# Planning Inspections of Fatigued Aircraft Structures via Damage Tolerance Approach

Nicholas A. Nechval, Maris Purgailis, Kaspars Cikste, and Konstantin N. Nechval

*Abstract*— **Fatigue is one of the most important problems of aircraft arising from their nature as multiple-component structures, subjected to random dynamic loads. For guaranteeing safety, the structural life ceiling limits of the fleet aircraft are defined from three distinct approaches: Safe-Life, Fail-Safe, and Damage Tolerance approaches. The common objectives to define fleet aircraft lives by the three approaches are to ensure safety while at the same time reducing total ownership costs. In this paper, the Damage Tolerance approach is considered and the focus is on the inspection scheme with decreasing intervals between inspections. The paper proposes an analysis methodology to determine appropriate decreasing intervals between inspections of fatigue-sensitive aircraft structures (as alternative to constant intervals between inspections often used in practice), so that risk of catastrophic accident during flight is minimized. The suggested approach is unique and novel in that it allows one to utilize judiciously the results of earlier inspections of fatigued aircraft structures for the purpose of determining the time of the next inspection and estimating the values of several parameters involved in the problem that can be treated as uncertain. An illustrative example is given.**

*Index Terms*—**Aircraft, Fatigue crack, Inspection, Planning.**

## I. Introduction

In spite of decades of investigation, fatigue response of materials is yet to be fully understood. This is partially due to the complexity of loading at which two or more loading axes fluctuate with time. Examples of structures experiencing such complex loadings are automobile, aircraft, off-shores, railways and nuclear plants. Fluctuations of stress and/or strains are difficult to avoid in many practical engineering situations and are very important in design against fatigue failure. There is a worldwide need to rehabilitate civil infrastructure. New materials and methods are being broadly investigated to alleviate current problems and provide better and more reliable future services. While most industrial failures involve fatigue, the assessment of the fatigue reliability of industrial components being subjected to various dynamic loading situations is one of the most difficult engineering problems. This is because material degradation processes due to fatigue depend upon material characteristics, component geometry, loading history and environmental conditions. The traditional analytical method of engineering fracture mechanics (EFM) usually assumes that crack size, stress level, material property and crack growth rate, etc. are all deterministic values which will lead to conservative or very conservative outcomes. However, according to many experimental results and field data, even in well-controlled laboratory conditions, crack growth results usually show a considerable statistical variability (as shown in Fig. 1).

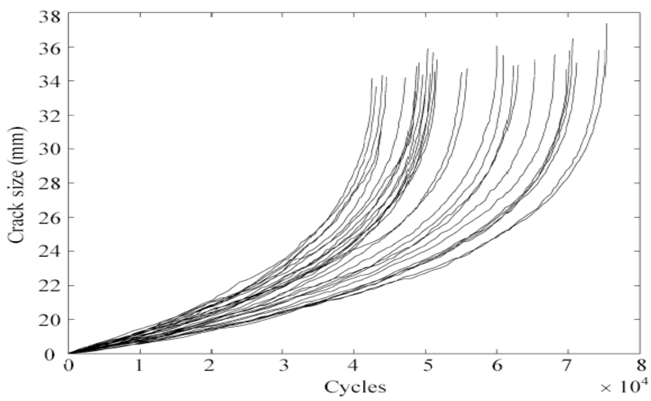


**Fig. 1.** Constant Amplitude Loading Fatigue Test Data Curves

Fatigue is one of the most important problems of aircraft arising from their nature as multiple-component structures, subjected to random dynamic loads. The analysis of fatigue crack growth is one of the most important tasks in the design and life prediction of aircraft fatigue-sensitive structures (for instance, wing, fuselage) and their components (for instance, aileron or balancing flap as part of the wing panel, stringer, etc.). An example of in-service cracking from B727 aircraft [1] (year of manufacture 1981; flight hours not available; flight cycles 39,523) is given on Fig. 2.

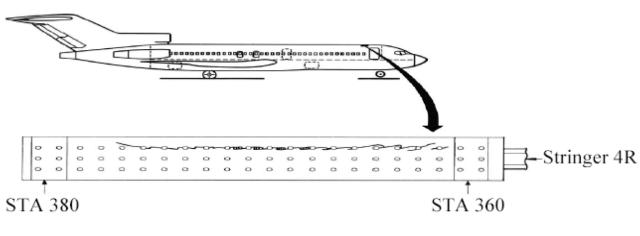


**Fig. 2.** Example of In-service Cracking from B727 Aircraft

Several probabilistic or stochastic models have been

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employed to fit the data from various fatigue crack growth experiments. Among them, the Markov chain model [2], the second-order approximation model [3], and the modified second-order polynomial model [4]. Each of the models may be the most appropriate one to depict a particular set of fatigue growth data but not necessarily the others. All models can be improved to depict very accurately the growth data but, of course, it has to be at the cost of increasing computational complexity. Yang's model [3] and the polynomial model [4] are considered more appropriate than the Markov chain model [2] by some researchers through the introduction of a differential equation which indicates that fatigue crack growth rate is a function of crack size and other parameters. The parameters, however, can only be determined through the observation and measurement of many crack growth samples. If fatigue crack growth samples are observed and measured, descriptive statistics can then be applied directly to the data to find the distributions of the desired random quantities. Thus, these models still lack prediction algorithms. Moreover, they are mathematically too complicated for fatigue researchers as well as design engineers. A large gap still needs to be bridged between the fatigue experimentalists and researchers who use probabilistic methods to study the fatigue crack growth problems.

