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Lower Asymptotics for the Finite Time Ruin Probability in the Renewal Model with Subexponential Claims

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Abstract—**For the renewal risk model with subexponential claim sizes, we establish for the finite time ruin probability a lower asymptotic estimate as initial surplus increases, subject to the demand that it should hold uniformly over all time horizons in an infinite interval. This extends a recent work partly on the topic from the case of Pareto-type claim sizes to the case of subexponential claim sizes and, simplifies the proof of lower bound in Leipus and Siaulys (2006).**

Keywords: Asymptotics; finite time ruin probability; strongly subexponential distributions; the compound Poisson/renewal model; uniform convergence.

1 Introduction

Consider the renewal risk model, in which the claim sizes Z_i, $i = 1, 2, \ldots$, form a sequence of independent, identically distributed (i.i.d.), nonnegative random variables with common distribution B, while the inter-occurrence times θ_i, $i = 1, 2, \ldots$, form another sequence of i.i.d. positive random variables with common finite mean $1/\lambda$. We assume that the two sequences $\{Z_i, i = 1, 2, \ldots\}$ and $\{\theta_i, i = 1, 2, \ldots\}$ are mutually independent. The locations of claims $\tau_k = \sum_{i=1}^{k} \theta_i$, $k = 1, 2, \ldots$, constitute a renewal counting process

$$N(t) = \#\{k = 1, 2, \ldots : \tau_k \in (0, t]\}, \qquad t \geq 0, \qquad (1.1)$$

with a mean function $\lambda(t) = \mathrm{E}N(t) \sim \lambda t$ as $t \to \infty$. The surplus process is then defined as

$$R(t) = x + ct - \sum_{i=1}^{N(t)} Z_i, \qquad t \geq 0, \qquad (1.2)$$

where $R(0) = x \geq 0$ denotes the initial surplus, $c > 0$ denotes the constant premium rate, and a summation over an empty set of index is 0 by convention.

We write

$$\psi(x; t) = \Pr\left(\inf_{0 \leq s \leq t} R(s) < 0 \;\middle|\; R(0) = x \right), \qquad t \geq 0,$$

and

$$\begin{aligned} \psi(x; \infty) &= \lim_{t \to \infty} \psi(x; t) \\ &= \Pr\left(\inf_{0 \leq s < \infty} R(s) < 0 \;\middle|\; R(0) = x \right), \end{aligned}$$

which are, respectively, the probabilities of ruin by time t and of ultimate ruin. In order for the ultimate ruin not to be certain, it is natural to assume the safety loading condition

$$\mu = \frac{c}{\lambda} - \mathrm{E}Z_1 > 0. \qquad (1.3)$$

We refer readers to Asmussen (1984, 2000) for a nice reviews on the study of the finite time ruin probability and to Tang (2004b) for a list of references devoted to this study. Our goal in the current paper is to derive an asymptotic lower estimate as the initial surplus x increases for the finite time ruin probability $\psi(x; t)$, subject to the requirement that the asymptotic result should hold uniformly over all time horizons t in an infinite interval.

Hereafter, all limit relationships are for $x \to \infty$ unless stated otherwise. For two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(x) \lesssim b(x)$ if $\limsup a(x)/b(x) \leq 1$, write $a(x) \gtrsim b(x)$ if $\liminf a(x)/b(x) \geq 1$, and write $a(x) \sim b(x)$ if both. As done in the main result of this paper, we shall assign a certain uniformity property to some asymptotic relations under discussion. Let us take an example to clarify the meaning of uniformity. For two positive bivariate functions $a(\cdot;\cdot)$ and $b(\cdot;\cdot)$, we say that the asymptotic relation $a(x; t) \sim b(x; t)$ holds uniformly over all t in a nonempty set Δ if

$$\lim_{x \to \infty} \sup_{t \in \Delta} \left| \frac{a(x; t)}{b(x; t)} - 1 \right| = 0.$$

