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Stability of a Numerical Discretisation Scheme for the SIS Epidemic Model with a Delay

Ekkachai Kunnawuttipreechachan *

Abstract—**This paper deals with stability properties of the discrete numerical scheme for the SIS epidemic model with maturation delay. We provide the sufficient conditions of the numerical step-size for the numerical solutions to be asymptotically stable. These will be useful for choosing a suitable numerical step-size when we simulate problems with the provided numerical scheme.**

Keywords: SIS epidemic model, delay differential equations, discrete numerical scheme, asymptotic stability

1 Introduction

In this paper we aim to analyse a numerical scheme for a class of the SIS epidemic models. In a general SIS model, the population size $N(t)$ is divided into two groups: $I(t)$ the infected population size, and $S(t)$ the susceptible population size. Here we continue further the results from Cooke *et al.* [1], who have developed the SIS epidemic model with maturation delay:

$$\begin{aligned} I'(t) &= \mu(N(t)-I(t))\tfrac{I(t)}{N(t)} - (\delta+\varepsilon+\gamma)I(t), \\ S'(t) &= B\big(N(t-\tau)\big)N(t-\tau)e^{-\delta_1\tau} \\ &\qquad -\delta S(t) - \tfrac{\mu S(t)I(t)}{N(t)} + \gamma I(t), \\ N'(t) &= B\big(N(t-\tau)\big)N(t-\tau)e^{-\delta_1\tau} \\ &\qquad -\delta N(t) - \varepsilon I(t). \end{aligned} \tag{1}$$

Here $B(N)$ is a birth rate function, $\varepsilon \geq 0$ is the disease induced death constant rate, $\gamma \geq 0$ is the recovery constant rate (i.e. $1/\gamma$ is the average infective time), $\mu > 0$ is the contact constant rate, δ is the death rate constant, and δ_1 is the transfer rate constant for the life stage prior to the adult stage. The standard incidence function $\mu I/N$ refers to the average number of adequate contacts with infectives of one susceptible per unit time. The delay τ is considered as the development or the maturation time.

In addition, model (1) is based as follows on the assumptions by Cooke *et al.* [1]:

- the transmission of the disease occurs due to a contact between the susceptible class $S(t)$ and the infective class $I(t)$;
- there is no vertical transmission;
- the disease presents no immunity against reinfection, i.e., on recovery, an infective individual returns to the susceptible class.

Moreover, in Cooke *et al.* [1], they also identified the basic reproduction number

$$R_0 = \frac{\mu}{\delta+\varepsilon+\gamma}, \tag{2}$$

which generates the average number of new infective individuals produced by one infective during the mean death adjusted infective.

Many authors have studied dynamics of the system (1) by considering R_0 as a parameter [1, 2, 3]. Cooke *et al.* [1] showed the existence of the equilibria of (1). They proved that if $R_0 < 1$, there exists a unique nontrivial equilibrium $(\bar{I},\bar{S},\bar{N}) = (0,\bar{N}_0,\bar{N}_0)$, called *disease-free equilibrium*, which is globally asymptotic stable (see Cooke *et al.* [1], Theorems 4.2 and Theorem 4.3). When $R_0 > 1$, there also exists another nontrivial equilibrium $(\bar{I},\bar{S},\bar{N}) = (\bar{I}_+,\bar{S}_+,\bar{N}_+)$ or, so called, *endemic equilibrium.* Analysis of the system in this case becomes harder. For the non-delay problem ($\tau = 0$), Cooke *et al.* [1] showed that the reproduction number R_0 acts as a *sharp threshold.* Their results show that, for nonnegative solutions, when $R_0 < 1$, the disease dies out, i.e. $I(t) \to \bar{I}_0 = 0$ as $t \to \infty$. On the contrary, if $R_0 > 1$ and $I(0) > 0$, then the disease remains endemic with $I(t) \to \bar{I}_+$, $S(t) \to \bar{S}_+$ and $N(t) \to \bar{N}_+$ as $t \to \infty$.

In addition, for the positive delay ($\tau > 0$) with $\varepsilon = 0$, Cooke *et al.* [1] obtained the globally asymptotic stability for the nontrivial equilibrium in this case. They also indicated that if $R_0 > 1$ and τ is sufficiently large, the positive solutions of (1) oscillate about the positive equilibrium, i.e. they are periodic solutions. Later, Zhao and Zou [3] provided a proof where ε is very small using the perturbation technique. They obtained sufficient conditions for the stability of (1). Moreover, Wei and Zou [2] determined the bifurcation analysis of (1) with p as the bifurcation parameter when $\varepsilon = 0$. However, the dynamics of the SIS model (1) still largely remain undetermined, especially for $\varepsilon > 0$ [3].

*Department of Mathematics, King Mongkut's University of Technology North Bangkok (KMUTNB), 1518 Pibulsongkram Road, Bangkok, THAILAND 10800; Tel:+66(0)81-828-3100/Fax:+66(0)28-587-8258. Email: ekc@kmutnb.ac.th, or ekc.math@gmail.com.

