A New Approach to the Divergence of a Tensor and Application to the Curvature Tensor in the General Theory of Relativity

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Abstract—**In the General Theory of Relativity, the expression of the Einstein tensor is deduced by requiring the tensor to fulfil three conditions, among which the value zero of its divergence. The proof of the value zero of the divergence, has been carried out in literature, in fact, in two manners: either by a considered to be a direct simple verification in a book of Einstein or using the Bianchi identity by several authors. In both cases the known solutions show certain drawbacks.**

In the present paper, the author aims at: a) to precise the manner of calculating the divergence of a tensor, taking into account its character, namely covariant and contravariant in the respective index and certain required restrictions for avoiding errors, what has not been examined in literature; b) to show drawbacks of known proofs of the divergence of the Einstein tensor; c) to bring a new proof avoiding the usage of Bianchi identity what is also not found in the known literature.

Index Terms—**Curvature tensor, Divergence of a tensor, General theory of relativity, Field theory.**

I. Introduction

The calculation of the divergence of a tensor occurs in many problems of Physical Mathematics. We consider that the notion of divergence of a tensor has not been enough defined and analysed in literature. We shall try to establish a consistent definition of a tensor including the calculation procedure, and to apply it to the calculation of the divergence of the curvature tensor of the four-dimensional continuum space-time. In order to avoid any confusion, we shall disregard the cosmological term, introduced by Einstein, having in view his explanations in the book [1, Appendix 1, pp. 129-130], and moreover that it has no relation with the mathematical analysis for calculating the divergence of a tensor.

Several studies on the divergence of tensors have been carried out for a long time, [1]-[10]. In paper [10], a deep study on the differential operations on tensors has been performed without using the tensor calculus, but instead the general method concerning the differential quadratic forms, from the classical mathematical analysis, has been utilized. The methods used in the other papers will be examined further on.

In order to facilitate the understanding, and have a uniform system of symbols, the paper includes three appendices: the first containing several tensor formulae frequently used in the paper; the second presenting a relatively simple deduction of the Riemann curvature; and the third containing a very short deduction of the Bianchi identity.

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II. The Divergence of Covariant and Contravariant Tensors

The calculation of the divergence of a tensor, as known, is based on the calculation of the derivative of that tensor.

The divergence of a vector in a Cartesian system of co-ordinates is:

$$\nabla_k A^i = \frac{\partial A^i}{\partial x^k}, \quad \forall k = i; \quad i \in [0, 3]. \tag{1}$$

We recall the symbols of the covariant and contravariant derivatives, namely ∇_k and ∇^k, respectively. Correspondingly, we may use the denominations of covariant and contravariant divergence, respectively.

The expression of the covariant derivative of a contravariant tensor of rank 1, i.e., a contravariant vector, is:

$$\nabla_k A^i = \frac{\partial A^i}{\partial x^k} + \Gamma^i_{lk} A^l, \quad \forall i, l, k \in [0, 3], \tag{2}$$

where $\Gamma_{v,lk}$, and $\Gamma^i_{lk} = g^{iv}\Gamma_{v,lk}$ are the Christoffel symbols of the first and second kind, respectively. The expression of the divergence of a vector in any system of co-ordinates is obtained starting from the relation (2), contracted in indices i, k:

$$\nabla_i A^i = \frac{\partial A^i}{\partial x^i} + \Gamma^i_{li} A^l, \quad \forall i, l \in [0, 3], \tag{3}$$

and represents a tensor of rank zero, i.e., a scalar. Relation (3) may also be written:

$$\operatorname{Div} A^i = \nabla_i A^i = \frac{1}{\sqrt{\pm g}} \cdot \frac{\partial}{\partial x^i}\left(\sqrt{\pm g}\, A^i\right), \quad \forall i \in [0, 3], \tag{3 a}$$

where g represents the expression of the determinant of the fundamental tensor, and the plus and minus signs from denominator and numerator correspond to each other, the sign minus being taken if g is negative. The symbol Div,

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instead of div, has been used in order to emphasize the general system of reference.