Airworthiness regulations require proof that aircraft can be operated safely. This implies that critical components must be replaced or repaired before safety is compromised. For guaranteeing safety, the structural life ceiling limits of the fleet aircraft are defined from three distinct approaches: Safe-Life, Fail-Safe, and Damage-Tolerant approaches. In this paper, the Damage Tolerance approach is considered and the focus is on the inspection scheme with decreasing intervals between inspections.

From an engineering standpoint the fatigue life of a component or structure consists of two periods (this concept is shown schematically in Fig. 3):

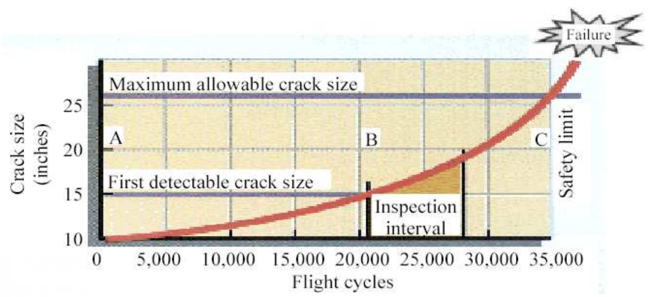


**Fig. 3.** Schematic Fatigue Crack Growth Curve
(Crack initiation period (A-B); Crack propagation period (B-C))

(i) crack initiation period, which starts with the first load cycle and ends when a technically detectable crack is present, and

(ii) crack propagation period, which starts with a technically detectable crack and ends when the remaining cross section can no longer withstand the loads applied and fails statically.

Periodic inspections of aircraft are common practice in order to maintain their reliability above a desired minimum level. The appropriate inspection intervals are determined so that the fatigue reliability of the entire aircraft structure remains above the minimum reliability level throughout its service life.

## II. Inspection Scheme under Fatigue Crack Initiation

At first, we consider in this section the problem of estimating the minimum time to crack initiation (warranty period or time to the first inspection) for a number of aircraft structure components, before which no cracks (that may be detected) in materials occur, based on the results of previous warranty period tests on the structure components in question. If in a fleet of $k$ aircraft there are $km$ of the same individual structure components, operating independently, the length of time until the first crack initially formed in any of these components is of basic interest, and provides a measure of assurance concerning the operation of the components in question. This leads to the consideration of the following problem. Suppose we have observations $X_1$, ..., $X_{\mathrm{n}}$ as the results of tests conducted on the components; suppose also that there are $km$ components of the same kind to be put into future use, with times to crack initiation $Y_1$, ..., $Y_{km}$. Then we want to be able to estimate, on the basis of $X_1$, ..., $X_n$, the shortest time to crack initiation $Y_{(1,km)}$ among the times to crack initiation $Y_1$, ..., $Y_{km}$. In other words, it is desirable to construct lower simultaneous prediction limit, $L_\gamma$, that is exceeded with probability $\gamma$ by observations or functions of observations of all $k$ future samples, each consisting of $m$ units. In this section, the problem of estimating $Y_{(1,km)}$, the smallest of all $k$ future samples of $m$ observations from the underlying distribution, based on an observed sample of $n$ observations from the same distribution, is considered.

### A. *Assigning the Interval Time until the First Inspection*

Experiments show that the number of flight cycles (hours) at which a technically detectable crack will appear in a fatigue-sensitive component of aircraft structure follows the two-parameter Weibull distribution. The probability density function for the random variable $X$ of the two-parameter Weibull distribution is given by

$$f(x\,|\,\beta,\delta)=\frac{\delta}{\beta}\left(\frac{x}{\beta}\right)^{\delta-1}\exp\left[-\left(\frac{x}{\beta}\right)^{\delta}\right]\quad(x>0),\qquad(1)$$

where $\delta>0$ and $\beta>0$ are the shape and scale parameters, respectively. The following theorem is used to assign the interval time until the first inspection (warranty period).