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Our approach in this paper is to study a special class of (1). Assume that the death rate in each stage prior to the adult stage is negligible compared with the death rate of the adult stage, then we can assume $\delta_1 = 0$. Using the birth rate function $B(N(t)) = pe^{-aN(t)}$. Hence (1) becomes

$$\begin{aligned} I'(t) &= \mu(N(t) - I(t))\tfrac{I(t)}{N(t)} - (\delta + \varepsilon + \gamma)I(t), \\ S'(t) &= pN(t-\tau)e^{-aN(t-\tau)} \\ &\qquad -\delta S(t) - \tfrac{\mu S(t)I(t)}{N(t)} + \gamma I(t), \\ N'(t) &= pN(t-\tau)e^{-aN(t-\tau)} - \delta N(t) - \varepsilon I(t). \end{aligned} \tag{3}$$

Like with the model (1), the dynamical behaviour of the SIS model (3) is still largely undetermined because there are many open problems which are still unsolved at present [2]. In this paper we provide alternatively a numerical scheme for (3). Our aim is to analyse further the dynamical properties of a numerical scheme for (3). The main objective in this study is to determine the asymptotic stability of the equilibria of the given numerical scheme and illustrate numerical solutions of (3). We obtain the sufficient conditions for the numerical stepsize causing the numerical solutions are asymptotically stable.

This paper is organised as follows. In the next section we analyse the stability of (3). For any nonnegative solutions, our result in Theorem 2 shows conditions for the existence of the equilibria, which are called *disease-free equilibrium* and *endemic equilibrium*. Most results for the continuous system have already been studied, but we restate some related results with our own proof in Theorem 3 and Theorem 4 for the stability of (3). Later, in Section 3, we describe our main problems for this paper. Here we develop the numerical scheme for (3), and then investigate the stability of the numerical solutions. Our main contributions are presented in Theorem 6 and Theorem 7 for the stability of the disease-free equilibrium and the endemic equilibrium, respectively. Finally, the numerical results are also illustrated in Section 4.

2 Dynamics of the Equilibria

Since the total number of population $N(t)$ is the sum of the infective class $I(t)$ and the susceptible class $S(t)$, we have

$$N(t) = S(t) + I(t).$$

It is adequate to reduce the dimension of the system to

$$I'(t) = \mu(N(t) - I(t))\frac{I(t)}{N(t)} - (\delta + \varepsilon + \gamma)I(t), \tag{4}$$

$$N'(t) = pe^{-aN(t-\tau)}N(t-\tau) - \delta N(t) - \varepsilon I(t). \tag{5}$$

Note that $\varepsilon = 0$, then (5) becomes the *Nicholson's blowflies equation*:

$$N'(t) = pe^{-aN(t-\tau)}N(t-\tau) - \delta N(t), \tag{6}$$

which proposed by Gurney *et al.* [4]. The dynamics of (6) have been studied by many authors (see for examples [5, 6, 7, 8, 9]). It is not difficult to show that (6) has the trivial equilibrium $\bar{N}_0 = 0$ and the positive equilibrium $\bar{N}_+ = \frac{1}{a}\ln(p/\delta)$. Here we state the theorem for stability properties of (6) resulted by [9].

Theorem 1. (Stability of the blowflies equation; [9]) *Let $\bar{N}_0$ and $\bar{N}_+$ be the trivial equilibrium and positive equilibrium of (6), respectively.*
(1) If $p < \delta$, then the trivial equilibrium $\bar{N}_0$ is (locally) asymptotically stable.
(2) If the ratio p/δ satisfies

$$1 < \frac{p}{\delta} < e^2, \tag{7}$$

then the positive equilibrium $\bar{N}_+$ of (6) is (locally) asymptotically stable.

Proof. See [4, 9] for more details on the proof. □

2.1 The equilibria

To analyse the dynamical behaviour of (4)-(5), we investigate in a first stage all equilibria. Suppose that $I(t) \equiv \bar{I}$ and $N(t) \equiv \bar{N}$. Let $I'(t) = N'(t) = 0$, then the system (4)-(5) becomes

$$\mu(\bar{N} - \bar{I})\frac{\bar{I}}{\bar{N}} - (\delta + \varepsilon + \gamma)\bar{I} = 0, \tag{8}$$

$$pe^{-a\bar{N}}\bar{N} - \delta\bar{N} - \varepsilon\bar{I} = 0. \tag{9}$$

Here, we only consider the equilibria in the positive quadrant, i.e. $\bar{N} > 0$ and $\bar{I} \geq 0$. Note that the case that $N = 0$ is not our interested. We state the following theorem to show conditions for the existence of the equilibria with our proof.

Theorem 2. (The equilibria of (4)-(5)) *Suppose that $\varepsilon, \gamma, a \geq 0$ and $\mu > 0$. If $R_0 \leq 1$ and $p > \delta \geq 0$, there exists only a semi-trivial equilibrium $(\bar{I}_0, \bar{N}_0)$, namely*

$$\bar{I}_0 = 0 \quad \textit{and} \quad \bar{N}_0 = \frac{1}{a}\ln\frac{p}{\delta}. \tag{10}$$

This equilibrium is called the **disease-free equilibrium**. *On the other hand, if $R_0 > 1$ and $p > \delta + \varepsilon(1 - 1/R_0)$, there exists also another positive equilibrium $(\bar{I}_+, \bar{N}_+)$, namely*

$$\bar{I}_+ = \left(1 - \frac{1}{R_0}\right)\bar{N}_+ \quad \textit{and} \quad \bar{N}_+ = \frac{1}{a}\ln\frac{p}{\delta + \varepsilon(1 - 1/R_0)}, \tag{11}$$

called the **endemic equilibrium**.