That is, for each fixed $\varepsilon > 0$, there exists some $x_0 > 0$ irrespective to t such that the two-sided inequality

$$(1 - \varepsilon) b(x; t) \leq a(x; t) \leq (1 + \varepsilon) b(x; t)$$

holds for all $x \geq x_0$ and $t \in \Delta$. This is further equivalent to that both $a(x) \lesssim b(x)$ and $a(x) \gtrsim b(x)$ hold uniformly over all $t \in \Delta$. Admittedly, results that hold with such a uniformity property are of higher theoretical and practical interest.

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We shall only consider the case of heavy-tailed claim sizes. The most important class of heavy-tailed distributions is the subexponential class. By definition, a distribution F on $[0,\infty)$ is said to be subexponential, written as $F \in \mathcal{S}$, if its right tail $\overline{F} = 1 - F$ satisfies $\overline{F}(x) > 0$ for all x and the relation

$$\overline{F^{*2}}(x) \sim 2\overline{F}(x) \tag{1.4}$$

holds, where F^{*2} denotes the convolution of F with itself. More generally, a distribution F on $(-\infty,\infty)$ is still said to be subexponential if the distribution $F^{+}(x) = F(x)1_{(0\le x<\infty)}$ is subexponential, where 1_A denotes the indicator function of A. It is well known that every subexponential distribution F is long tailed, written as $F \in \mathcal{L}$, in the sense that the relation

$$\overline{F}(x+y) \sim \overline{F}(x) \tag{1.5}$$

holds for each fixed real number y; see, for example, Embrechts et al. (1997, Lemma 1.3.5).

Very often the class $\mathcal{S}$ appears to be too wide to possess desirable probabilistic properties. For this reason, researchers in applied probability have introduced many subclasses of $\mathcal{S}$ to meet certain special requirements. In this regard, Korshunov (2001) introduced the class of strongly subexponential distributions. For a distribution F on $(-\infty,\infty)$ with $0 < m = \int_0^\infty \overline{F}(u)\mathrm{d}u < \infty$ and for each fixed $l \in (0,\infty]$, we write

$$\overline{F_l}(x) = \begin{cases} \min\left\{1, \int_x^{x+l} \overline{F}(u)\mathrm{d}u\right\}, & x \ge 0, \\ 1, & x < 0. \end{cases}$$

Clearly, for each $l \in (0,\infty]$ the function F_l defines a standard distribution on $[0,\infty)$. In the terminology of Korshunov (2001), the distribution F is said to be strongly subexponential, denoted by $F \in \mathcal{S}_*$, if the relation

$$\lim_{x\to\infty} \frac{\overline{F_l^{*2}}(x)}{\overline{F_l}(x)} = 2 \tag{1.6}$$

holds uniformly over all $l \in [1,\infty]$. It is easy to check that relation (1.6) with an arbitrarily fixed number $l \in [1,\infty)$ implies $F \in \mathcal{S}$; see Kaas and Tang (2003). Hence, $\mathcal{S}_*$ is a subclass of $\mathcal{S}$. From the discussions of Korshunov (2001), we see that the class $\mathcal{S}_*$ covers almost all useful subexponential distributions with $m < \infty$. Specifically, the class $\mathcal{S}_*$ contains all Pareto-like distributions with $m < \infty$, all lognormal-like distributions, and all heavy-tailed Weibull-like distributions.

The main result of this paper is the following:

Theorem 1.1. *Consider the renewal model with the safety loading condition (1.3), which is introduced at the very beginning of this paper. If $B \in \mathcal{L}$, then for every positive function $f(\cdot)$ with $f(x) \to \infty$, it holds uniformly over all $t \in [f(x),\infty]$ that*

$$\psi(x;t) \gtrsim \frac{1}{\mu}\int_x^{x+\mu\lambda t} \overline{B}(u)\mathrm{d}u. \tag{1.7}$$

When $t = \infty$, formula (1.7) is reduced to

$$\psi(x;\infty) \gtrsim \frac{1}{\mu}\int_x^{\infty} \overline{B}(u)\mathrm{d}u, \tag{1.8}$$

which is well known, first established by Veraverbeke (1977) and Embrechts and Veraverbeke (1982).