The covariant derivative of a covariant tensor of rank 1, i.e., a covariant vector, is given by the following relation, and its divergence results by contracting the expression in indices *i* and *k*:

$$\nabla_k A_i = \frac{\partial A_i}{\partial x^k} - g^{vl}\Gamma_{l,ik}\, A_v\,, \quad k = i\,. \tag{4}$$

The contravariant derivative of the same tensor is given by the following relation, and its divergence results by contracting the expression in indices *i* and *k*:

$$\nabla^k A_i = g^{pk}\nabla_p A_i = g^{pk}\left(\frac{\partial A_i}{\partial x^p} - g^{vl}\Gamma_{l,ip}\, A_v\right), \quad k = i\,, \quad \forall\, i, l, p \in [0, 3]. \tag{5}$$

The expression of the divergence of a contravariant tensor of rank 2 may be written similarly, starting from the formula of the covariant derivative of a twice contravariant tensor G^{ij}, contracted in indices *j* and *k*:

$$\nabla_k G^{ij} = \frac{\partial G^{ij}}{\partial x^k} + \Gamma^i_{lk}\, G^{lj} + \Gamma^j_{lk}\, G^{il}\,, \quad \forall\, i, j \in [0, 3], \quad k = j\,. \tag{6}$$

Let us to consider a covariant tensor of rank 2, the divergence can be expressed in several manners, as below:

$$\nabla^k G_{ik} = g^{ks}\nabla_s G_{ik} = \nabla_s g^{ks} G_{ik} = \nabla_s G_i^s\,. \tag{6 a}$$

Remarks on the calculation of the divergence of a tensor

From the relations above, we may realize the following properties:

a. The divergence of a given contravariant tensor results from the expression of the covariant derivative of that tensor, and due to the contraction, the divergence will be a tensor of a rank less by two units with respect to that of the mentioned expression. In this case, the contraction is performed in each pair of concerned indices, one subscript and the other superscript. It may be called covariant divergence.

b. The divergence of a given covariant tensor results like for the preceding case, *a*, but in this case, *b*, the contraction must be performed for each pair of concerned indices placed in the same position, both subscript. It may also be called covariant divergence.

c. For to keep the same manner of contraction as in case *a*, the divergence of a given covariant tensor results from the expression of the contravariant derivative of that tensor. It may be called contravariant divergence. The contraction will be performed as for case *a*. The divergences of a tensor, calculated by procedures *b* and *c*, are expressions which differ by a factor representing the fundamental tensor.

d. A convenient manner for simplifying the calculation of the divergence of a tensor, in a geodesic system of co-ordinates, in the case in which the fundamental tensor occurs as a factor in any term, is the transport of the fundamental tensor before or behind the differential operator. However, attention must be paid in order to avoid the transport if in the respective term including the differential operator, besides the differential operator, there are tensors with the same index as the differential operator. If the respective index appears three times or more, the transport may produce errors.

Each of the procedures from *b* and *c* may be advantageous in certain applications.

The calculation of the divergence in the case of a tensor of any rank greater than unity, say of rank 2, yields not a unique result as in the case of a contravariant tensor of rank 1, because the calculation has to be carried out with respect to the pair of indices formed by that of the divergence operator and by one of the tensor. Also, by choosing various pairs of indices we may obtain different results for the divergence.

III. Application on the Case of the Riemann Curvature Tensor and Einstein Tensor

The equations of the classical General Theory of Relativity contain the Einstein tensor developed starting from the Riemann tensor of the four-dimensional continuum space-time. For more clarity, in Appendix II, we have given a relatively simple demonstration of the Riemann curvature tensor for the four-dimensional continuum space-time.