Proof. Rearranging (8), we have

$$\bar{I}\left(\mu - \frac{\mu\bar{I}}{\bar{N}} - (\delta + \varepsilon + \gamma)\right) = 0.$$

For $\bar{N} > 0$, then the solutions of $\bar{I}$ are divided into two cases:

$$\bar{I} = 0,$$

or

$$\bar{I} = \left(1 - \frac{\delta + \varepsilon + \gamma}{\mu}\right) \bar{N} = \left(1 - \frac{1}{R_0}\right) \bar{N},$$

where R_0 is defined by (2).

Case I: For $\bar{I} = 0$, then (9) becomes

$$\bar{N}\left(pe^{-a\bar{N}} - \delta\right) = 0.$$

Since $\bar{N} > 0$, there exists only the nonnegative equilibrium $(\bar{I}, \bar{N}) = (\bar{I}_0, \bar{N}_0)$, where

$$\bar{I}_0 = 0 \quad \text{and} \quad \bar{N}_0 = \frac{1}{a} \ln \frac{p}{\delta}.$$

Clearly, $(\bar{I}_0, \bar{N}_0)$ exists for all values of R_0 whenever $p > \delta$. Hence $(\bar{I}_0, \bar{N}_0)$ is the equilibrium of (4)-(5) satisfying (8)-(9) for both $R_0 \le 1$ and $R_0 > 1$.

Case II: For $\bar{I} = (1 - 1/R_0)\bar{N}$, it can be seen that there exists also a positive equilibrium if and only if $R_0 > 1$. Then (9) becomes

$$\bar{N}\left(pe^{-a\bar{N}} - \delta - \varepsilon(1 - \frac{1}{R_0})\right) = 0.$$

Since $N(t) > 0$, then $pe^{-a\bar{N}} - \delta - \varepsilon\left(1 - \frac{1}{R_0}\right) = 0$. It yields that

$$\bar{N} = \frac{1}{a} \ln\left(\frac{p}{\delta + \varepsilon(1 - \frac{1}{R_0})}\right).$$

Clearly, when $R_0 > 1$ it is easy to verify that there exists another equilibrium, $(\bar{I}, \bar{N}) = (\bar{I}_+, \bar{N}_+)$, defined by

$$\bar{I}_+ = \left(1 - \frac{1}{R_0}\right) \bar{N}_+$$

and

$$\bar{N}_+ = \frac{1}{a} \ln\left(\frac{p}{\delta + \varepsilon(1 - 1/R_0)}\right).$$

It can be seen that if $p > \delta + \varepsilon(1 - 1/R_0)$ and $R_0 > 1$, then $\bar{I}_+$ and $\bar{N}_+$ are positive. As the results, it concludes that when $R_0 > 1$, there exist two positive equilibria, which are the disease-free equilibrium $(\bar{I}_0, \bar{N}_0)$, and the endemic equilibrium $(\bar{I}_+, \bar{N}_+)$. On the other hand, if $R_0 \le 1$, the disease-free equilibrium $(\bar{I}_0, \bar{N}_0)$ is only a nonnegative equilibrium of (4)-(5). □

Note that the disease-free equilibrium $(\bar{I}_0, \bar{N}_0) = (0, \frac{1}{a} \ln(p/\delta))$ is sometimes called the *semi-trivial equilibrium*. It means that there will be no infective population in the system as time tends to infinity, and the number of population will reach a constant value $\bar{N}_0$. Throughout this paper we use the words *disease-free equilibrium* to represent $(\bar{I}_0, \bar{N}_0)$ and *endemic equilibrium* to represent $(\bar{I}_+, \bar{N}_+)$.

2.2 Stability analysis

To analyse the stability properties of (4)-(5), we use R_0 as a parameter. We also investigate how the reproduction number R_0 and the delay τ affect the behaviour of the system. Our study is divided into two cases: $R_0 \le 1$ and $R_0 > 1$. First, we consider $R_0 \le 1$. Cooke *et al.* [1] gave the result in this case for the general model (1). For (4)-(5), the following theorem shows the sufficient conditions for the stability of the equilibria.

Theorem 3. (Stability of the equilibria when $R_0 \le 1$) *Let $R_0 < 1$. Consider the SIS model (4)-(5) with the positive initial values where $N(t) \ge I(t) > 0$ on $[-\tau, 0]$. Then $I(t)$ tends to zero as t tends to infinity. If, in addition, the condition $1 < p/\delta < e^2$ holds, then $\bar{N}_0 = \frac{1}{a}\ln(p/\delta)$ is asymptotically stable.*

Proof. Let $I(t) = \bar{I} + y(t)$ and $N(t) = \bar{N} + x(t)$. Using the linearisation method. The SIS system (4)-(5) becomes the linearised system

$$\begin{aligned} y'(t) &= \left(\mu - (\delta + \varepsilon + \gamma) - 2\mu\frac{\bar{I}}{\bar{N}}\right) y(t) + \mu\frac{\bar{I}^2}{\bar{N}^2}x(t), \\ x'(t) &= -\varepsilon y(t) - \delta x(t) + pe^{-a\bar{N}}(1 - a\bar{N})x(t - \tau), \end{aligned}$$

where $(\bar{I}, \bar{N})$ is the equilibrium of (4)-(5).