*Theorem 1* (*Lower one-sided prediction limit for the lth order statistic of the Weibull distribution*). Let $X_1<\ldots<X_r$ be the first $r$ ordered past observations from a sample of size $n$ from the distribution (1). Then a lower one-sided conditional $(1-\alpha)$ prediction limit $h$ on the $l$th order statistic $Y_l$ of a set of $m$ future ordered observations $Y_1<\ldots<Y_m$ is given by

$$\Pr\{Y_l\ge h\,|\,\mathbf{z}\}=\Pr\left\{\widehat{\delta}\ln\left(\frac{Y_l}{\widehat{\beta}}\right)\ge\widehat{\delta}\ln\left(\frac{h}{\widehat{\beta}}\right)\Big|\,\mathbf{z}\right\}$$

$$=\Pr\{W_l\ge w_h\,|\,\mathbf{z}\}$$

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$$
=\frac{\sum_{j=0}^{l-1}\left[\binom{l-1}{j}\frac{(-1)^{l-1-j}}{m-j}\int_0^\infty v^{r-2}e^{v\hat{\delta}\sum_{i=1}^{r}\ln(x_i/\hat{\beta})}\times\left((m-j)e^{vw_{(1-\alpha)}}+\sum_{i=1}^{r}e^{v\hat{\delta}\ln(x_i/\hat{\beta})}+(n-r)e^{v\hat{\delta}\ln(x_r/\hat{\beta})}\right)^{-r}dv\right]}{\sum_{j=0}^{l-1}\left[\binom{l-1}{j}\frac{(-1)^{l-1-j}}{m-j}\int_0^\infty v^{r-2}e^{v\hat{\delta}\sum_{i=1}^{r}\ln(x_i/\hat{\beta})}\times\left(\sum_{i=1}^{r}e^{v\hat{\delta}\ln(x_i/\hat{\beta})}+(n-r)e^{v\hat{\delta}\ln(x_r/\hat{\beta})}\right)^{-r}dv\right]}
$$

$$
=1-\alpha, \qquad (2)
$$

where $\hat{\beta}$ and $\hat{\delta}$ are the maximum likelihood estimators of $\beta$ and $\delta$ based on the first $r$ ordered past observations ($X_1$, …, $X_r$) from a sample of size $n$ from the Weibull distribution, which can be found from solution of

$$
\hat{\beta}=\left(r^{-1}\sum_{i=1}^{r}x_i^{\hat{\delta}}+(n-r)x_r^{\hat{\delta}}\right)^{1/\hat{\delta}}, \qquad (3)
$$

and

$$
\hat{\delta}=\left[\left(\sum_{i=1}^{r}x_i^{\hat{\delta}}\ln x_i+(n-r)x_r^{\hat{\delta}}\ln x_r\right)\times\left(\sum_{i=1}^{r}x_i^{\hat{\delta}}+(n-r)x_r^{\hat{\delta}}\right)^{-1}-\frac{1}{r}\sum_{i=1}^{r}\ln x_i\right]^{-1}, \qquad (4)
$$

$$
\mathbf{z}=(z_1, z_2, ..., z_{r-2}) \qquad (5)
$$

$$
Z_i=\hat{\delta}\ln\left(\frac{X_i}{\hat{\beta}}\right),\quad i=1,...,r-2, \qquad (6)
$$

$$
W_l=\hat{\delta}\ln\left(\frac{Y_l}{\hat{\beta}}\right),\quad w_h=\hat{\delta}\ln\left(\frac{h}{\hat{\beta}}\right). \qquad (7)
$$

(Observe that an upper one-sided conditional $\alpha$ prediction limit $h$ on the $l$th order statistic $Y_l$ may be obtained from a lower one-sided conditional (1-$\alpha$) prediction limit by replacing 1-$\alpha$ by $\alpha$.)

*Proof.* The proof is given by Nechval *et al.* [5] and so it is omitted here. □

*Corollary 1.1.* A lower one-sided conditional ($1-\alpha$) prediction limit $h$ on the minimum $Y_1$ of a set of $m$ future ordered observations $Y_1 \le \ldots \le Y_m$ is given by

$$
\Pr\{Y_1\ge h\,|\,\mathbf{z}\}=\Pr\left\{\hat{\delta}\ln\left(\frac{Y_1}{\hat{\beta}}\right)\ge\hat{\delta}\ln\left(\frac{h}{\hat{\beta}}\right)\Big|\,\mathbf{z}\right\}
$$

$$
=\Pr\{W_1\ge w_h\,|\,\mathbf{z}\}
$$

$$
=\frac{\int_0^\infty v^{r-2}e^{v\hat{\delta}\sum_{i=1}^{r}\ln(x_i/\hat{\beta})}\left(me^{vw_h}+\sum_{i=1}^{r}e^{v\hat{\delta}\ln(x_i/\hat{\beta})}+(n-r)e^{v\hat{\delta}\ln(x_r/\hat{\beta})}\right)^{-r}dv}{\int_0^\infty v^{r-2}e^{v\hat{\delta}\sum_{i=1}^{r}\ln(x_i/\hat{\beta})}\left(\sum_{i=1}^{r}e^{v\hat{\delta}\ln(x_i/\hat{\beta})}+(n-r)e^{v\hat{\delta}\ln(x_r/\hat{\beta})}\right)^{-r}dv}
$$

$$
=1-\alpha. \qquad (8)
$$

Thus, when $l$ = 1, (2) reduces to formula (8).