Tang (2004b) established (1.7) in the form of equivalence in the renewal model under the assumption, among others, that the distribution B is consistently varying tailed in the sense that

$$\lim_{l\nearrow 1}\limsup_{x\to\infty} \frac{\overline{B}(lx)}{\overline{B}(x)} = 1.$$

Hence, his result works essentially only for the case of Pareto-like claim sizes. Recently, under the three assumptions as following:

(1) There exists a nonnegative function q: $R_+ \to R_+$ such that

$$Q(u) = \int_0^u q(v)dv, \quad u \in R_+$$
$$\text{and} \quad \limsup_{u\to\infty} \frac{uq(u)}{Q(u)} =: r \;\; is \;\; finite;$$

(2) The hazard rate $q(u)$ satisfies $\liminf_{u\to\infty} uq(u) \ge \max\left\{1, \frac{1}{1-r}\right\}$;

(3) The random variable θ is such that $P(0 \le \theta < \epsilon)$ and $P(\theta = 0) = 1$ for every positive $\epsilon > 0$,

Leipus and Siaulys (2006) obtained that, in the renewal risk model with the safety loading condition (1.3), if $B \in \mathcal{S}_*$, then for every positive function $f(\cdot)$ with $f(x) \to \infty$, it holds uniformly over all $t \in [f(x), \gamma x]$ that

$$\psi(x;t) \sim \frac{1}{\mu}\int_x^{x+\mu\lambda t} \overline{B}(u)\mathrm{d}u. \tag{1.9}$$

Analyzing this result carefully, one could see that, firstly, the assumptions it demands seem to be too strong to be suitable for more general case. Secondly, the proof of their results is too complicated to be pretty mathematically. Finally, that fact that class $\mathcal{L}L$ is much bigger than class S_* illustrates the wider range of usage of Theorem 1.1.

Below is the extension to the Theorem of Korshunov (2001); See also Tang (2004a):

Lemma 1.1. *Let $\{X_i, i = 1,2,\ldots\}$ be a sequence of i.i.d. random variables with common distribution F and finite mean $\mathrm{E}X_1 = -\mu < 0$. If $F \in \mathcal{L}$, then it holds uniformly over all $n = 1,2,\ldots$ that*

$$\Pr\left(\max_{1\le k\le n} \sum_{i=1}^{k} X_i > x\right) \gtrsim \frac{1}{\mu}\int_x^{x+\mu n} \overline{F}(u)\mathrm{d}u. \tag{1.10}$$

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2 The proof of the main result

Proof. Since, by definition, $B \in \mathcal{L}$ implies $B_I \in \mathcal{S}$, by virtue of relation (1.8), it suffices to prove the uniformity of (1.7) over all $t \in [f(x), \infty)$. For arbitrarily fixed $\delta > 0$, we write

$$M_-(\delta) = \min_{0\le k<\infty} \left(\frac{(1+\delta)k}{\lambda} - \tau_k \right),$$

which is nonpositive and finite almost surely. From the equivalent definition of finite time ruin probability

$$\psi(x;t) = \Pr\left(\max_{0\le k\le N(t)} \left(\sum_{i=1}^{k} Z_i - c\tau_k \right) > x \right), \qquad t > 0, \tag{2.1}$$