In the work [1, pp. 99-101, formula (96)], Einstein gives directly the tensor considered to satisfy the three required conditions the value zero of the divergence included. Let us consider the Einstein tensor:

$$R_{ij} + a\, g_{ij} R\,, \tag{7}$$

where R_{ij} is the Riemann tensor of rank 2, also called Ricci tensor, and R is given by the relation:

$$R = g^{ij} R_{ij} = R_i^i\,, \tag{7 a}$$

and a of (7) is a constant. For the third condition to be satisfied, the constant a should be equal to $\frac{-1}{2}$. The same expression is given in many works, among which [3, Chapter III, par. 44, p. 248], [4, p. 200], [5, p. 99]. Let us examine this solution. For calculating the expression of the divergence, we shall use the contravariant derivative. We shall assume that the following relation exists:

$$\begin{aligned} \mathrm{Div}\left(R_{ij} - \frac{1}{2} g_{ij} R\right) &= \nabla^i \left(R_{ij} - \frac{1}{2} g_{ij} R\right) \\ &= g^{ip}\nabla_p \left(R_{ij} - \frac{1}{2} g_{ij} R\right) \\ &= \nabla_p \left(g^{ip} R_{ij} - \frac{1}{2} g^{ip} g_{ij} R\right) \\ &= \nabla_p \left(R_j^p - \frac{1}{2} g_j^p R\right) = 0\,, \end{aligned} \tag{8}$$

where for the transformations, we have used the relations of Appendix I. We shall expand the last term of expression (8), around any point, having in view the usage of geodesic co-ordinates, also recalled in Appendix I. It follows:

$$\nabla_p\left(R_j^p - \frac{1}{2} g_j^p R\right) = \left(\frac{\partial}{\partial x^0} R_j^0 + \frac{\partial}{\partial x^1} R_j^1 + \frac{\partial}{\partial x^2} R_j^2 + \frac{\partial}{\partial x^3} R_j^3\right) - \frac{1}{2} \cdot \frac{\partial}{\partial x^j} R. \tag{8 a}$$

We may examine in which way the divergence of the tensor of the left-hand side of (8 a) can be calculated. In the right-hand side, the first term (within parentheses) contains certain terms of the form $\frac{\partial}{\partial x^2} R_3^2$, which cannot appear in the second term, which contains only terms of the form $\frac{\partial}{\partial x^j} R_p^p$. A simple method could be to transform this relation by multiplying it with certain tensor factors. For instance, by multiplying (8 a) with g_p^j and having in view that g_p^j is different from zero only for $p = j$, it is difficult to reach a useful result because in each term the same index p occurs three times, not twice. Therefore, other methods have to be applied in order to calculate the divergence.

IV. Calculation of the Divergence Given by Einstein

In his book [1, after formula (97)], he considers a certain tensor containing the Riemann tensor of rank 2, and the corresponding scalar quantity, at any point in a four-dimensional continuum space-time. He has used a system of co-ordinates for which the fundamental tensor g_{ij} and g^{ij} have the value ± 1 if $i = j$, and the value zero if $i \neq j$. Although no mention is made, one may suppose that a geodesic co-ordinate system has been in view. Further, he considered that for the proof of above, namely the value zero of the divergence of the considered tensor, it suffices to prove that the divergence expression of the considered tensor is zero, namely:

$$\frac{\partial}{\partial x_s}\left[\sqrt{-g}\; g^{vs}\left(R_{uv} - \frac{1}{2} g_{uv} R\right)\right] = 0, \tag{9}$$

where g represents the expression of the determinant of the fundamental tensor.

The expression of Riemann tensor of rank 2, hence a Ricci tensor, he used in relation (9), has been:

$$R_{uv} = -\frac{\partial \Gamma_{uv}^r}{\partial x^r} + \Gamma_{uw}^r \Gamma_{vr}^w + \frac{\partial \Gamma_{ur}^r}{\partial x^v} - \Gamma_{uv}^r \Gamma_{rw}^w, \tag{10}$$

where the capital letters gamma represent the Christoffel symbols. Further, the explanations lead to the conclusion that this expression is indeed zero. However, no complete explanations are given.

The subsequent authors have used in their works geodesic systems of reference and resorted to certain corresponding general tensor relations, including a deeper analysis.