For the disease-free equilibrium $(\bar{I}_0, \bar{N}_0) = \left(0, \frac{1}{a}\ln(p/\delta)\right)$, the linearised equations are

$$y'(t) = (\mu - (\delta + \varepsilon + \gamma))\, y(t) = \mu\left(1 - \frac{1}{R_0}\right) y(t) \quad (12)$$

and

$$x'(t) = -\varepsilon y(t) - \delta x(t) + pe^{-a\bar{N}}(1 - a\bar{N})x(t - \tau). \quad (13)$$

It can be seen that $y(t)$ is asymptotically stable provided that

$$\mu\left(1 - \frac{1}{R_0}\right) < 0.$$

Since $\mu > 0$, it implies that $R_0 < 1$. Hence, $y(t) \to 0$ as $t \to \infty$. Note that if $R_0 = 0$, then (12) becomes $y'(t) = 0$ and it is asymptotically stable.

Moreover, for a sufficiently large of time t, we can ignore the term of $y(t)$ and (13) becomes

$$x'(t) = -\delta x(t) + pe^{-a\bar{N}}(1 - a\bar{N})x(t - \tau). \quad (14)$$

From Theorem 1, when $p > \delta$, the equilibrium $(\bar{I}_0, \bar{N}_0)$ is (locally) asymptotically stable provided that

$$1 < \frac{p}{\delta} < e^2. \quad (15)$$

As the result, if $R_0 < 1$, then $I(t) \to I_0 = 0$ as $t \to \infty$. In addition, $N(t)$ is (locally) asymptotically stable when the condition (15) holds. □

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If there is no disease-related death ($\varepsilon = 0$), then $\bar{N}_+ = \bar{N}_0 = \frac{1}{a}\ln(p/\delta)$. The system (4)-(5) becomes

$$\begin{aligned} I'(t) &= \mu(N(t) - I(t))\frac{I(t)}{N(t)} - (\delta+\gamma)I(t), && (16)\\ N'(t) &= pe^{-aN(t-\tau)}N(t-\tau) - \delta N(t). && (17) \end{aligned}$$

Note that, in this case, $N(t)$ in (17) satisfies the Nicholson's blowflies equation and its properties is provided in Theorem 1. Moreover, since $N(t)$ in (17) is independent from $I(t)$, the system (16)-(17) is called a *decoupled* or *non-interactive* system. The following theorem presents the dynamics of the equilibria in the decoupled system.

Theorem 4. (Stability of the equilibria when $R_0 > 1$) *Let $R_0 = \frac{\mu}{\delta+\gamma} > 1$. Consider the decoupled SIS system (16)-(17) with the positive initial values $N(t) \geq I(t) > 0$ on $[-\tau, 0]$. If, in addition, the condition $1 < p/\delta < e^2$ holds, then $\bar{N}_+ = \bar{N}_0 = \frac{1}{a}\ln(p/\delta)$ and $\bar{I}_+$ are asymptotically stable, while $\bar{I}_0$ is unstable.*

Proof. Since (17) is the Nicholson's blowflies equation, we can use the results from Theorem 1. In this case we know that if $p > \delta$ and $1 < p/\delta < e^2$, then $\bar{N} = \frac{1}{a}\ln(p/\delta)$ is asymptotically stable for all $\tau \geq 0$. If $N(t) \to \bar{N}$ as $t \to \infty$, then the long term behaviour of $I(t)$ is governed by

$$I'(t) = \mu\left(1 - \frac{\bar{I}}{\bar{N}}\right)I(t) - (\delta+\gamma)I(t). \quad (18)$$

Let $I(t) = \bar{I} + y(t)$. The linearised equation of (18) is

$$\begin{aligned} y'(t) &= \left(\mu - (\delta+\gamma) - 2\mu\frac{\bar{I}}{\bar{N}}\right)y(t)\\ &= \left(\mu\left(1 - \frac{1}{R_0}\right) - 2\mu\frac{\bar{I}}{\bar{N}}\right)y(t). && (19) \end{aligned}$$

First, if $\bar{I} = \bar{I}_0 = 0$, (19) becomes

$$y'(t) = \mu\left(1 - \frac{1}{R_0}\right)y(t),$$

which is unstable if $R_0 > 1$. Hence, $\bar{I}_0$ is unstable when $R_0 > 1$. Next, when $\bar{I} = \bar{I}_+ = (1 - 1/R_0)\bar{N}_+$, then (19) yields

$$y'(t) = -\mu\left(1 - \frac{1}{R_0}\right)y(t),$$

which is asymptotically stable provided that $R_0 > 1$. Hence, this shows that $\bar{I}_+$ is asymptotically stable if and only if $R_0 > 1$. □

Note that the dynamical analysis when $\varepsilon \neq 0$ has been studied by Zhao and Zou [3], but they can only provide a case when ε is sufficiently small. As in our knowledge, dynamics of the SIS model (4)-(5) for general values of ε is still undermined. In the next section we will show dynamics of the discretisation numerical scheme and compare them with the results for the continuous case.

