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New Solutions for the Three-Dimensional Electrical Impedance Equation and its Application to the Electrical Impedance Tomography Theory

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***Abstract*— Using a quaternionic reformulation of the three-dimensional Electrical Impedance Equation, and a generalization of the Bers generating sets, as well as the Beltrami equation, we introduce a new class of exact solutions for the case when the conductivity is represented by a separable-variables function depending upon three spacial variables. Finally, we discuss the possible contribution of these results into the Electrical Impedance Tomography Theory.**

***Index Terms*— Electrical Impedance Tomography, Pseudoanalytic Functions.**

I. Introduction

The Electrical Impedance Tomography problem was posed by A. Calderon in 1980 [5]. It is basically a boundary value problem for the equation

$$\operatorname{grad}(\sigma \operatorname{div} u) = 0, \tag{1}$$

where σ represents the conductivity function depending upon three spacial variables, and u is the electric potential. The goal is to reconstruct σ inside a domain surrounded by a boundary Γ, when the values of u in the boundary are given. This problem is specially important for medical image reconstruction, because it constitutes an auxiliary noninvasive technique for the diagnosis of several diseases.

Yet, by more than twenty years, the mathematical complexity of (1) restricted the use of the Electrical Impedance Tomography because the achieved images were deficient compared to such provided by (for instance) the Positron Emission Tomography, or the Magnetic Resonance Imaging.

Then, in 2006, through the path of relating (1) with a Vekua equation [22], K. Astala and L. Päivärinta [2] gave a positive answer for the two-dimensional Electrical Impedance Tomography problem. And in 2007, V. Kravchenko and H. Oviedo [16] obtained what it could be considered the first general solution of (1) in analytic form, employing elements of the Pseudoanalytic Function Theory [3] for a certain class of σ. Two years latter, adopting new results in Complex Analysis [13], it was possible to pose the general solution for the two-dimensional case of (1) in terms of Taylor series in formal powers, when σ is a separable-variables function [20].

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Using a quaternionic generalization of the Bers generating sets [17], we pose the structure of the general solution for the three-dimensional Electrical Impedance Equation, and bias a quaternionic Beltrami equation, we introduce a new class of exact solutions for (1), when the conductivity σ is a separable-variables function depending upon three spacial variables.

Finally, we discuss the possible contribution of these results into the field of the Electrical Impedance Tomography Theory.

II. Elements of Quaternionic Analysis and Pseudoanalytic Function Theory

The algebra of real quaternions (see e.g. [9],[15]) will be represented by $\mathbb{H}(\mathbb{R})$. The elements belonging to $\mathbb{H}(\mathbb{R})$ are written as $q = q_0 + \overrightarrow{q}$, where q_0 is known as the *scalar part* of the quaternion q, and $\overrightarrow{q} = \sum_{k=1}^{3} e_k q_k$ is the *vectorial part* of q. The elements q_k; $k = \overline{0,3}$ are all real-valued functions, whereas e_1, e_2 and e_3 are the standard quaternionic units possessing the properties

$$e_1 e_2 e_3 = -1,$$
$$e_1^2 = e_2^2 = e_3^2 = -1.$$

The subset of purely vectorial quaternions $q = \overrightarrow{q}$ can be identified with the set of three-dimensional vectors. In other words, every vector $\overrightarrow{E} = (E_1, E_2, E_3)$ is associated with a vectorial quaternion $\overrightarrow{E} = E_1 e_1 + E_2 e_2 + E_3 e_3$. The correspondence is one-to-one.

By virtue of this isomorphism, the multiplication between two vectorial quaternions $\overrightarrow{q}$ and $\overrightarrow{p}$ can be represented as

$$\overrightarrow{q}\ \overrightarrow{p} = -\langle \overrightarrow{q}, \overrightarrow{p} \rangle + [\overrightarrow{q} \times \overrightarrow{p}], \tag{2}$$

where $\langle \overrightarrow{q}, \overrightarrow{p} \rangle$ denotes the classical Cartesian scalar product, and $[\overrightarrow{q} \times \overrightarrow{p}]$ is the vectorial product, written in the quaternionic sense (see e.g. [15],[20]). The reader can notice that in general $\overrightarrow{q}\ \overrightarrow{p} \neq \overrightarrow{p}\ \overrightarrow{q}$. Because of this, we will use the notation

$$M^{\overrightarrow{p}} \overrightarrow{q} = \overrightarrow{q}\ \overrightarrow{p} \tag{3}$$

to indicate the *multiplication by the right-hand side* of the quaternion $\overrightarrow{q}$ by the quaternion $\overrightarrow{p}$.