V. Remarks Concerning the Calculation of the Expression Involving Riemann Tensor

In order to clarify the calculation of the tensor divergence, avoiding certain errors, we shall consider an example concerning the mixed Riemann curvature tensor of rank four and its transformation into a covariant tensor of the same rank. Although well known, we shall remake a simple deduction in order to emphasize a very important property accepted but never explicitly mentioned in the known works. According to (A.II.5), we have:

$$R_{rsu}^{\ \cdots i} = \frac{\partial \Gamma_{ur}^i}{\partial x^s} - \frac{\partial \Gamma_{us}^i}{\partial x^r} + \Gamma_{qs}^i \Gamma_{ur}^q - \Gamma_{qr}^i \Gamma_{us}^q. \tag{11}$$

Let us look for expressing the formula:

$$R_{rsup} = g_{ip}\, R_{rsu}^{\ \cdots i}. \tag{12}$$

There follows

$$g_{ip}\, R_{rsu}^{\ \cdots i} = g_{ip}\left(\frac{\partial \Gamma_{ur}^i}{\partial x^s} - \frac{\partial \Gamma_{us}^i}{\partial x^r}\right) + g_{ip}\left(\Gamma_{qs}^i \Gamma_{ur}^q - \Gamma_{qr}^i \Gamma_{us}^q\right)$$
$$= \frac{\partial\left(g_{ip}\, g^{iw} \Gamma_{w,ur}\right)}{\partial x^s} - \frac{\partial\left(g_{ip} g^{iw} \Gamma_{w,us}\right)}{\partial x^r} - \Gamma_{ur}^i \frac{\partial g_{ip}}{\partial x^s}$$
$$+ \Gamma_{us}^i \frac{\partial g_{ip}}{\partial x^r} + g_{ip}\left(\Gamma_{qs}^i \Gamma_{ur}^q - \Gamma_{qr}^i \Gamma_{us}^q\right). \tag{13}$$

We shall calculate the parentheses, having in view that $g_{ip} g^{iw} = g_p^w$, w taking all values from zero to 3, and p being fixed, we have $g_p^w = g_p^p = 1$. We shall obtain:

$$g_{ip}\, R_{rsu}^{\ \cdots i} = \frac{\partial \Gamma_{p,ur}}{\partial x^s} - \frac{\partial \Gamma_{p,us}}{\partial x^r} - \Gamma_{ur}^i\left(\Gamma_{p,is} + \Gamma_{i,ps}\right)$$
$$+ \Gamma_{us}^i\left(\Gamma_{p,ir} + \Gamma_{i,pr}\right) + g_{ip}\left(\Gamma_{qs}^i \Gamma_{ur}^q - \Gamma_{qr}^i \Gamma_{us}^q\right) \tag{13 a}$$

or

$$g_{ip}\, R_{rsu}^{\ \cdots i} = \frac{\partial \Gamma_{p,ur}}{\partial x^s} - \frac{\partial \Gamma_{p,us}}{\partial x^r} - \Gamma_{ur}^i\left(\Gamma_{p,is} + \Gamma_{i,ps}\right) \tag{13 b}$$
$$+ \Gamma_{us}^i\left(\Gamma_{p,ir} + \Gamma_{i,pr}\right) + g_{ip}\, g^{iw}\left(\Gamma_{w,qs} \Gamma_{ur}^q - \Gamma_{w,qr} \Gamma_{us}^q\right)$$

and, with the same remark as after (13), there follows:

$$g_{ip}\, R_{rsu}^{\ \cdots i} = \frac{\partial \Gamma_{p,ur}}{\partial x^s} - \frac{\partial \Gamma_{p,us}}{\partial x^r} - \Gamma_{ur}^i\left(\Gamma_{p,is} + \Gamma_{i,ps}\right)$$
$$+ \Gamma_{us}^i\left(\Gamma_{p,ir} + \Gamma_{i,pr}\right) + g_p^w\left(\Gamma_{w,qs} \Gamma_{ur}^q - \Gamma_{w,qr} \Gamma_{us}^q\right). \tag{13 c}$$

Further on, we have:

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$$g_{ip}\, R_{rsu}^{\dots i} = \frac{\partial \Gamma_{p,ur}}{\partial x^s} - \frac{\partial \Gamma_{p,us}}{\partial x^r} - \Gamma_{ur}^q \left(\Gamma_{p,qs} + \Gamma_{q,ps}\right)$$
$$+ \Gamma_{us}^q \left(\Gamma_{p,qr} + \Gamma_{q,pr}\right) + g_p^p \left(\Gamma_{p,qs}\Gamma_{ur}^q - \Gamma_{p,qr}\Gamma_{us}^q\right). \quad (13\text{ d})$$