*Theorem 2* (*Lower one-sided prediction limit for the lth order statistic of the exponential distribution*). Under conditions of Theorem 1, if $\delta$=1, we deal with the exponential distribution, the probability density function of which is given by

$$
f(x\,|\,\beta)=\frac{1}{\beta}\exp\left(-\frac{x}{\beta}\right)\quad (x>0). \qquad (9)
$$

Then a lower one-sided conditional ($1-\alpha$) prediction limit $h$ on the $l$th order statistic $Y_l$ of a set of $m$ future ordered observations $Y_1 < \ldots < Y_m$ is given by

$$
\Pr\{Y_l\ge h\,|\,S_\beta=s_\beta\}=\Pr\left\{\frac{Y_l}{S_\beta}\ge\frac{h}{s_\beta}\,\Big|\,S_\beta=s_\beta\right\}
$$

$$
=\Pr\{W_l>w_h\}=\frac{1}{\mathrm{B}(l,m-l+1)}\sum_{j=0}^{l-1}\binom{l-1}{j}(-1)^j
$$

$$
\times\frac{1}{(m-l+1+j)[1+w_l(m-l+1+j)]^r}=1-\alpha. \qquad (10)
$$

where

$$
W_l=\frac{Y_s}{S_\beta}, \qquad (11)
$$

$$
S_\beta=\sum_{i=1}^{r}X_i+(m-r)X_r. \qquad (12)
$$

*Proof.* It follows readily from standard theory of order statistics that the distribution of the $l$th order statistic $Y_l$ from a set of $m$ future ordered observations $Y_1 \le \ldots \le Y_m$ is given by

$$
f(y_l\,|\,\beta)dx_l=\frac{1}{\mathrm{B}(l,m-l+1)}
$$

$$
\times[F(x_l\,|\,\beta)]^{l-1}[1-F(x_l\,|\,\beta)]^{m-l}dF(x_l\,|\,\beta), \qquad (13)
$$

where

$$
F(x\,|\,\beta)=1-\exp(-x/\beta). \qquad (14)
$$

The factorization theorem gives

$$
S_\beta=\sum_{i=1}^{r}X_i+(n-r)X_r \qquad (15)
$$

sufficient for $\beta$. The density of $S_\beta$ is given by

$$
g(s_\beta\,|\,\beta)=\frac{1}{\Gamma(r)\beta^r}s_\beta^{r-1}\exp\left(-\frac{s_\beta}{\beta}\right),\quad s_\beta\ge 0. \qquad (16)
$$

Since $Y_l$, $S_\beta$ are independent, we have the joint density of $Y_l$ and $S_\beta$ as

$$
f(y_l,s_\beta\,|\,\beta)=\frac{1}{\mathrm{B}(l,m-l+1)}\frac{1}{\Gamma(r)}
$$

$$
\times[1-e^{-x_l/\beta}]^{l-1}[e^{-x_l/\beta}]^{m-l+1}\frac{1}{\beta^{r+1}}s_\beta^{r-1}e^{-s_\beta/\beta}. \qquad (17)
$$

Making the transformation $w_l = y_l/s_\beta$, $s_\beta$= $s_\beta$, and integrating out $s_\beta$, we find the density of $W_l$ as the beta density

$$
f(w_l)=\frac{r}{\mathrm{B}(l,m-l+1)}\sum_{j=0}^{l-1}\binom{l-1}{j}(-1)^j
$$

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$$\frac{1}{[(m-l+1+j)w_l+1]^{r+1}}, \quad 0<w_l<\infty. \qquad (18)$$

This ends the proof. □

*Corollary 2.1.* A lower one-sided conditional $(1-\alpha)$ prediction limit $h$ on the minimum $Y_1$ of a set of $m$ future ordered observations $Y_1 \le \ldots \le Y_m$ is given by

$$\Pr\{Y_1 \ge h \mid S_\beta = s_\beta\} = \Pr\left\{\frac{Y_1}{S_\beta} \ge \frac{h}{s_\beta} \,\middle|\, S_\beta = s_\beta\right\}$$

$$= \Pr\{W_1 > w_h\} = \frac{1}{(1+mw_h)^r} = 1-\alpha. \qquad (19)$$

## B. Example

Consider the data of fatigue tests on a particular type of structural components (stringer) of aircraft IL-86. The data are for a complete sample of size $r = n = 5$, with observations of time to crack initiation (in number of $10^4$ flight hours): $X_1$=5, $X_2$=6.25, $X_3$=7.5, $X_4$=7.9, $X_5$=8.1.