In the set of at least once differentiable quaternionic-valued functions, we can define the Moisil-Theodoresco differential operator $D = e_1\partial_1 + e_2\partial_2 + e_3\partial_3$, where $\partial_k = \frac{\partial}{\partial x_k}$. According to the classical notations, when applying D to a vectorial quaternion $\overrightarrow{q}$, we have

$$D\overrightarrow{q} = -\operatorname{div}\overrightarrow{q} + \operatorname{rot}\overrightarrow{q}. \tag{4}$$

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Notice also that any scalar function (let us say σ) can be considered a purely scalar quaternion, and for this case

$$D\sigma = \operatorname{grad} \sigma. \tag{5}$$

We shall now introduce some concepts of the Pseudoanalytic Function Theory. Following [3], let F and G be two complex-valued functions that satisfy

$$\operatorname{Im}\left(\overline{F}G\right) > 0, \tag{6}$$

where $\overline{F}$ denotes the complex conjugation of $F : \overline{F} = \operatorname{Re} F - i \operatorname{Im} F$, and i denotes the standard complex unit $i^2 = -1$. Indeed, (6) is an alternative way for indicating that the functions F and G are linearly independent, hence they can be used as a base for representing any complex-valued function. In other words, every complex function W can be expressed as

$$W = \phi F + \psi G,$$

where ϕ and ψ are real-valued functions. When a pair of functions (F, G) satisfies the condition (6), it is called a *Bers generating pair.*

The idea of representing the set of complex functions by means of a generating pair (F, G) instead of the traditional base $(1, i)$ could seem a simple change at the first look, but when we analyze the implications into the Complex Function Theory, the results are amazing.

This idea was used by Lipman Bers to develop the Pseudoanalytic Function Theory [3], which in recent years has become of great importance for the Electrical Impedance Tomography Theory.

Let us start by introducing the notion of the *derivative in the sense of Bers* (or the (F, G)-*derivative*) of a complex-valued function W:

$$\partial_{(F,G)} W = (\partial_z \phi) F + (\partial_z \psi) G, \tag{7}$$

where $\partial_z = \partial_x - i\partial_y$. This derivative will exist if and only if

$$(\partial_{\overline{z}} \phi) F + (\partial_{\overline{z}} \psi) G = 0, \tag{8}$$

where $\partial_{\overline{z}} = \partial_x + i\partial_y$.

In order to express (7) and (8) in terms of W, let us introduce the notations

$$A_{(F,G)} = \frac{\overline{F}\partial_z G - \overline{G}\partial_z F}{F\overline{G} - \overline{F}G}, \quad a_{(F,G)} = \frac{\overline{G}\partial_{\overline{z}} F - \overline{F}\partial_{\overline{z}} G}{F\overline{G} - \overline{F}G}, \tag{9}$$
$$B_{(F,G)} = \frac{F\partial_z G - G\partial_z F}{F\overline{G} - \overline{F}G}, \quad b_{(F,G)} = \frac{F\partial_{\overline{z}} G - G\partial_{\overline{z}} F}{F\overline{G} - \overline{F}G}.$$

L. Bers called this expressions the *characteristic coefficients* of the generating pair (F, G). Applying these notations, the equation (7) becomes

$$\partial_{(F,G)} W = \partial_z W - A_{(F,G)} W - B_{(F,G)} \overline{W}, \tag{10}$$

and (8) turns into

$$\partial_{\overline{z}} W - a_{(F,G)} W - b_{(F,G)} \overline{W} = 0. \tag{11}$$

The equation (11) has special significance in the Electrical Impedance Tomography Theory, and it is known as the *Vekua equation,* because it was deeply studied by Ilia Vekua when he developed the Theory of Generalized Analytic Functions [22], closely related with the Pseudoanalytic Function Theory of L. Bers, but with a strong orientation to the Operational Analysis.