3 Numerical Scheme and Analysis

This section contains our main contributions to the analysis of numerical solutions of the SIS model (4)-(5). Firstly, we introduce a numerical scheme for the problem, and then analyse its stability near the equilibria. Our main target is to find sufficient conditions for the equilibria of the numerical scheme for (4)-(5) to be asymptotically stable or unstable.

3.1 The numerical scheme

From the forward difference formula:

$$y'(t) \simeq \frac{y_{n+1} - y_n}{h},$$

where $n = 0, 1, 2, \ldots$, $y_n = y(t_n)$, $t_n = t_0 + nh$ and h is the numerical step-size of the derivative approximation. Let $I_n = I(t_n)$ and $N_n = N(t_n)$. We construct the numerical scheme for the SIS model (4)-(5) as follows:

$$\begin{aligned} \frac{I_{n+1} - I_n}{h} &= \mu(N_n - I_n)\frac{I_n}{N_n} - (\delta+\varepsilon+\gamma)I_n,\\ \frac{N_{n+1} - N_n}{h} &= pN_{n-k}e^{-aN_{n-k}} - \delta N_n - \varepsilon I_n. \end{aligned}$$

Here, we use the equal step-size $h = \tau/k$, where k is a positive integer. After rearranging the system above, we get the explicit scheme:

$$\begin{aligned} I_{n+1} &= (1 + h\mu - h(\delta+\varepsilon+\gamma))\,I_n - h\mu\frac{I_n^2}{N_n}, && (20)\\ N_{n+1} &= -\varepsilon hI_n + (1 - h\delta)N_n\\ &\qquad + hpN_{n-k}e^{-aN_{n-k}}. && (21) \end{aligned}$$

The initial conditions become

$$I_0 = I(t_0) \quad \text{and} \quad N_i = \varphi_i, \quad (22)$$

where $\varphi_i = \varphi(t_i)$ for $i = -k, -k+1, ..., 0$.

The equilibria of (20)-(21) can be calculated by putting $I = \bar{I}$ and $N = \bar{N}$. So we have

$$\begin{aligned} \bar{I} &= (1 + h\mu - h(\delta+\varepsilon+\gamma))\,\bar{I} - h\mu\frac{\bar{I}^2}{\bar{N}}, && (23)\\ \bar{N} &= -\varepsilon h\bar{I} + (1 - h\delta)\bar{N} + hp\bar{N}e^{-a\bar{N}}. && (24) \end{aligned}$$

Solving the nonlinear system (23)-(24), the equilibria are the same as in Theorem 2, i.e. if $R_0 \leq 1$, there exists only the disease-free equilibrium $(\bar{I}_0, \bar{N}_0)$:

$$\bar{I}_0 = 0; \quad \bar{N}_0 = \frac{1}{a}\ln\frac{p}{\delta}. \quad (25)$$

On the other hand, if $R_0 > 1$, there exist both the disease-free equilibrium $(\bar{I}_0, \bar{N}_0)$ and also the endemic equilibrium $(\bar{I}_+, \bar{N}_+)$ which defined by

$$\bar{I}_+ = \left(1 - \frac{1}{R_0}\right)\bar{N}_+; \quad \bar{N}_+ = \frac{1}{a}\ln\frac{p}{\delta + \varepsilon(1 - \frac{1}{R_0})}. \quad (26)$$

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Note that the equilibria of the numerical scheme (20)-(21) are the same as in the continuous problem (4)-(5). We will next investigate sufficient conditions for the asymptotic stability of the equilibria. Our analysis is focused on both the disease-free equilibrium and the endemic equilibrium.

3.2 Dynamical analysis of the equilibria for the numerical scheme

First, we use the linearisation method on the system (23)-(24) about the equilibrium $(\bar{I}, \bar{N})$. Let $I(t) = \bar{I} + u(t)$ and $N(t) = \bar{N} + v(t)$, the linearised system becomes

$$\begin{aligned} u_{n+1} &= \left(1 + h\mu(1 - \frac{1}{R_0}) - 2h\mu\frac{\bar{I}}{\bar{N}})\right) u_n \\ &\qquad + \left(h\mu\frac{\bar{I}^2}{\bar{N}^2}\right) v_n, \quad (27) \\ v_{n+1} &= -\varepsilon h u_n + (1 - h\delta)v_n + hpe^{-a\bar{N}}(1 - a\bar{N})v_{n-k}. \quad (28) \end{aligned}$$

Suppose that $x_n^0 = v_n$, and let

$$\begin{aligned} x_n^1 &= v_{n-1}, \\ x_n^2 &= x_{n-1}^1 = v_{n-2}, \\ &\vdots \\ x_n^k &= x_{n-1}^{k-1} = x_{n-2}^{k-2} = \ldots = x_{n-k-1}^1 = v_{n-k}, \end{aligned}$$

then (28) becomes

$$x_{n+1}^0 = -\varepsilon h u_n + (1 - h\delta)x_n^0 + hpe^{-a\bar{N}}(1 - a\bar{N})x_n^k.$$