There follows:

$$R_{rsup} = g_{ip}\, R_{rsu}^{\dots i} = \frac{\partial \Gamma_{p,ur}}{\partial x^s} - \frac{\partial \Gamma_{p,us}}{\partial x^r}$$
$$- \Gamma_{ur}^q\, \Gamma_{q,ps} + \Gamma_{us}^q\, \Gamma_{q,pr}\,. \quad (13\text{ e})$$

Replacing the Christoffel symbols, we obtain:

$$R_{rsup} = \frac{\partial}{\partial x^s}\frac{1}{2}\left(\frac{\partial g_{pr}}{\partial x^u} + \frac{\partial g_{pu}}{\partial x^r} - \frac{\partial g_{ur}}{\partial x^p}\right)$$
$$- \frac{\partial}{\partial x^r}\frac{1}{2}\left(\frac{\partial g_{ps}}{\partial x^u} + \frac{\partial g_{pu}}{\partial x^s} - \frac{\partial g_{us}}{\partial x^p}\right) \quad (14)$$
$$- \Gamma_{ur}^q\, \Gamma_{q,ps} + \Gamma_{us}^q\, \Gamma_{q,pr}\,.$$

After reducing the like terms, we obtain:

$$R_{rsup} = \frac{1}{2}\left(\frac{\partial^2 g_{pr}}{\partial x^s\,\partial x^u} - \frac{\partial^2 g_{ur}}{\partial x^s\,\partial x^p} - \frac{\partial^2 g_{ps}}{\partial x^r\,\partial x^u} + \frac{\partial^2 g_{us}}{\partial x^r\,\partial x^p}\right)$$
$$- \Gamma_{ur}^q\, \Gamma_{q,ps} + \Gamma_{us}^q\, \Gamma_{q,pr}\,. \quad (15)$$

We shall make here a very useful remark, not found in the known works. In general, the quantity g^{ij} is expressed starting from a Galilean system of reference by formula:

$$g_{ij} = \sum_{u=0}^{u=3} \frac{\partial y^u}{\partial x^i} \cdot \frac{\partial y^u}{\partial x^j}. \quad (16)$$

In this case:

$$\frac{\partial g_{ij}}{\partial x^k} = \frac{\partial g_{ik}}{\partial x^j}. \quad (16\text{ a})$$

If we accepted this relation, then all curvatures would be zero. The reason is that according to relation (16 a), the obtained fundamental tensor may always be reduced to a Galilean one, therefore the curvature will be zero. Hence, for domains with curvature, the usage of relation (16 a) is not allowed. This property although accepted in calculations, in the known works, is never explicitly stated.

VI. Calculation of the Divergence by the Authors Using the Bianchi Identity

The other known authors have used the Bianchi identity for this purpose [1]-[5], [8]. In this case, they start from a tensor of rank four, in order to establish the property of a tensor of rank 2, what seems to be an indirect and more complicated way. Further, a set of transformations bring the expression from a vector of rank four to a vector of rank 2, namely a mixed vector.

For more facility in following this paper, in Appendix III, we have given a very simple deduction of the Bianchi identity.

These proofs, aiming the calculation of the divergence of the curvature tensor are based on Bianchi identity and on the usage of the geodesic co-ordinates, like in works [1]-[5].