Every complex-valued function W that fulfills the equation (11) will be called (F, G)-*pseudoanalytic*.

In this work it will be more convenient to give the further explanations according to the definitions posed by L. Bers. The following statements were originally presented in [3].

Remark 1: The complex-valued functions that constitute the generating pair (F, G) are (F, G)-pseudoanalytic, and their (F, G)-derivatives are $\partial_{(F,G)} F = \partial_{(F,G)} G = 0$.

Definition 2: Let (F, G) and (F_1, G_1) be two generating pairs, and let their characteristic coefficients, defined in (9), satisfy the relations

$$a_{(F,G)} = a_{(F_1,G_1)} \quad \text{and} \quad B_{(F,G)} = -b_{(F_1,G_1)}. \tag{12}$$

Hence, the pair (F_1, G_1) will be called the *successor pair* of (F, G), as well (F, G) will be named the *predecessor pair* of (F_1, G_1).

Theorem 3: Suppose the complex-valued function W is (F, G)-pseudoanalytic, and let (F_1, G_1) be a successor pair of (F, G). Then the (F, G)-derivative of W

$$\partial_{(F,G)} W = \partial_z W - A_{(F,G)} W - B_{(F,G)} \overline{W},$$

will be (F_1, G_1)-pseudoanalytic. This is

$$\partial_{\overline{z}} \left(\partial_{(F,G)} W\right) - a_{(F_1,G_1)} \left(\partial_{(F,G)} W\right) - b_{(F_1,G_1)} \overline{\left(\partial_{(F,G)} W\right)} = 0$$

Definition 4: Let (F, G) be a generating pair. Its *adjoint pair* will be denoted by (F^*, G^*), and defined by the formulas

$$F^* = -\frac{2\overline{F}}{F\overline{G} - \overline{F}G}, \qquad G^* = \frac{2\overline{G}}{F\overline{G} - \overline{F}G}.$$

L. Bers also posed the (F, G)-*integral* of a complex-valued function W:

$$\int_{z_0}^{z_1} W \ d_{(F,G)} z = \tag{13}$$
$$= F(z_1) \operatorname{Re} \int_{z_0}^{z_1} G^* W \ dz + G(z_1) \operatorname{Re} \int_{z_0}^{z_1} F^* W \ dz.$$

If $W = \phi F + \psi G$ is an (F, G)-pseudoanalytic function, then

$$\int_{z_0}^{z} \partial_{(F,G)} W \ d_{(F,G)} z = \tag{14}$$
$$= W(z) - \phi(z_0) F(z) - \psi(z_0) G(z),$$

and since

$$\partial_{(F,G)} F = \partial_{(F,G)} G = 0,$$

the integral expression (14) represents the *antiderivative in the sense of Bers* of $\partial_{(F,G)} W$.

Notice that a continuous complex function w is (F, G)-*integrable* if and only if

$$\operatorname{Re} \oint G^* w dz + i \operatorname{Re} \oint F^* w dz = 0.$$

Theorem 5: The derivative in the sense of Bers $\partial_{(F,G)} W$ of a (F, G)-pseudoanalytic function W will be (F, G)-integrable.

Theorem 6: Suppose that the generating pair (F, G) is a predecessor of (F_1, G_1). A complex-valued function W

will be (F_1, G_1)-pseudoanalytic if and only if it is (F, G)-integrable.

Definition 7: Let $\{(F_m, G_m)\}$, $m = 0, \pm 1, \pm 2, \pm 3, ...$ be a sequence of generating pairs. If every (F_{m+1}, G_{m+1}) is a successor pair of (F_m, G_m), we will call $\{(F_m, G_m)\}$ a *generating sequence*. Specifically, if the generating pair $(F_0, G_0) = (F, G)$ we say that (F, G) is *embedded* in the generating sequence $\{(F_m, G_m)\}$.