The system (27)-(28) can be rewritten in a system of $k+2$ equations;

$$\begin{aligned} u_{n+1} &= \left(1 + h\mu(1 - \tfrac{1}{R_0}) - 2h\mu\tfrac{\bar{I}}{\bar{N}})\right) u_n \\ &\qquad + \left(h\mu\tfrac{\bar{I}^2}{\bar{N}^2}\right) x_n^0, \\ x_{n+1}^0 &= -\varepsilon h u_n + (1 - h\delta)x_n^0 \\ &\qquad + hpe^{-a\bar{N}}(1 - a\bar{N})x_n^k, \quad (29) \\ x_{n+1}^1 &= x_n^0, \\ &\vdots \\ x_{n+1}^k &= x_n^{k-1}, \end{aligned}$$

for $n = 1, 2, 3, \ldots$; and

$$\mathbf{x}_n = (u_n, x_n^0, x_n^1, \ldots, x_n^{k-1}, x_n^k)^T.$$

The system (29) can be written in matrix form

$$\mathbf{x}_{n+1} = \mathbf{A}(\bar{\mathbf{x}})\ \mathbf{x}_n, \quad (30)$$

where $\bar{\mathbf{x}}$ is the equilibrium of the system, i.e.

$$\bar{\mathbf{x}} = (\bar{I}, \bar{N}, \bar{N}, \ldots, \bar{N})^T.$$

Here, $\mathbf{A}$ is the constant matrix defined by

$$\mathbf{A}(\bar{\mathbf{x}}) = \begin{pmatrix} A_{1,1} & h\mu\frac{\bar{I}^2}{\bar{N}^2} & 0 & \cdots & 0 & 0 \\ -\varepsilon h & 1 - h\delta & 0 & \cdots & 0 & A_{2,k+2} \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \quad (31)$$

where

$$A_{1,1} = 1 + h\mu\left(1 - \frac{1}{R_0}\right) - 2h\mu\frac{\bar{I}}{\bar{N}}$$

and

$$A_{2,k+2} = hpe^{-a\bar{N}}(1 - a\bar{N}).$$

The matrix $\mathbf{A}$ has dimension $(k+2)\times(k+2)$. The system (30) is a linear system, so the stability of the system can be determined from its eigenvalues. To find the eigenvalues λ of $\mathbf{A}$, we need to solve $|\mathbf{A} - \lambda\mathbf{I}| = 0$, where $\mathbf{I}$ is the identity matrix. It is not difficult to simplify that the characteristic polynomial of $\mathbf{A}$ is

$$\begin{aligned} \Phi(\lambda) &= (A_{1,1} - \lambda)\left((1 - h\delta - \lambda)\lambda^k + A_{2,k+2}\right) \\ &\qquad + \varepsilon h^2\mu\frac{\bar{I}^2}{\bar{N}^2}\lambda^k = 0. \quad (32) \end{aligned}$$

For the system (29) to be stable at its equilibrium $(\bar{I}, \bar{N})$, we need all roots (eigenvalues) of (32) at the equilibrium to lie in the unit disk, i.e. $|\lambda| < 1$. The following lemma gives the sufficient conditions so that all roots of a polynomial equation lie inside the open unit disk.

Lemma 5. [10]
Consider the polynomial of degree k

$$\lambda^k + p_1\lambda^{k-1} + p_2\lambda^{k-2} + \ldots + p_{k-1}\lambda + p_k = 0. \quad (33)$$

If $\sum_{i=1}^k |p_i| \le 1$, then all roots of (33) lie inside the open unit disk, i.e. $|\lambda| < 1$.

Proof. See [10] and references therein for more details. □

Next, using advantages of Lemma 5, we show the theorem for sufficient conditions for the disease-free equilibrium, $(\bar{I}_0, \bar{N}_0)$, to be asymptotically stable.

Theorem 6. *Consider the numerical scheme (20)-(21), with the positive initial condition in (22). Suppose that $1 < p/\delta < e^2$. If, in addition, the condition*

$$h < \min\left\{\frac{1}{\delta}, \frac{2}{\delta + \varepsilon + \gamma - \mu}\right\} \quad (34)$$

holds and $R_0 < 1$, then $\bar{I}_0$ and $\bar{N}_0$ are asymptotically stable.

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Proof. For $\bar{I}_0 = 0$ and $\bar{N}_0 = \frac{1}{a}\ln(p/\delta)$, the characteristic equation (32) becomes

$$(A_{1,1} - \lambda)\left((1 - h\delta - \lambda)\lambda^k + A_{2,k+2}\right) = 0, \tag{35}$$

where $A_{1,1} = 1 + h\mu(1 - \frac{1}{R_0})$ and $A_{2,k+2} = h\delta(1 - \ln\frac{p}{\delta})$.

Solving (35), we have two factors:

$$\lambda = A_{1,1} = 1 + h\mu\left(1 - \frac{1}{R_0}\right), \tag{36}$$

and

$$(1 - h\delta - \lambda)\lambda^k + h\delta(1 - \ln\frac{p}{\delta}) = 0. \tag{37}$$

For the stability of the numerical scheme, we need to show that all solutions of (36)-(37) lie in the unit disk, i.e. $|\lambda| < 1$.

Case I: Consider (36). The solution is asymptotically stable when

$$|\lambda| = \left|1 + h\mu\left(1 - \frac{1}{R_0}\right)\right| < 1.$$

Solving the inequality above, it yields that

$$-2 < h\mu\left(1 - \frac{1}{R_0}\right) < 0.$$

Because $R_0 < 1$, so $1 - 1/R_0 < 0$, we have the condition

$$0 < h\mu < \frac{2}{\frac{1}{R_0} - 1} = \frac{2\mu}{\delta + \varepsilon + \gamma - \mu}.$$

Hence,

$$h < \frac{2}{\delta + \varepsilon + \gamma - \mu}, \tag{38}$$

for all $h > 0$.