According to relation (A.III.2) and the inversion of indices, the following identity may be written:

$$\nabla_k R_{rsu}^i - \nabla_s R_{rku}^i - \nabla_r R_{ksu}^i = 0\,. \quad (17)$$

By putting $i = r$, and contracting with respect to index i, there follows:

$$\nabla_k R_{su} - \nabla_s R_{ku} - \nabla_i R_{ksu}^i = 0\,. \quad (18\text{ a})$$

Multiplying the last relation by g^{su}, and taking into account relation (A.I.4 c), one will obtain:

$$-\nabla_k R_s^s + \nabla_s R_k^s + \nabla_i R_k^i = 0\,, \quad (18\text{ b})$$

and replacing the index s by i in the second term of the left-hand side of relation (18 b), both taking all values from zero to 3, one obtains:

$$-\nabla_k R_s^s + 2\nabla_i R_k^i = 0\,. \quad (18\text{ c})$$

From here on, we shall modify the known procedure [4, p. 201], avoiding the Kronecker symbol. Starting from relation (18 c) and changing the first index of the first term, we get:

$$-\nabla_i\, g_k^i R_s^s + 2\nabla_i R_k^i = 0\,, \quad (19)$$

and taking into account (7 a), we obtain

$$\nabla_i\left(R_k^i - \frac{1}{2} g_k^i R\right) = 0\,, \quad (20)$$

where the term within parentheses corresponds to that of Einstein tensor, but in a mixed form.

III. The Proposed New Proof

Unlike the other methods, we shall consider the tensor we are looking for as not known. Let us start from the general formula of the Riemann curvature of rank 2, hence the Ricci form, of Appendix II, in geodesic co-ordinates:

$$R_{su} = \left(\frac{\partial \Gamma_{ur}^r}{\partial x^s} - \frac{\partial \Gamma_{us}^r}{\partial x^r}\right), \quad \forall\, r \in [0, 3], \quad (21)$$

and s and u are fixed values chosen from the set $[0, 3]$. Then, from formula (21), also using geodesic co-ordinates, we get:

$$\nabla_k R_{su} = \nabla_k\left(\frac{\partial \Gamma_{ur}^r}{\partial x^s} - \frac{\partial \Gamma_{us}^r}{\partial x^r}\right) = \frac{\partial}{\partial x^k}\left(\frac{\partial \Gamma_{ur}^r}{\partial x^s} - \frac{\partial \Gamma_{us}^r}{\partial x^r}\right)$$
$$= \frac{\partial}{\partial x^s}\cdot\frac{\partial \Gamma_{ur}^r}{\partial x^k} - \frac{\partial}{\partial x^r}\cdot\frac{\partial \Gamma_{us}^r}{\partial x^k}, \quad \forall\, r \in [0, 3], \quad (22)$$

and k, s, u are fixed values chosen from the set $[0, 3]$. Therefore, according to formula (21) or (22), by adding and subtracting the same term, we can write:

$$\nabla_k R_{su} = \frac{\partial}{\partial x^s}\left(\frac{\partial \Gamma^r_{ur}}{\partial x^k} - \frac{\partial \Gamma^r_{uk}}{\partial x^r}\right) + \frac{\partial}{\partial x^r}\left(\frac{\partial \Gamma^r_{uk}}{\partial x^s} - \frac{\partial \Gamma^r_{us}}{\partial x^k}\right), \quad \forall r \in [0, 3], \tag{23}$$

and k, s, u are fixed values, like in the preceding relation. Consequently:

$$\nabla_k R_{su} = \frac{\partial}{\partial x^s}\left(\frac{\partial \Gamma^r_{ur}}{\partial x^k} - \frac{\partial \Gamma^r_{uk}}{\partial x^r}\right) + \frac{\partial}{\partial x^w}\left(\frac{\partial \Gamma^w_{uk}}{\partial x^s} - \frac{\partial \Gamma^w_{us}}{\partial x^k}\right), \quad \forall r \in [0, 3] \tag{23 a}$$

or, with the symbols of Appendix II, we have:

$$\nabla_k R_{su} = \nabla_s R_{ku} + \nabla_w R^w_{ksu}, \quad \forall w \in [0, 3]. \tag{24}$$

After rearranging the terms, we obtain:

$$\nabla_k R_{su} - \nabla_s R_{ku} - \nabla_w R^w_{ksu} = 0, \quad \forall w \in [0, 3]. \tag{24 a}$$

By multiplying both sides with g^{su}, it follows:

$$-\nabla_k R^s_s + \nabla_s R^s_k + \nabla_w R^w_k = 0, \quad \forall s, w \in [0, 3]. \tag{25}$$