Now, let W be a (F, G)-pseudoanalytic function, and suppose that $\{(F_m, G_m)\}$ is a generating sequence in which (F, G) is embedded. Hence we will be able to express the higher derivatives in the sense of Bers of W as

$$W^{[0]} = W; \quad W^{[m+1]} = \partial_{(F_m, G_m)} W^{[m]}; \; m = 0, 1, 2, 3, ...$$

Definition 8: The complex function $Z_m^{(0)}(a, z_0; z)$ will be called *formal power with center at z_0, coefficient a and exponent* 0. This function will depend upon a complex variable z, and it will be defined as the linear combination of the elements of the generating pair (F_m, G_m), with two real constant coefficients λ and μ such that

$$\lambda F_m(z_0) + \mu G_m(z_0) = a.$$

The *formal powers* with exponents $n = 1, 2, 3, ...$ will be defined according to the recursive formulas

$$Z_m^{(n)}(a, z_0; z) = n \int_{z_0}^{z} Z_{m+1}^{(n-1)}(a, z_0; \xi) \, d_{(F_m, G_m)} \xi,$$

where the integral operators are integrals in the sense of Bers, described in (13).

The proofs of the following statements can be found in [3]:

1) $Z_m^{(n)}(a, z_0; z)$ is an (F_m, G_m)-pseudoanalytic function.
2) If a_1 and a_2 are real constants, then

$$Z_m^{(n)}(a_1 + i a_2, z_0; z) = \\ = a_1 Z_m^{(n)}(1, z_0; z) + a_2 Z_m^{(n)}(i, z_0; z).$$

3) The following relations hold

$$\partial_{(F_m, G_m)} Z_m^{(n)}(a, z_0; z) = Z_{m+1}^{(n-1)}(a, z_0; z).$$

4) And finally,

$$Z_m^{(n)}(a, z_0; z) \to a (z - z_0)^n \text{ when } z \to z_0.$$

Remark 9: Every complex-valued function W, solution of (11), can be expanded in Taylor series in formal powers

$$W = \sum_{n=0}^{\infty} Z^{(n)}(a_n, z_0; z), \tag{15}$$

where the absence of the subindex m means that all formal powers correspond to the same generating pair. In other words, this *Remark* explains that the expansion (15) *is an analytic representation of the general solution of (11).*

It is also important to mention that the Taylor coefficients a_n are obtained according to the formulas

$$a_n = \frac{W^{[n]}(z_0)}{n!}.$$

III. New Solutions for the Electrical Impedance Equation when σ is a separable-variables function depending upon three spacial variables

Following [17], let us consider the following inhomogeneous Vekua equation

$$\partial_{\overline{z}}(\phi F) - a(\phi F) - b\left(\overline{\phi F}\right) = (\partial_{\overline{z}} \phi) F, \tag{16}$$

where ϕ is a real-valued function, and F is a complex-valued function. It is possible to check that this equality holds if and only if F is a particular solution of the homogeneous Vekua equation

$$\partial_{\overline{z}} F - aF - b\overline{F} = 0.$$

In the same way, let us consider now the equality

$$\partial_{\overline{z}}(\psi G) - a(\psi G) - b\left(\overline{\psi G}\right) = (\partial_{\overline{z}} \psi) G, \tag{17}$$

where ψ is a real function and G is a complex function such that, together with F, fulfills the inequality (6) $\operatorname{Im}\left(\overline{F}G\right) > 0$. Again, the relation (17) will be true iff

$$\partial_{\overline{z}} G - aG - b\overline{G} = 0.$$

By denoting $W = \phi F + \psi G$, it is evident that the equation

$$\partial_{\overline{z}} W - aW - b\overline{W} = 0$$

will be fulfilled if and only if

$$(\partial_{\overline{z}} \phi) F + (\partial_{\overline{z}} \psi) G = 0.$$

Additional calculations show that the coefficients a and b are indeed the characteristic coefficients defined in (9). Hence, this is an alternative way for introducing the notion of an (F, G)-pseudoanalytic function W, and according to [17], we can use the same idea for posing the structure of the general solution for the three-dimensional quaternionic Electrical Impedance Equation.