Case II: Consider (37). It can be rearranged into

$$\lambda^{k+1} - (1 - h\delta)\lambda^k - h\delta(1 - \ln\frac{p}{\delta}) = 0. \tag{39}$$

From Lemma 5, the system is asymptotically stable if $|1 - h\delta| + |h\delta(1 - \ln\frac{p}{\delta})| < 1$. It provides that

$$h < \frac{1}{\delta} \text{ and } 1 < \frac{p}{\delta} < e^2. \tag{40}$$

Combining the conditions (38) and (40), we conclude that if $1 < p/\delta < e^2$, with $R_0 < 1$, and the condition (34) holds, then the disease-free equilibrium $(\bar{I}_0, \bar{N}_0)$ is asymptotically stable. □

Note that, for the disease-free equilibrium, the sufficient conditions are depended only on the ratio p/δ and the numerical step-size h. The results are similar as in Theorem 3.

Next, we will consider the case of no disease related death ($\varepsilon = 0$), and analyse dynamics of the selected SIS system at the endemic equilibrium, $(\bar{I}_+, \bar{N}_+)$. The following theorem shows the conditions for which $\bar{I}_+$ and $\bar{N}_+$ are asymptotically stable.

Theorem 7. *Consider the numerical method (20)-(21) with the positive initial condition in (22). Let $\varepsilon = 0$, then $R_0 = \lambda/(\delta + \gamma)$. Assume that $1 < p/\delta < e^2$. If $R_0 > 1$ and the condition*

$$h < \min\left\{\frac{1}{\delta}, \frac{2}{\mu - (\delta + \gamma)}\right\} \tag{41}$$

holds, then the endemic equilibrium $(\bar{I}_+, \bar{N}_+)$ of (20)-(21), where $\varepsilon = 0$, is asymptotically stable.

Proof. Let $\varepsilon = 0$, with $\bar{N}_+ = \bar{N}_0 = \frac{1}{a}\ln(p/\delta)$, and $\bar{I}_+ = (1 - 1/R_0)\bar{N}_+$ or $\bar{I}_+/\bar{N}_+ = 1 - 1/R_0$. Then (32) becomes

$$\left(1 - h\mu(1 - \frac{1}{R_0}) - \lambda\right)\left((1 - h\delta - \lambda)\lambda^k + h\delta(1 - \ln\frac{p}{\delta})\right) = 0. \tag{42}$$

Solving for λ in (42), we have

$$\lambda = 1 - h\mu\left(1 - \frac{1}{R_0}\right), \tag{43}$$

or

$$(1 - h\delta - \lambda)\lambda^k + h\delta(1 - \ln\frac{p}{\delta}) = 0. \tag{44}$$

For the asymptotic stability of the equilibria of the numerical scheme, we need to show that all solutions of (43)-(44) lie inside the unit disk, i.e. $|\lambda| < 1$.

Case I: in (43), the solution is asymptotically stable provided that

$$\left|1 - h\mu\left(1 - \frac{1}{R_0}\right)\right| < 1.$$

Hence

$$-2 < -h\mu\left(1 - \frac{1}{R_0}\right) < 0.$$

Since $R_0 > 1$, we have $1 - 1/R_0 > 0$. It yields

$$0 < h\mu < \frac{2}{1 - \frac{1}{R_0}} = \frac{2\mu}{\mu - (\delta + \gamma)},$$

or

$$h < \frac{2}{\mu - (\delta + \gamma)}, \tag{45}$$

for all $h > 0$.

Case II: consider (44), it can be rearranged to

$$\lambda^{k+1} - (1 - h\delta)\lambda^k - h\delta(1 - \ln\frac{p}{\delta}) = 0. \tag{46}$$

Like with (37), using Lemma 5, all roots of (46) lie inside the unit disk $|\lambda| < 1$ provided that

$$h < \frac{1}{\delta} \text{ and } 1 < \frac{p}{\delta} < e^2. \tag{47}$$

Combining the conditions (45) and (47), we conclude that if (41) holds for $1 < p/\delta < e^2$ and $R_0 > 1$, then $\bar{I}_+$ and $\bar{N}_+ = \bar{N}_0$ are asymptotically stable. □

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Compare the results for the numerical analysis in Theorem 7 with the analytical analysis in Theorem 4. We see that the conditions for stability are the same in general. There is only one exception. The conditions for the stability of the numerical solutions need an additional condition on the step-size h.

4 Numerical Results

The numerical experiments given in this part are divided into two cases: $R_0 < 1$ and $R_0 > 1$. This is to support our main contributions in Theorems 6 and Theorem 7.

4.1 Case $R_0 < 1$

Figure 1 shows the numerical solutions of the SIS model with $p = e^2, \delta = 1.1$ $(p/\delta < e^2), a = 2.0, \mu = 2.0, \gamma = \varepsilon = 1.0$; hence $R_0 < 1$. In this case $\bar{I}_0 = 0$ and $\bar{N}_0) \simeq 0.9523$. In Figure 1, both I $(\cdots$ dot) and N $(-$ solid) are attracted by the disease-free equilibrium, i.e. $(\bar{I}, \bar{N}) \to (\bar{I}_0, \bar{N}_0) = (0, 0.9523)$.

