(y)}{\partial y} = 0$ yields

$2Gy + Hx + \mathrm{M} + \mathrm{K} = 0$,

from which we obtain the optimal control

$$y_t^* = y = -\frac{1}{2G}(K + Hx + M).$$

Substituting the last equality into the equation (3.5), we obtain

$$Fx^2 + \frac{1}{4G}(K + Hx + M)^2 - \frac{Hx+M}{2G}(K + Hx + M) + Lx + N - \rho(Kx + \eta) - \frac{K}{2G}(K + Hx + M) = 0.$$

Therefore the value function satisfies the following partial differential equation

$$\left(F - \frac{H^2}{4G}\right)x^2 - \left(\frac{HM}{2G} - L + \rho K + \frac{KH}{2G}\right)x - \frac{1}{2G}\left[M\left(K + \frac{M}{2}\right) + \frac{K^2}{2}\right] + N - \rho\eta = 0.$$

The following example can be solved by using the equation of optimality (2.4):

Example 3.3 *A special fuzzy differential equation based on a linear Liu's fuzzy control system*

$$dx_t = (\alpha x_t + \beta y_t + \eta)dt + (\gamma x_t + \delta y_t + \vartheta)dL_t$$

Where $\alpha, \beta, \gamma, \delta, \eta$ *and* ϑ *are deterministic real numbers.*

IV. AN APPLICATION TO FUZZY PRODUCTION INVENTORY CONTROL MODEL

In this section, we discuss a fuzzy production inventory planning model with fuzzy factors. We consider a factory producing a single homogeneous good and having a finished goods warehouse. In order to state the model we define the following notations:

x_t : the inventory level at time *t* (state variable),
y_t : the production rate at time t (control variable),
u_t : the constant demand rate at time t,
$\hat{x}$: the inventory goal level,
$\hat{y}$: the production goal level,
$x(0)$: initial inventory level,
q : the inventory holding cost coefficient; ($q > 0$),
r : the production cost coefficient; ($r > 0$),
ρ : the constant nonnegative discount rate; $\rho \geq 0$,
σ : the constant diffusion coefficient.

The inventory dynamics can be described as the fuzzy differential equation

$$dx_t = [y(t) - u(t)]dt + \sigma x_t dL_t, \quad x(0) = x, \tag{4.1}$$

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where L_t is a white noise which can be interpreted as sales returns or inventory spoilage. Note that x_t and y_t are fuzzy variables for each fixed time. The objective function of the model is:

$$J(x,y,t)=\frac{1}{2}E\left[\int_0^\infty e^{-\rho t}\left\{q\left(x_t-\hat{x}\right)^2+r\left(y_t-y\right)^2\right\}dt\right]$$
(4.2)

The purpose of this objective function is to keep the inventory *x* as close as possible to its goal level $\hat{x}$, and also keep the production rate *y* as close as possible to its goal level $\hat{y}$. The quadratic $\frac{q}{2}\left(x_t-\hat{x}\right)^2$ and $\frac{r}{2}\left(y_t-\hat{y}\right)^2$ impose penalties for having either *x* or *y* not being close to its corresponding goal level.

Then fuzzy production inventory panning model is formulated as follows

$$V\left(x,y,t)\right)=\min_{y} J\left(x,y,t\right) \tag{4.3}$$

subject to (4.1).

It follows from the equation of optimality (2.4) that

$$\begin{aligned}\rho V\left(t,x\right)&=\min_{y}\left\{\frac{1}{2}\left\{q\left(x_t-\hat{x}\right)^2+r\left(y_t-y\right)^2\right\}+\right.\\&\left.\frac{\partial V\left(t,x\right)}{\partial t}+\left[y\left(t\right)-u\left(t\right)\right]\frac{\partial V\left(t,x\right)}{\partial x}\right.\\&=\min_{y}U\left(y\right)\end{aligned} \tag{4.4}$$

where $U(y)$ denotes the term in the braces. Setting $\frac{\partial U\left(y\right)}{\partial y}=0$, we obtain the necessary condition for optimality

$$y_t=\hat{y}-\frac{1}{r}\frac{\partial V\left(t,x\right)}{\partial x}. \tag{4.5}$$

Substitute (4.5) to (4.4), we obtain the following partial differential equation

$$\begin{aligned}\rho V\left(t,x\right)&=\frac{1}{2}q\left(x_t-\hat{x}\right)^2+\frac{1}{2r}\left(\frac{\partial V\left(t,x\right)}{\partial x}\right)^2+\frac{\partial V\left(t,x\right)}{\partial t}+\\&\left[\hat{y}-\frac{1}{r}\frac{\partial V\left(t,x\right)}{\partial x}-u(t)\right]\frac{\partial V\left(t,x\right)}{\partial x}\end{aligned} \tag{4.6}$$

From (4.5) we find $\frac{\partial V\left(t,x\right)}{\partial x}=r\left(\hat{y}-y_t\right)$. The solution of $\frac{\partial V\left(t,x\right)}{\partial x}=r\left(\hat{y}-y_t\right)$ is $V=r\left(\hat{y}-y_t\right)x+C$, where C is a constant and $\frac{\partial V\left(t,x\right)}{\partial t}=0$. Substituting these result in (4.6), we obtain

$$\begin{aligned}&r\rho\left(\hat{y}-y_t\right)x+C\rho=\\&\frac{q}{2}x_t^2-q\hat{x}x_t+\frac{q}{2}x^2+\frac{r}{2}\left(\hat{y}-y_t\right)^2+r\left(y_t-u(t)\right)\left(y-y_t\right).\end{aligned}$$

Since $q>0$ and now matching the coefficients of the last polynomial in *x*, we require

$-q\hat{x}=r\rho\left(y-y_t\right)$ and

$C\rho=r\left(\hat{y}-y_t\right)\left\{\frac{1}{2}\left(y_t+y\right)-u\left(t\right)\right\}+\frac{q}{2}\hat{x}^2$. These give

$y_t^*=\hat{y}+\frac{q\hat{x}}{r\rho}$, and

$$C=\frac{r}{\rho}\left(\hat{y}-y_t\right)\left\{\frac{1}{2}\left(y_t+y\right)-u\left(t\right)\right\}+\frac{q}{2\rho}\hat{x}^2.$$

The optimal control is to set $y_t^*=\hat{y}+\frac{q\hat{x}}{r\rho}$ and the optimal value of the objective function is

$$V\left(x\right)=r\left(\hat{y}-y_t\right)\left[x+\frac{1}{\rho}\left\{\frac{1}{2}\left(y_t+y\right)-u\left(t\right)\right\}\right]+\frac{q}{2\rho}\hat{x}^2$$

where $x_t=x$.

V. CONCLUSION

A. *Figures and Tables*

We studied a fuzzy optimal control problem minimizing the expected value of discounted objective function for an infinite horizon subject to a fuzzy differential equation. We developed an equation of optimality in case of fuzzy optimal control problem. Following the equation of optimality we obtain an optimal feedback control in context of fuzzy linear quadratic control problem for fuzzy control system and we proved that the value function satisfied the partial differential equation associated with fuzzy linear quadratic control problem. It was also carried out with an example. We gave an application to fuzzy production inventory planning problem by using the equation of optimality.

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