*Goodness-of-fit testing.* It is assumed that $X_i$, $i$=1(1)5, follow the two-parameter Weibull distribution (1), where the parameters $\beta$ and $\delta$ are unknown. We assess the statistical significance of departures from the Weibull model by performing empirical distribution function goodness-of-fit test. We use the $S$ statistic (Kapur and Lamberson [6]). For censoring (or complete) datasets, the $S$ statistic is given by

$$S = \frac{\sum_{i=[r/2]+1}^{r-1}\left(\frac{\ln(x_{i+1}/x_i)}{M_i}\right)}{\sum_{i=1}^{r-1}\left(\frac{\ln(x_{i+1}/x_i)}{M_i}\right)} = \frac{\sum_{i=3}^{4}\left(\frac{\ln(x_{i+1}/x_i)}{M_i}\right)}{\sum_{i=1}^{4}\left(\frac{\ln(x_{i+1}/x_i)}{M_i}\right)} = 0.184, \quad (20)$$

where $[r/2]$ is a largest integer $\le r/2$, the values of $M_i$ are given in Table 13 (Kapur and Lamberson [6]). The rejection region for the $\alpha$ level of significance is $\{S > S_{n;\alpha}\}$. The percentage points for $S_{n;\alpha}$ were given by Kapur and Lamberson [6]. For this example,

$$S = 0.184 < S_{n=5;\alpha=0.05} = 0.86. \qquad (21)$$

Thus, there is not evidence to rule out the Weibull model. The maximum likelihood estimates of the unknown parameters $\beta$ and $\delta$ are $\hat{\beta} = 7.42603$ and $\hat{\delta} = 7.9081$, respectively. It follows from (8) that

$$\Pr\{Y_1 \ge h \mid \mathbf{z}\} = \Pr\left\{\hat{\delta}\ln\left(\frac{Y_1}{\hat{\beta}}\right) \ge \hat{\delta}\ln\left(\frac{h}{\hat{\beta}}\right) \,\middle|\, \mathbf{z}\right\}$$

$$= \Pr\{W_1 \ge w_h \mid \mathbf{z}\}$$

$$= \Pr\{W_1 > -8.4378; \mathbf{z}\} = \frac{0.0000141389}{0.0000148830} = 0.95 \qquad (22)$$

and a lower 0.95 prediction limit for $Y_1$ is $h$=2.5549 ($\times 10^4$) flight hours, i.e., we have obtained the interval time until the first inspection (or warranty period) equal to 25549 flight hours with confidence level $\gamma = 1-\alpha = 0.95$.

## C. Inspection Policy after Warranty Period

Let us assume that in a fleet of $m$ aircraft there are $m$ of the same individual structure components, operating independently. Suppose an inspection is carried out at time $\tau_j$, and this shows that initial crack (which may be detected) has not yet occurred. We now have to schedule the next inspection. Let $Y_1$ be the minimum time to crack initiation in the above components. In other words, let $Y_1$ be the smallest observation from an independent second sample of $m$ observations from the distribution (1). Then the inspection times can be calculated (from (25) using (24)) as

$$\tau_j = \hat{\beta}\exp(w_{\tau_j}/\hat{\delta}), \quad j \ge 2, \qquad (23)$$

where it is assumed that $\tau_0$=0, $\tau_1$ is the time until the first inspection (warranty period), $w_{\tau_j}$ is determined from

$$\Pr\{Y_1 > \tau_j \mid Y_1 > \tau_{j-1}, \mathbf{z}\}$$

$$= \Pr\left\{\hat{\delta}\ln\left(\frac{Y_1}{\hat{\beta}}\right) > \hat{\delta}\ln\left(\frac{\tau_j}{\hat{\beta}}\right) \,\middle|\, \hat{\delta}\ln\left(\frac{Y_1}{\hat{\beta}}\right) > \hat{\delta}\ln\left(\frac{\tau_{j-1}}{\hat{\beta}}\right), \mathbf{z}\right\}$$

$$= \Pr\{W_1 > w_{\tau_j} \mid W_1 > w_{\tau_{j-1}}, \mathbf{z}\} = \frac{\Pr\{W_1 > w_{\tau_j} \mid \mathbf{z}\}}{\Pr\{W_1 > w_{\tau_{j-1}} \mid \mathbf{z}\}} = 1-\alpha, \qquad (24)$$

where

$$W_1 = \hat{\delta}\ln\left(\frac{Y_1}{\hat{\beta}}\right), \quad w_{\tau_j} = \hat{\delta}\ln\left(\frac{\tau_j}{\hat{\beta}}\right), \qquad (25)$$

$\hat{\beta}$ and $\hat{\delta}$ are the MLE's of $\beta$ and $\delta$, respectively, and can be found from solution of (3) and (4), respectively.