Subsequently, we will use this proposal for obtaining a new class of solutions for the case when the conductivity σ is a separable-variables function depending upon three spacial variables.

A. The Quaternionic Electrical Impedance Equation and the structure of its general solution

Consider the Electrical Impedance Equation (1)

$$\operatorname{grad}(\sigma \operatorname{div} u) = 0,$$

where the conductivity function σ has the form

$$\sigma(x_1, x_2, x_3) = \alpha(x_1) \beta(x_2) \gamma(x_3), \tag{18}$$

and α, β, and γ are real-valued functions at least once differentiable. Let us introduce the notations (see e.g. [15], [19])

$$\overrightarrow{\mathcal{E}} = \sqrt{\sigma} \operatorname{grad} u, \qquad \overrightarrow{\sigma} = \frac{\operatorname{grad} \sqrt{\sigma}}{\sqrt{\sigma}}, \tag{19}$$

where the gradient is applied in the sense of (5). The equation (1) will turn into

$$\left(D + M^{\overrightarrow{\sigma}}\right) \overrightarrow{\mathcal{E}} = 0, \tag{20}$$

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where D is the Moisil-Theodoresco differential operator (4) and $M^{\overrightarrow{\sigma}}$ is the operator of multiplication by the right-hand side (3). Since σ is a separable-variables function, it will be useful to introduce the auxiliary notations

$$\overrightarrow{\sigma} = \sigma_1 e_1 + \sigma_2 e_2 + \sigma_3 e_3,$$

where

$$\sigma_1 = \frac{\partial_1 \sqrt{\alpha}}{\sqrt{\alpha}},\ \sigma_2 = \frac{\partial_2 \sqrt{\beta}}{\sqrt{\beta}},\ \sigma_3 = \frac{\partial_3 \sqrt{\gamma}}{\sqrt{\gamma}}. \quad (21)$$

Following the idea posed in (16) and (17), let us consider the equality

$$\left(D + M^{\overrightarrow{\sigma}}\right)\left(\varphi \overrightarrow{\mathcal{E}}\right) = (D\varphi)\overrightarrow{\mathcal{E}},$$

where φ is a real-valued function. This equation will be valid iff $\overrightarrow{\mathcal{E}}$ is a particular solution of (20). Hence the next statement is in order.

Theorem 10: [17] Let $\overrightarrow{\mathcal{E}}_1$, $\overrightarrow{\mathcal{E}}_2$ and $\overrightarrow{\mathcal{E}}_3$ be a set of linearly independent solutions of (20). Then the quaternionic-valued function

$$\overrightarrow{\mathcal{E}} = \sum_{k=1}^{3} \varphi_k \overrightarrow{\mathcal{E}}_k$$

will be the general solution of (20), where the real-valued functions φ_1, φ_2 and φ_3 are all solutions of the equation

$$\sum_{k=1}^{3} (D\varphi_k)\overrightarrow{\mathcal{E}}_k = 0. \quad (22)$$

Indeed, the quaternionic equation (20) can be considered a generalization of the Vekua equation (see e.g. [15]), as well as (22) could represent a quaternionic generalization of the Beltrami equation (a different generalization, developed from a purely mathematical point of view, was posed by U. Kähler in [10]).

In order to introduce new solutions for (20) through the path of solving (22), we will construct a set of three linearly independent solutions $\overrightarrow{\mathcal{E}}_1$, $\overrightarrow{\mathcal{E}}_2$ and $\overrightarrow{\mathcal{E}}_3$ for (20). Let us assume $\overrightarrow{\mathcal{E}}_1 = \mathcal{E}_1 e_1$, where $\mathcal{E}_1$ is a real-valued function. Substituting it into (20) we will obtain the differential system

$$\begin{gathered}\partial_1 \mathcal{E}_1 + \mathcal{E}_1 \sigma_1 = 0,\ \partial_2 \mathcal{E}_1 - \mathcal{E}_1 \sigma_2 = 0, \\ \partial_3 \mathcal{E}_1 - \mathcal{E}_1 \sigma_3 = 0.\end{gathered} \quad (23)$$