for each fixed $L > 0$, we have

$$\begin{aligned}
&\psi(x;t)\\
&= \Pr\left(\max_{0\le k\le N(t)} \left(\sum_{i=1}^{k} \left(Z_i - \frac{c(1+\delta)}{\lambda} \right) \right.\right.\\
&\qquad \left.\left. + c\left(\frac{(1+\delta)k}{\lambda} - \tau_k \right) \right) > x \right)\\
&\ge \Pr\left(\max_{0\le k\le N(t)} \sum_{i=1}^{k} \left(Z_i - \frac{c(1+\delta)}{\lambda} \right) > x + cL, \right.\\
&\qquad \left. + M_-(\delta) > -L \right)\\
&= \sum_{n=1}^{\infty} \Pr\left(\max_{0\le k\le n} \sum_{i=1}^{k} \left(Z_i - \frac{c(1+\delta)}{\lambda} \right) \right.\\
&\qquad \left. > x + cL \right) \Pr\left(N(t) = n, M_-(\delta) > -L \right).
\end{aligned} \tag{2.2}$$

We write

$$\mu_2(\delta) = \frac{c(1+\delta)}{\lambda} - \mathrm{E}Z_1 > 0.$$

Applying Lemma 1.1, it holds uniformly over all $n = 1, 2, \ldots$ that

$$\begin{aligned}
&\Pr\left(\max_{0\le k\le n} \sum_{i=1}^{k} \left(Z_i - \frac{c(1+\delta)}{\lambda} \right) > x + cL \right)\\
&\gtrsim \frac{1}{\mu_2(\delta)} \int_x^{x+\mu_2(\delta)n} \overline{B}(u + cL)\, \mathrm{d}u.
\end{aligned}$$

Substituting this into (2.2) and considering an arbitrarily fixed number $0 < l < 1$, we have that, uniformly over all $t \in [f(x), \infty)$,

$$\begin{aligned}
&\psi(x;t)\\
&\gtrsim \frac{1}{\mu_2(\delta)} \sum_{n=1}^{\infty} \int_x^{x+\mu_2(\delta)n} \overline{B}(u + cL)\, \mathrm{d}u \cdot \Pr\left(N(t) = n, \right.\\
&\qquad \left. M_-(\delta) > -L \right)\\
&\ge \frac{1}{\mu_2(\delta)} \sum_{n\ge(1-l)\lambda t} \int_x^{x+\mu n} \overline{B}(u + cL)\, \mathrm{d}u \cdot \Pr\left(N(t) = n, \right.\\
&\qquad \left. M_-(\delta) > -L \right)\\
&\ge \frac{1}{\mu_2(\delta)} \int_x^{x+(1-l)\mu\lambda t} \overline{B}(u + cL)\, \mathrm{d}u \cdot \Pr\left(\frac{N(t)}{\lambda t} \ge 1 - l, \right.\\
&\qquad \left. M_-(\delta) > -L \right).
\end{aligned} \tag{2.3}$$

We apply an elementary inequality, $\Pr(AB) \ge \Pr(A) + \Pr(B) - 1$, to obtain that

$$\begin{aligned}
&\Pr\left(\frac{N(t)}{\lambda t} \ge 1 - l,\ M_-(\delta) > -L \right)\\
\ge\ &\Pr\left(\frac{N(t)}{\lambda t} \ge 1 - l \right) + \Pr\left(M_-(\delta) > -L \right) - 1.
\end{aligned}$$

As $t \to \infty$, it is well known that $N(t)/\lambda t \to 1$ holds almost surely; see, for example, Section 2.5 of Embrechts et al. (1997). Hence for each $\varepsilon > 0$, we may find some $x_0 > 0$ and $L_0 > 0$ such that the inequality

$$\Pr\left(\frac{N(t)}{\lambda t} \ge 1 - l, M_-(\delta) > -L_0 \right) \ge 1 - \varepsilon$$

holds for all $t \in [f(x_0), \infty)$. Substitution of this into (2.3) with $L = L_0$ gives that, uniformly over all $t \in [f(x), \infty)$,