It will be noted that if $\delta$=1, then it follows from Corollary 2.1 that (23) reduces to

$$\tau_j = w_{\tau_j} s_\beta, \quad j \ge 2, \qquad (26)$$

where $w_{\tau_j}$ is determined from

$$\Pr\{Y_1 > \tau_j \mid Y_1 > \tau_{j-1}, S_\beta = s_\beta\}$$

$$= \Pr\left\{\frac{Y_1}{S_\beta} > \frac{\tau_j}{s_\beta} \,\middle|\, \frac{Y_1}{S_\beta} > \frac{\tau_{j-1}}{s_\beta}, S_\beta = s_\beta\right\}$$

$$= \Pr\{W_1 > w_{\tau_j} \mid W_1 > w_{\tau_{j-1}}\} = \frac{\Pr\{W_1 > w_{\tau_j}\}}{\Pr\{W_1 > w_{\tau_{j-1}}\}} = 1-\alpha, \qquad (27)$$

where

$$W_1 = \frac{Y_1}{S_\beta}, \quad w_{\tau_j} = \frac{\tau_j}{s_\beta}, \qquad (28)$$

$S_\beta$ is given by (12). Since it is assumed that $\tau_0$=0, $\tau_1$ is the time of the first inspection (warranty period), which is found as

$$\tau_1 = w_{\tau_1} s_\beta, \qquad (29)$$

where $w_{\tau_1}$ is determined from (19),

$$w_{\tau_1} = \arg\left(1/(1+mw_{\tau_1})^r = 1-\alpha\right), \qquad (30)$$

it follows from (27) that

$$\Pr\{W_1 > w_{\tau_j}\} = 1/(1+mw_{\tau_j})^r = (1-\alpha)^j. \qquad (31)$$

Thus, we have from (26) and (31) that

$$\tau_j = \frac{s_\beta}{m}\left[(1-\alpha)^{-j/r} - 1\right], \quad j \ge 1. \qquad (32)$$

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But again, for instance, consider the data of fatigue tests on a particular type of structural components of aircraft IL-86: $X_1$=5, $X_2$=6.25, $X_3$=7.5, $X_4$=7.9, $X_5$=8.1 (in number of $10^4$ flight hours) given above, where $r$=$n$=5 and the maximum likelihood estimates of unknown parameters $\beta$ and $\delta$ are $\hat{\beta} = 7.42603$ and $\hat{\delta} = 7.9081$, respectively. Thus, using (23) and (24) with $\tau_1$=2.5549 (×$10^4$ flight hours) (the time of the first inspection), we have obtained the following inspection time sequence (see Table 1).

**Table 1.** Inspection Time Sequence

| $w_{\tau_j} \equiv w_j$ | Inspection time $\tau_j$ (×$10^4$ flight hours) | Interval $\tau_{j+1}-\tau_j$ (flight hours) |
|---|---|---|
| – | $\tau_0$= 0 | – |
| $w_1$ = −8.4378 | $\tau_1$= 2.5549 | 25549 |
| $w_2$ = −6.5181 | $\tau_2$= 3.2569 | 7020 |
| $w_3$ = −5.5145 | $\tau_3$= 3.6975 | 4406 |
| $w_4$ = −4.8509 | $\tau_4$= 4.0212 | 3237 |
| $w_5$ = −4.3623 | $\tau_5$= 4.2775 | 2563 |
| $w_6$ = −3.9793 | $\tau_6$= 4.4898 | 2123 |
| $w_7$ = −3.6666 | $\tau_7$= 4.6708 | 1810 |
| $w_8$ = −3.4038 | $\tau_8$= 4.8287 | 1579 |
| ⋮ | ⋮ | ⋮ |

## III. Optimization of Inspection Policy

Consider the case where an optimal inspection policy has to be computed with linear costs. Let $c_1$ be the cost of each of the inspections. If crack occurs at time $t$ and is detected at the $j$th inspection time $\tau_j$, so that $\tau_j \geq t$, let the cost due to undetected crack be $c_2(\tau_j - t)$. It will be noted that under small failure probability one can be restricted by the first term of the Taylor series as a presentation of a proper loss function $c_2(.)$. Then an optimal inspection policy is one, which minimizes the expected value of the total cost

$$C = jc_1 + c_2(\tau_j - t). \tag{33}$$

Taking into account (25), we obtain for the Weibull case:

$$E\{C \mid \mathbf{z}\} = \frac{c_1}{\alpha} + c_2\left(\sum_{j=1}^{\infty} \tau_j [1-\alpha]^{j-1}\alpha - E\{T \mid \mathbf{z}\}\right)$$

$$= \frac{c_1}{\alpha} + c_2\left(\sum_{j=1}^{\infty} \hat{\sigma} \exp\left(\frac{w_{\tau_j}}{\hat{\delta}}\right)[1-\alpha]^{j-1}\alpha - E\{T \mid \mathbf{z}\}\right), \tag{34}$$

Now an optimal $\alpha$ has to be found such that minimizes (34). The optimal value of $\alpha$ has to be determined numerically. The optimal $\tau_j$, $j \geq 1$, are found from (25) using (24).