It is easy to check that

$$\mathcal{E}_1 = A_1 e^{-\int \sigma_1 dx_1 + \int \sigma_2 dx_2 + \int \sigma_3 dx_3},$$

where A_1 is a real constant, conforms a solution of (23). Applying the same idea, we can verify that

$$\begin{aligned}\overrightarrow{\mathcal{E}}_1 &= e_1 A_1 e^{-\int \sigma_1 dx_1 + \int \sigma_2 dx_2 + \int \sigma_3 dx_3}, \\ \overrightarrow{\mathcal{E}}_2 &= e_2 A_2 e^{\int \sigma_1 dx_1 - \int \sigma_2 dx_2 + \int \sigma_3 dx_3}, \\ \overrightarrow{\mathcal{E}}_3 &= e_3 A_3 e^{\int \sigma_1 dx_1 + \int \sigma_2 dx_2 - \int \sigma_3 dx_3},\end{aligned}$$

constitute a set of linearly independent solutions of (20). Here A_2 and A_3 are real constants.

Nonetheless it is not clear how to solve the general case of (22) in order to obtain the general solution of (20), we can introduce a wide class of new solutions using the following method (a similar analysis was posed in [19], for the case when σ depends upon only two spacial variables).

Case 11: Consider in (22) the particular case when $\varphi_3 = 0$. Thus we have

$$(D\varphi_1)\overrightarrow{\mathcal{E}}_1 + (D\varphi_2)\overrightarrow{\mathcal{E}}_2 = 0.$$

Expanding this quaternionic equation, we obtain

$$\begin{gathered}A_1 e^{-\int \sigma_1 dx_1 + \int \sigma_2 dx_2 + \int \sigma_3 dx_3}\partial_1\varphi_1 + \\ + A_2 e^{\int \sigma_1 dx_1 - \int \sigma_2 dx_2 + \int \sigma_3 dx_3}\partial_2\varphi_2 = 0; \\ A_1 e^{-\int \sigma_1 dx_1 + \int \sigma_2 dx_2 + \int \sigma_3 dx_3}\partial_2\varphi_1 - \\ - A_2 e^{\int \sigma_1 dx_1 - \int \sigma_2 dx_2 + \int \sigma_3 dx_3}\partial_1\varphi_2 = 0, \\ \partial_3\varphi_1 = \partial_3\varphi_2 = 0;\end{gathered}$$

and by introducing the notation

$$p = \frac{A_1}{A_2} e^{-2\int \sigma_1 dx_1 + 2\int \sigma_2 dx_2},$$

the system takes the form

$$\partial_1\varphi_1 = -\frac{1}{p}\partial_2\varphi_2, \quad \partial_2\varphi_1 = \frac{1}{p}\partial_1\varphi_2. \quad (24)$$

This system of equations is not other than the so-called p-analytic system [18].

Theorem 12: [13] The real-valued functions φ_1 and φ_2 will be solutions of the system (24) if and only if the function

$$W = \varphi_1 p + i\frac{\varphi_2}{p}$$

is a solution of the Vekua equation

$$\partial_{\overline{\zeta}} W - \frac{\partial_{\overline{\zeta}} p}{p}\overline{W} = 0, \quad (25)$$

where $\partial_{\overline{\zeta}} = \partial_2 + i\partial_1$.

This Vekua equation has precisely the form of the one studied in the previous work [20], where it was posed the structure of the general solution for the two-dimensional Electrical Impedance Equation.

Noticing that p is in fact a separable-variables function, an explicit generating sequence [13] can be introduced in order to express the general solution of (25) in terms of Taylor series in formal powers (see e.g. [13])

$$W = \sum_{n=0}^{\infty} Z^n(a, \zeta_0; \zeta),$$

where $\zeta = x_2 + ix_1$.

But, using a particular property of (25) pointed out in [8], we are able to obtain an exact solution for (25) that will give us the possibility of better explaining the contribution of the Pseudoanalytic Function Theory to the Electrical Impedance Tomography Theory.