$$\begin{aligned}
&\psi(x;t)\\
&\gtrsim \frac{1-\varepsilon}{\mu_2(\delta)} \int_x^{x+(1-l)\mu\lambda t} \overline{B}(u + cL_0)\, \mathrm{d}u\\
&\sim \frac{1-\varepsilon}{\mu_2(\delta)} \left(\int_x^{x+\mu\lambda t} - \int_{x+(1-l)\mu\lambda t}^{x+\mu\lambda t} \right) \overline{B}(u)\, \mathrm{d}u\\
&\ge \frac{1-\varepsilon}{\mu_2(\delta)} \int_x^{x+\mu\lambda t} \overline{B}(u)\, \mathrm{d}u \left(1 - \frac{\int_{x+(1-l)\mu\lambda t}^{x+\mu\lambda t} \overline{B}(u)\, \mathrm{d}u}{\int_x^{x+(1-l)\mu\lambda t} \overline{B}(u)\, \mathrm{d}u} \right)\\
&\ge \frac{1-\varepsilon}{\mu_2(\delta)} \int_x^{x+\mu\lambda t} \overline{B}(u)\, \mathrm{d}u \Big(1\\
&\qquad - \frac{l\mu\lambda t \overline{B}(x + (1-l)\mu\lambda t)}{(1-l)\mu\lambda t \overline{B}(x + (1-l)\mu\lambda t)} \Big)\\
&= \frac{1-\varepsilon}{\mu_2(\delta)} \frac{1-2l}{1-l} \int_x^{x+\mu\lambda t} \overline{B}(u)\, \mathrm{d}u.
\end{aligned}$$

Since the constants $\delta > 0$, $0 < l < 1$, and $\varepsilon > 0$ can be arbitrarily small, we finally obtain the desired relation (1.7) with the indicated uniformity property. □

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References

[1] Asmussen, S. Approximations for the probability of ruin within finite time. Scand. Actuar. J. (1984), no. 1, 31–57.

[2] Asmussen, S. Ruin probabilities. World Scientific Publishing Co., Inc., River Edge, NJ, 2000.

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[3] Embrechts, P.; Klüppelberg, C.; Mikosch, T. Modelling extremal events for insurance and finance. Springer-Verlag, Berlin, 1997.

[4] Embrechts, P.; Veraverbeke, N. Estimates for the probability of ruin with special emphasis on the possibility of large claims. Insurance Math. Econom. 1 (1982), no. 1, 55–72.

[5] Kaas, R.; Tang, Q. Note on the tail behavior of random walk maxima with heavy tails and negative drift. N. Am. Actuar. J. 7 (2003), no. 3, 57–61.

[6] Klüppelberg, C.; Mikosch, T. Large deviations of heavy-tailed random sums with applications in insurance and finance. J. Appl. Probab. 34 (1997), no. 2, 293–308.

[7] Korshunov, D. A. Large deviation probabilities for the maxima of sums of independent summands with a negative mean and a subexponential distribution. (Russian) Teor. Veroyatnost. i Primenen. 46 (2001), no. 2, 387–397; translation in Theory Probab. Appl. 46 (2003), no. 2, 355–365.

[8] Leipus R.; Siaulys J. Asymptotic behaviour of the finite-time ruin probability under subexponential claim sizes. Insurance Math. Econom. 7 (2006), no. 7, 1016-1027.

[9] Ross, S. M. Stochastic processes. John Wiley & Sons, Inc., New York, 1983.

[10] Tang, Q. Uniform estimates for the tail probability of maxima over finite horizons with subexponential tails. Probab. Engrg. Inform. Sci. 18 (2004a), no. 1, 71–86.

[11] Tang, Q. Asymptotics for the finite time ruin probability in the renewal model with consistent variation. Stoch. Models 20 (2004b), no. 3, 281–297.

[12] Veraverbeke, N. Asymptotic behaviour of Wiener-Hopf factors of a random walk. Stochastic Processes Appl. 5 (1977), no. 1, 27–37.