## IV. Inspections Scheme Under Fatigue Crack Propagation

### A. Probabilistic Model of Fatigue Crack Growth

Many probabilistic models of fatigue crack growth are based on the deterministic crack growth equations. The most well known equation is

$$da(t)/dt = q(a(t))^b \tag{35}$$

in which $q$ and $b$ are constants to be evaluated from the crack growth observations. The independent variable $t$ can be interpreted as either stress cycles, flight hours, or flights depending on the applications [7]. It is noted that the power-law form of $q(a(t))^b$ at the right hand side of (35) can be used to fit some fatigue crack growth data appropriately and is also compatible with the concept of Paris–Erdogan law. The service time for a crack to grow from size $a(t_0)$ to $a(t)$ (where $t > t_0$) can be found by performing the necessary integration

$$\int_{t_0}^{t} dt = \int_{a(t_0)}^{a(t)} \frac{dv}{qv^b} \tag{36}$$

to obtain

$$t - t_0 = \frac{[a(t_0)]^{-(b-1)} - [a(t)]^{-(b-1)}}{q(b-1)}. \tag{37}$$

In this paper, we consider a stochastic version of (37),

$$\frac{1}{a_0^{b-1}} - \frac{1}{a^{b-1}} = (b-1)q(t-t_0) + V, \tag{38}$$

where $a_0 \equiv a(t_0)$, $a \equiv a(t)$. If $V \sim N(0,[(b-1)\sigma(t-t_0)^{1/2}]^2)$, then the probability that crack size $a(t)$ will exceed any given (say, maximum allowable) crack size $a^\bullet$ can be derived and expressed as

$$\Pr\{a(t) \geq a^\bullet\}$$

$$= 1 - \Phi\left(\left[\frac{(a_0^{-(b-1)} - (a^\bullet)^{-(b-1)}) - (b-1)q(t-t_0)}{(b-1)\sigma(t-t_0)^{1/2}}\right]\right), \tag{39}$$

where $\Phi(.)$ is the standard normal distribution function. In this case, the conditional probability density function of $a$ is given by

$$f(a,t \mid b,q,\sigma) = \frac{a^{-b}}{\sigma[2\pi(t-t_0)]^{1/2}}$$

$$\exp\left(-\frac{1}{2}\left[\frac{(a_0^{-(b-1)} - a^{-(b-1)}) - (b-1)q(t-t_0)}{(b-1)\sigma(t-t_0)^{1/2}}\right]^2\right). \tag{40}$$

This model allows one to characterize the random properties that vary during crack growth [8-9].

### B. Inspection Policy under Parametric Certainty

Let us assume that all the parameters of the crack exceedance probability (39) are known. Then the inspection times can be calculated recursively from

$$\Pr\{a(\tau_j) < a^\bullet \mid a(\tau_{j-1}) < a^\bullet\}$$

$$= \frac{\Pr\{a(\tau_j) < a^\bullet\}}{\Pr\{a(\tau_{j-1}) < a^\bullet\}} = 1 - \alpha, \quad j \geq 2, \tag{41}$$

where

$$\Pr\{a(\tau_j) < a^\bullet\}$$

$$= \Phi\left(\left[\frac{(a_0^{-(b-1)} - (a^\bullet)^{-(b-1)}) - (b-1)q(\tau_j - \tau_0)}{(b-1)\sigma(\tau_j - \tau_0)^{1/2}}\right]\right), \tag{42}$$

$\tau_0$=0, $\tau_1$ is the time of the inspection when the initial crack was detected. It is assumed that cracks start growing from the time

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the aircraft entered service. For typical aircraft metallic materials, an initial discontinuity size ($a_0$) found through quantitative fractography is approximately between 0.02 and 0.05 mm. Choosing a typical value for initial discontinuity state (e.g., 0.02 mm) is more conservative than choosing an extreme value (e.g., 0.05 mm). This implies that if the lead cracks can be attributed to unusually large initiating discontinuities then the available life increases.