Remark 13: We can use the procedure of the *Case 11* considering $\varphi_1 = 0$ in (22), and subsequently $\varphi_2 = 0$. These cases will provide a wider class of new solutions for (20) and, in consequence, for (1).

B. A particular solution for the Vekua equation bias the stationary Schrödinger equation

A simple calculation will show that

$$\frac{\partial_{\overline{\zeta}} p}{p} = 2\left(-\sigma_2 + i\sigma_1\right),$$

where σ_1 and σ_2 have the form (21). Indeed, we can write

$$\frac{\partial_{\overline{\zeta}} p}{p} = 2\left(-\sigma_2 + i\sigma_1\right) = -\frac{\partial_{\zeta}\left(\alpha\beta\right)}{\alpha\beta},$$

where $\partial_{\zeta} = \partial_2 - i\partial_1$, and α and β are defined in (18). Thus the equation (25) turns into

$$\partial_{\overline{\zeta}} W + \frac{\partial_{\zeta}\left(\alpha\beta\right)}{\alpha\beta}\overline{W} = 0,$$

or using the operational notation

$$\left(\partial_{\overline{\zeta}} + \frac{\partial_{\zeta}\left(\alpha\beta\right)}{\alpha\beta} C\right) W = 0, \tag{26}$$

where C denotes the complex conjugation operator, acting upon a complex-valued function $W = W_0 + iW_1$ as $C\left(W_0 + iW_1\right) = W_0 - iW_1$.

A detail that was not mentioned in [20] is that the differential operator of (26) possesses a very particular property. This will be pointed out in the following proposition.

Proposition 14: [14] Let the function f_0 be a non-vanishing particular solution of the two-dimensional stationary Schrödinger equation

$$\Delta f - vf = 0, \tag{27}$$

where $\Delta = \partial_1^2 + \partial_2^2$. Then, its differential operator $(\Delta - v)$ can be factorized in the following way

$$\left(\partial_{\overline{\zeta}} + \frac{\partial_{\zeta} f_0}{f_0} C\right)\left(\partial_{\zeta} - \frac{\partial_{\zeta} f_0}{f_0}\right) f = 0. \tag{28}$$

The reader can notice the first differential operator of (28) has the same structure that the corresponding one to (26). Therefore, it is clear that possessing two particular solutions of (27), f_0 and f, the complex-valued function

$$W = \left(\partial_{\zeta} - \frac{\partial_{\zeta} f_0}{f_0}\right) f \tag{29}$$

will be a solution of

$$\partial_{\overline{\zeta}} W + \frac{\partial_{\zeta} f_0}{f_0}\overline{W} = 0. \tag{30}$$

In order to use this property for solving (26), let us consider the particular case of (27) when v is a real constant. This is, the equation (27) becomes the Yukawa equation (see e.g. [4],[13]).

We can check that one solution for (27) is

$$f = Ke^{m_1 x_1 + m_2 x_2}, \tag{31}$$

where K is a real constant and m_1, m_2 are real numbers such that $m_1^2 + m_2^2 = v$; and

$$f_0 = e^{n_1 x_1 + n_2 x_2}, \tag{32}$$

where n_1 and n_2 are also real numbers fulfilling $n_1^2 + n_2^2 = v$, is another solution. Substituting (31) and (32) into (29) we obtain

$$W = \left[m_2 - im_1 - (n_2 - in_1)\right] Ke^{m_1 x_1 + m_2 x_2}, \tag{33}$$

that will be an exact solution of (30), and consequently, of (25). By identifying f_0 in (30) with $\alpha\beta$ of (26) we have

$$\alpha = e^{n_1 x_1} \text{ and } \beta = e^{n_2 x_2}.$$

Despite the mathematical simplicity of the functions α and β, to posses an exact solution W of the Vekua equation (25), that is not simply an element of the corresponding generating pair (F, G), allows us to expand such solution in terms of Taylor series in formal powers, and in consequence, to analyze the behavior of the formal powers (this would not be possible if the exact solution W was simply $W = F$ or $W = G$, since the expansion in formal powers requires the n^{th}-derivative in the sense of Bers of W, but by virtue of *Remark 1*, the first derivative in the sense of Bers of F and of G is equal to zero: $\partial_{(F,G)} F = \partial_{(F,G)} G = 0$).