Therefore:

$$-\nabla_k R^s_s + 2\nabla_s R^s_k = 0. \tag{26}$$

Let us use the contravariant derivative:

$$-\nabla^p g_{kp} R^s_s + 2\nabla^p g_{sp} R^s_k = 0. \tag{27 a}$$

Taking into account relation (7 a), and changing the order of terms, relation (27 a) yields:

$$\nabla^p\left(R_{kp} - \frac{1}{2} g_{kp} R\right) = 0, \quad \forall p \in [0, 3]. \tag{28}$$

Therefore, we have deduced, within parentheses, just the Einstein tensor, by a say inductive way, without resorting to the Bianchi identity. Due to its form, relation (28) represents the contravariant divergence.

APPENDIX I

Tensor Formulae

According to formulae of Appendices A.1, A.2, A.3 of [6], and starting from those formulae, we shall obtain:

$$g^{uj} g_{ui} = g^j_i; \quad g^j_i = 1, \quad \forall i = j; \qquad g^j_i = 0, \quad \forall i \neq j. \tag{A.I.1 a, b}$$

$$g^{ij} g_{ij} = g^i_i, \quad i = \text{const}; \qquad g^{ij} g_{ij} = 4, \quad \forall i, j \in [0, 3]. \tag{A.I.2 a, b}$$

$$g^j_i g^i_j = 1, \quad i = \text{const}; \qquad g^j_i g^i_j = 4, \quad \forall i, j \in [0, 3]. \tag{A.I.3 a, b}$$

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma_{i,jk} + \Gamma_{j,ik}; \qquad \nabla_k g_{ij} = 0; \quad \nabla_k g^{ij} = 0, \quad \forall i, j, k \in [0, 3]. \tag{A.I.4 a-c}$$

$$\nabla_k g^j_i = 0, \quad \forall i, j, k \in [0, 3]. \tag{A.I.5}$$

$$\nabla_k C^i_j = \frac{\partial C^i_j}{\partial x^k} + \Gamma^i_{lk} C^l_j - \Gamma^l_{jk} C^i_l. \tag{A.I.6}$$

All formulae of this Appendix are useful for the transformations occurring in the previous treatment.

Besides the relations above, the geodesic co-ordinate system has been used in certain applications [3, Chapter III, par. 42, p. 232]. We recall, more detailed than usually, that a geodesic reference system, generally, could be considered in a small space around any chosen point. The elements of the fundamental tensor are assumed to be functions of point, the first derivatives of which have, at the chosen point, the value zero, but their derivatives of higher orders may exist. If a tensor is zero at that point, it will keep the value zero, at that point, in any other system of reference.

The corresponding expressions of the co-ordinates when passing from any co-ordinate system, to a geodesic system, can be obtained from the formulae of transformation of the Christoffel symbols when passing from a system to another one. Also, for this scope, we can use the expressions of the covariant derivative of a covariant vector, when passing from the former system to the latter.

It is to be mentioned that at any point there is an infinity of geodesic reference systems [9, p. 77], and it is possible to choose the more convenient for the purpose we are looking for.

APPENDIX II

The Expression of the Riemann Curvature Tensor

Let us consider the covariant derivatives:

$$\nabla_r A^i = \frac{\partial A^i}{\partial x^r} + \Gamma^i_{ur} A^u, \tag{A.II.1}$$

where the left-hand side of each relation is a tensor of rank 2, once covariant and once contravariant. The following covariant derivatives will be expressed by using relation (A.I.6):