### C. *Inspection Policy under Parametric Uncertainty*

Let us assume that the parameters $b$, $q$ and $\sigma$ of the crack exceedance probability (39) are unknown. Given the data describing a single crack, say a sequence $\{(a_j, \tau_j)\}_{j=1}^{n}$, it is easy to construct a log-likelihood using the density given by (40) and estimate the parameters $b$, $q$ and $\sigma$ by maximum likelihood. The log-likelihood is

$$L(b,q,\sigma \mid \{(a_j,\tau_j)\} = -b\sum_{j=1}^{n} \ln a_j - n\ln\sigma$$

$$-\frac{1}{2}\sum_{j=1}^{n}\left(\frac{a_0^{1-b} - a_j^{1-b} - (b-1)q(\tau_j - \tau_0)}{(b-1)\sigma(\tau_j - \tau_0)^{1/2}}\right)^2. \qquad (43)$$

Inspection shows that this differs from the standard least-squares equation only in the term $-b\sum\ln a$, where the subscript $i$ has been dropped. The likelihood estimators are obtained by solving the equations

$$dL/db = 0; \quad dL/dq = 0; \quad dL/d\sigma = 0. \qquad (44)$$

In this case the equations have no closed solution. However, it is easy to see that the estimators for $q$ and $\sigma$ given $b$ are the usual least-squares estimators for the coefficients in (37) conditioned on $b$,

$$\hat{q}(b) = \frac{1}{b-1}\left(na_0^{1-b} - \sum_{j=1}^{n} a_j^{1-b}\right)\left(\sum_{j=1}^{n}(\tau_j - \tau_0)\right)^{-1}, \qquad (45)$$

$$\hat{\sigma}^2(b) = \frac{1}{n(b-1)^2}\sum_{j=1}^{n}\frac{[a_0^{1-b} - a_j^{1-b} - \hat{q}(b)(b-1)(\tau_j - \tau_0)]^2}{\tau_j - \tau_0}, \qquad (46)$$

and on substituting these back in the log-likelihood gives a function of $b$ alone,

$$L(b) = -b\sum_{j=1}^{n}\ln a_j - n\ln[\hat{\sigma}(b)] - n/2. \qquad (47)$$

Thus the technique is to search for the value of $b$ that maximizes $L(b)$ by estimating $q$ and $\sigma$ as functions of $b$ and substituting in $L(b)$. In this study a simple golden-section search worked very effectively. It will be noted that if we deal with small sample of the data describing a single crack, say a sequence $\{(a_j, \tau_j)\}_{j=1}^{n}$, then the estimates of the unknown parameters $b$, $q$ and $\sigma$ can be obtained via the Generalized Likelihood Ratio Test as follows. Let us assume (without loss of generality) that there are available only two past samples of the data describing a single similar crack, say sequences $\{(a_j^{(1)}, \tau_j^{(1)})\}_{j=1}^{n_1}$ and $\{(a_j^{(2)}, \tau_j^{(2)})\}_{j=1}^{n_2}$ with the unknown parameters $(b_1,q_1,\sigma_1)$ and $(b_2,q_2,\sigma_2)$, respectively, where $n_1$, $n_2 > n$. Then the likelihood ratio statistic for testing the null hypothesis $H_1$: ($b$=$b_1$, $q$=$q_1$, $\sigma$=$\sigma_1$) versus the alternative hypothesis $H_2$: ($b$=$b_2$, $q$=$q_2$, $\sigma$=$\sigma_2$) is given by

$$\mathrm{LR} = \frac{\max\limits_{H_1}\prod_{j=1}^{n} f(a_j,\tau_j \mid b_1,q_1,\sigma_1)\prod_{i=1}^{2}\prod_{j=1}^{n_i} f(a_j^{(i)},\tau_j^{(i)} \mid b_i,q_i,\sigma_i)}{\max\limits_{H_2}\prod_{j=1}^{n} f(a_j,\tau_j \mid b_2,q_2,\sigma_2)\prod_{i=1}^{2}\prod_{j=1}^{n_i} f(a_j^{(i)},\tau_j^{(i)} \mid b_i,q_i,\sigma_i)}, \qquad (48)$$

and hypothesis $H_1$ or $H_2$ is favoured according to whether LR is greater or less than 1, i.e.

$$\mathrm{LR}\begin{cases} >1, \text{ then } H_1 \ \ (\hat{b} = \hat{b}_1, \hat{q} = \hat{q}_1, \ \hat{\sigma} = \hat{\sigma}_1) \\ \leq 1, \text{ then } H_2 \ \ (\hat{b} = \hat{b}_2, \hat{q} = \hat{q}_2, \hat{\sigma} = \hat{\sigma}_2). \end{cases} \qquad (49)$$

The parametric estimates obtained after each inspection are treated as if they were the true values in order to obtain from (41) an adaptive inspection time sequence.

## V. Conclusion

In this paper, to capture the scatter of the fatigue crack growth data, the stochastic model that adopted the solution of the crack growth equation, proposed by Paris and Erdogan, and randomized one by including random factors into it is suggested. This stochastic model allows us to obtain the crack exceedance probability as well as the probability of random time to reach a specified crack size. Once the appropriate stochastic model is established, it can be used for the fatigue reliability prediction of structures made of the tested material. As such the model presented here provides a fast and computationally efficient way to predict the fatigue lives of realistic structures.

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