For achieving the expansion of W in Taylor series we need first an explicit generating sequence. Following [13], it is possible to verify that the generating pair (F, G) corresponding to the Vekua equation (25), has the form

$$F = 2\frac{\alpha}{\beta} = 2e^{n_1 x_1 - n_2 x_2} \quad \text{and} \quad G = 2i\frac{\beta}{\alpha} = 2ie^{-n_1 x_1 + n_2 x_2}.$$

Moreover, the generating sequence is composed as follows:

$$F_m = 2^m e^{n_1 x_1 - n_2 x_2} \quad \text{and} \quad G_m = 2^m ie^{-n_1 x_1 + n_2 x_2},$$

when m is an even number, and

$$F_m = 2^m e^{n_1 x_1 + n_2 x_2} \quad \text{and} \quad G_m = 2^m ie^{-n_1 x_1 - n_2 x_2},$$

when m is odd.

It is worth of mention that there are not many works dedicated to the behavior of formal powers when we take points far away from their center z_0, as well as when different kinds of closed curves are considered for their evaluation. In fact, a proper study of the expansion in Taylor series in formal powers of the solution (33), obtained bias the Yukawa equation, would reach enough material for an independent paper.

Still, recent works have shown that the convergence speed of the Taylor series in formal powers is very good (see e.g. [6]), compared to other classical methods for expanding functions.

Remark 15: When considering the two-dimensional case of (20), by virtue of *Theorem 10*, the procedures exposed in this work will allow us to obtain the general solution of the two-dimensional Electrical Impedance Equation (1). This implies that, for the two-dimensional case, the present work poses an alternative method to the one exposed in [20].

IV. Conclusions

The study of the Electrical Impedance Equation (1) is the base for well understanding the Electrical Impedance Tomography problem. We shall remark the possible relevance of these new solutions by mentioning (for instance) that

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many Electrical Impedance Tomography Medical Devices are strongly connected with Neural Network Modeling (see e.g. [11],[21],[23] and [24]). Such models require an efficient training phase (see for example [1],[7]) in order to reach useful approaches of the conductivity function σ, using values of the electric potential u collected from the boundary Γ of the domain of interest. But since the inverse problem of (1) is considered very unstable [12], the techniques based onto Neural Networks have found several difficulties for improving the quality of the reconstructed images.

In this context, the knowledge of a wide class of analytic solutions u for (1) could well become significant. By employing parametric transformations, we can always obtain an explicit function $u|_\Gamma$ describing the values of u near the boundary Γ of the domain of interest.

Since for all physical cases the boundary Γ is a closed curve or a closed surface, the function $u|_\Gamma$ could be represented using quite standard expansions, such as the Fourier series. Hence, the Neural Network might be trained using only the most relevant Fourier coefficients of $u|_\Gamma$. We shall notice that, in this hypothetical case, the Fourier coefficients would not belong to a single exact solution u of (1), but to its general solution in analytic form for the two-dimensional case, or to a wide class of exact solutions for the three-dimensional case. Perhaps from this point of view, the Electrical Impedance Tomography problem could be considered more stable.

The new class of solutions of (20) offers another possibilities. For instance, the opportunity of better understanding the behavior of the electrical currents inside inhomogeneous media.

A brief analysis of $\overrightarrow{\mathcal{E}}$ in (20), taking into account the notations (19), will take us precisely to the definition of the *electrical current intensity vector* $\overrightarrow{j}$:

$$\overrightarrow{j} = \sqrt{\sigma}\,\overrightarrow{\mathcal{E}} = \sigma \overrightarrow{E}, \tag{34}$$

where $\overrightarrow{E}$ is the electrical vector field. The relation (34) is known as the *Differential Ohm's Law*, and it describes point by point the behavior of the electrical currents. This is specially useful, since it allows us to estimate all sort of physical effects, such as the electrical power dissipation in some particular tissues, a critical matter in medical applications.

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