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$$\nabla_s\left(\nabla_r A^i\right)=\frac{\partial^2 A^i}{\partial x^s \partial x^r}+\frac{\partial \Gamma_{ur}^i}{\partial x^s} A^u+\Gamma_{ur}^i \frac{\partial A^u}{\partial x^s}$$
$$+\Gamma_{qs}^i\left(\frac{\partial A^q}{\partial x^r}+\Gamma_{ur}^q A^u\right)-\Gamma_{rs}^q\left(\frac{\partial A^i}{\partial x^q}+\Gamma_{uq}^i A^u\right), \quad \text{(A.II.2 a)}$$
$$\forall q, u \in [0,3]; \quad \forall i, r, s \in [0,3],$$

and

$$\nabla_r\left(\nabla_s A^i\right)=\frac{\partial^2 A^i}{\partial x^r \partial x^s}+\frac{\partial \Gamma_{us}^i}{\partial x^r} A^u+\Gamma_{us}^i \frac{\partial A^u}{\partial x^r}$$
$$+\Gamma_{qr}^i\left(\frac{\partial A^q}{\partial x^s}+\Gamma_{us}^q A^u\right)-\Gamma_{sr}^q\left(\frac{\partial A^i}{\partial x^q}+\Gamma_{uq}^i A^u\right), \quad \text{(A.II.2 b)}$$
$$\forall q, u \in [0,3]; \quad \forall i, r, s \in [0,3].$$

Now, in order to obtain a simple expression we shall subtract the last two equations, side by side, and we shall take into consideration that the terms in which certain indices are denoted by different letters, but take the same values from zero to 3, are like terms. There follows:

$$\left(\nabla_s \nabla_r-\nabla_r \nabla_s\right) A^i=\left(\frac{\partial \Gamma_{ur}^i}{\partial x^s}-\frac{\partial \Gamma_{us}^i}{\partial x^r}\right) A^u$$
$$+\left(\Gamma_{qs}^i \Gamma_{ur}^q-\Gamma_{qr}^i \Gamma_{us}^q\right) A^u, \quad \forall i, q, r, s, u \in [0,3]. \quad \text{(A.II.3)}$$

The expression (A.II.3) may be written:

$$\left(\nabla_s \nabla_r-\nabla_r \nabla_s\right) A^i=R_{rsu}^i A^u, \quad \forall i, r, s, u \in [0,3]. \quad \text{(A.II.4)}$$

The right-hand side of expression (A.II.4), apart from the contravariant vector as factor, satisfies the first two conditions, and is just the Riemann curvature tensor, in which we have separated, by parentheses, two terms:

$$R_{rsu}^i=R_{rsu}^{\cdot\cdot\cdot i}=\left(\frac{\partial \Gamma_{ur}^i}{\partial x^s}-\frac{\partial \Gamma_{us}^i}{\partial x^r}\right)$$
$$+\left(\Gamma_{qs}^i \Gamma_{ur}^q-\Gamma_{qr}^i \Gamma_{us}^q\right), \quad \forall i, r, s, u \in [0,3]. \quad \text{(A.II.5)}$$

We should remark that the positioning of the superscript is not absolutely necessary. By contracting the expression (A.II.5) in indices *i* and *r*, we shall obtain:

$$R_{su}=\left(\frac{\partial \Gamma_{ur}^r}{\partial x^s}-\frac{\partial \Gamma_{us}^r}{\partial x^r}\right)+\left(\Gamma_{qs}^r \Gamma_{ur}^q-\Gamma_{qr}^r \Gamma_{us}^q\right), \quad \text{(A.II.6)}$$
$$\forall r, s, u \in [0,3].$$

It is to be noted that in geodesic co-ordinate reference system, the expression (A.II.6), also called Ricci tensor, becomes much simpler, as it will be explained in Appendix III.

APPENDIX III

The Bianchi Identity

We shall give a simplified proof starting from the expressions of Appendix II.

As known, in geodesic co-ordinate system, the Christoffel symbols are zero, but their derivatives with respect to co-ordinates may be different from zero. Therefore the covariant derivative of expression (A.II.5) will concern only the first parenthesis and will be:

$$\nabla_k R_{rsu}^i=\frac{\partial^2 \Gamma_{ur}^i}{\partial x^k \partial x^s}-\frac{\partial^2 \Gamma_{us}^i}{\partial x^k \partial x^r}, \quad \text{(A.III.1)}$$

and the obtained result is valid in any system of reference. Using relation (A.III.1), there follows:

$$\nabla_k R_{rsu}^i+\nabla_s R_{kru}^i+\nabla_r R_{sku}^i=0, \quad \text{(A.III.2)}$$

which is the Bianchi identity.

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