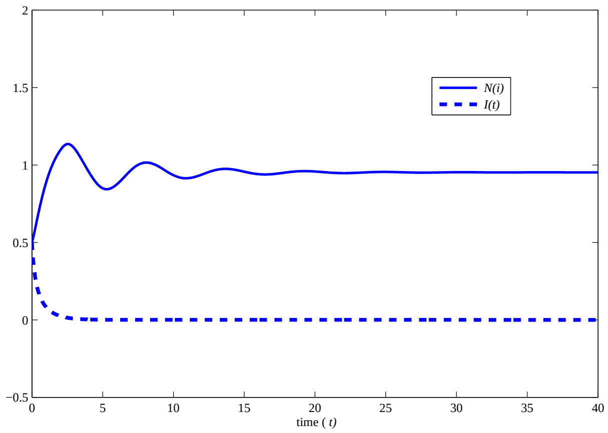

Figure 1: The numerical solutions of the SIS model with $p/\delta < e^2$ and $R_0 < 1$, where $\tau = 2.0$.

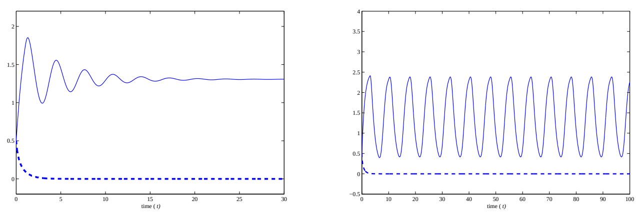

Figure 2: The numerical solutions of the SIS model with $p/\delta > e^2$ and $R_0 < 1$; where $\tau = 1.0$ (left) and $\tau = 3.0$ (right).

Next, Figure 2 compares the numerical solutions with different delay τ. Here we set $p = 15, \delta = 1.1$ $(p/\delta > e^2), a = 2.0, \mu = 2.0, \gamma = 1.0, \varepsilon = 1.0$. We can see that the delay affects the dynamical behaviour of the numerical solutions. We can see that the solution for $I(t)$ is attracted by $\bar{I}_0 = 0$, while the solution for $N(t)$ is changed its stability from asymptotic stability (Figure 2(left)) to periodic behaviour (Figure 2(right)).

4.2 Case $R_0 > 1$

For the case that $R_0 > 1$, Figure 3 shows the numerical experiments of the SIS epidemic model with $p = e^2, \delta = 1.1$ $(p/\delta < e^2), a = 2.0, \mu = 5.0, \gamma = 1.0, \varepsilon = 0$ when $\tau = 1.0$ and $\tau = 3.0$. The results indicate that if $R_0 > 1$, I and N are attracted by the endemic equilibrium, i.e. $(\bar{I}, \bar{N}) \to (\bar{I}_+, \bar{N}_+)$.

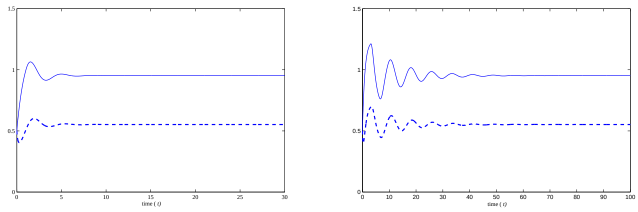

Figure 3: The numerical solutions of the SIS model with $p/\delta < e^2$ and $R_0 > 1$; where $\tau = 1.0$ (left) and $\tau = 3.0$ (right).

In addition, Figure 4 shows that when $p/\delta > e^2$. Here we set $p = 15, \delta = 1.1$ $(p/\delta > e^2), a = 2.0, \mu = 5.0, \gamma = 1.0, \varepsilon = 0$ The numerical solutions oscillate about the endemic equilibrium. Figure 4(left) and Figure 4(right) represent the solutions when $\tau = 1.5$ and $\tau = 1.65$, respectively. We can see that the behaviour of the solutions changes from a solution converging to the equilibrium to a periodic solution. Hence, a Hopf bifurcation occurs when τ is sufficiently large and $R_0 > 1$.

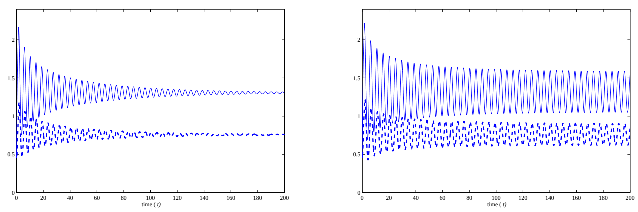

Figure 4: The numerical solutions of the SIS model with $p/\delta > e^2$ and $R_0 > 1$; where $\tau = 1.5$ (left) and $\tau = 1.65$ (right).

Note that in Figure 4, we can see that the numerical solution undergoes a Hopf bifurcation when τ is sufficiently large. Suppose that τ^* is the bifurcation point. We can see that $\tau^* \in (1.5, 1.65)$. The study of Hopf bifurcation is an interesting topic, and it can be a further study for this problem.

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