A Unified Model between the OWA Operator and the Weighted Average in Decision Making with Dempster-Shafer Theory

José M. Merigó, *Member*, *IAENG*, Kurt J. Engemann

***Abstract*—We present a new decision making model by using the Dempster-Shafer belief structure that uses probabilities, weighted averages and the ordered weighted averaging (OWA) operator. Thus, we are able to represent the decision making problem considering objective and subjective information and the attitudinal character of the decision maker. For doing so, we use the ordered weighted averaging – weighted average (OWAWA) operator. It is an aggregation operator that unifies the weighted average and the OWA in the same formulation. As a result, we form the belief structure – OWAWA (BS-OWAWA) aggregation. We study some of its main properties and particular cases. We also present an application of the new approach in a decision making problem concerning political management.**

***Index Terms*—Dempster-Shafer belief structure; Decision making; OWA operator; Weighted average; Aggregation operators.**

I. Introduction

The Dempster-Shafer (D-S) theory of evidence was introduced by Dempster [3] and by Shafer [10]. Since its introduction, this theory has been studied and applied in a lot of situations such as [4,8-11,14,17]. It provides a unifying framework for representing uncertainty because it includes as special cases the situations of risk (probabilistic uncertainty) and ignorance (imprecision). One of the key application areas of the D-S theory is in decision making because it allows to use risk and uncertain environments in the same framework. This framework can be carried out with a lot of aggregation operators [1-2,5-7,12-16]. Some authors [4,8,14] have considered the possibility of using the ordered weighted averaging (OWA) operator. The OWA operator [13] is an aggregation operator that provides a parameterized family of aggregation operators between the maximum and the minimum. Since its introduction, it has been applied in a wide range of situations [1-2,5-8,12-16].

Recently [6-7], Merigó has introduced the ordered weighted averaging – weighted average (OWAWA) operator. It is an aggregation operator that unifies the weighted average (WA) and the OWA operator in the same formulation considering the degree of importance that each concept has in the aggregation. The aim of this paper is to present a new decision making model with D-S theory by using the OWAWA operator. The main advantage of using this framework is that we are able to consider probabilistic information with WAs and OWAs. Thus, we are able to consider a decision making problem with objective and subjective information and considering the attitudinal character of the decision maker. For doing so, we present a new aggregation operator, the belief structure – OWAWA (BS-OWAWA) operator. It is a new aggregation operator that aggregates the belief structures with the OWAWA operator. We study some of its main properties and particular cases.

We also develop an illustrative example of the new approach in a decision making problem concerning the selection of policies. We study a problem where a government is planning the monetary policy for the next year. The main advantage of using this approach is that we are able to consider a wide range of scenarios and select the one closest with our interests.

This paper is organized as follows. In Section 2, we briefly review some basic concepts about the D-S theory, the WA, the OWA and the OWAWA operator. In Section 3 we present the new decision making approach. Section 4 introduces the BS-OWAWA operator and in Section 5 we develop an illustrative example. Section 6 summarizes the main conclusions of the paper.

II. Preliminaries

A. Dempster-Shafer Belief Structure

The D-S theory [3,10] provides a unifying framework for representing uncertainty as it can include the situations of risk and ignorance as special cases. Note that the case of certainty is also included as it can be seen as a particular case of risk and ignorance.

Definition 1. A D-S belief structure defined on a space X consists of a collection of n nonnull subsets of X, B_j for $j = 1,\ldots,n$, called focal elements and a mapping m, called the basic probability assignment, defined as, m: $2^X \rightarrow [0, 1]$ such that:

1) $m(B_j) \in [0, 1]$.
2) $\sum_{j=1}^{n} m(B_j) = 1$.
3) $m(A) = 0$, $\forall A \neq B_j$.

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J.M. Merigó is with the Department of Business Administration, University of Barcelona, Av. Diagonal 690, 08034 Barcelona, Spain (corresponding author: +34-93-4021962; fax: +34-93-4039882; e-mail: jmerigo@ ub.edu).
K.J. Engemann is with the Department of Information Systems, Hagan School of Business, IONA College, 10801 New Rochelle, NY, USA (email: kengemann@iona.edu).

As said before, the cases of risk and ignorance are included as special cases of belief structure in the D-S framework. For the case of risk, a belief structure is called Bayesian belief structure if it consists of n focal elements such that $B_j = \{x_j\}$, where each focal element is a singleton. Then, we can see that we are in a situation of decision making under risk environment as $m(B_j) = P_j = \text{Prob }\{x_j\}$.

The case of ignorance is found when the belief structure consists in only one focal element B, where $m(B)$ essentially is the decision making under ignorance environment as this focal element comprises all the states of nature. Thus, $m(B) = 1$. Other special cases of belief structures such as the consonant belief structure or the simple support function are studied in [10]. Note that two important evidential functions associated with these belief structures are the measures of plausibility and belief [10].

B. The OWA Operator

The OWA operator [13] is an aggregation operator that provides a parameterized family of aggregation operators between the minimum and the maximum. It can be defined as follows.

Definition 2. An OWA operator of dimension n is a mapping OWA: $R^n \to R$ that has an associated weighting vector W of dimension n with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$OWA\ (a_1, \ldots, a_n) = \sum_{i=1}^{n} w_j b_j \quad (1)$$

where b_j is the jth largest of the a_i.

Note that different properties could be studied such as the distinction between descending and ascending orders, different measures for characterizing the weighting vector and different families of OWA operators [1-2,6-8,12,15-16].

C. The Weighted Average

The weighted average (WA) is one of the most common aggregation operators found in the literature. It has been used in a wide range of applications. It can be defined as follows.

Definition 3. A WA operator of dimension n is a mapping WA: $R^n \to R$ that has an associated weighting vector V, with $v_j \in [0, 1]$ and $\sum_{i=1}^{n} v_i = 1$, such that:

$$WA\ (a_1, \ldots, a_n) = \sum_{j=1}^{n} v_i a_i \quad (2)$$

where a_i represents the argument variable.

The WA operator accomplishes the usual properties of the aggregation operators. For further reading on different extensions and generalizations of the WA, see for example [1-2,6].

D. The OWAWA Operator

The ordered weighted averaging – weighted average (OWAWA) operator [6-7] is a new model that unifies the OWA operator and the weighted average in the same formulation. Therefore, both concepts can be seen as a particular case of a more general one. It can be defined as follows.

Definition 4. An OWAWA operator of dimension n is a mapping $OWAWA$: $R^n \to R$ that has an associated weighting vector W of dimension n such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, according to the following formula:

$$OWAWA\ (a_1, \ldots, a_n) = \sum_{j=1}^{n} \hat{v}_j b_j \quad (3)$$

where b_j is the jth largest of the a_i, each argument a_i has an associated weight (WA) v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, $\hat{v}_j = \beta w_j + (1-\beta) v_j$ with $\beta \in [0, 1]$ and v_j is the weight (WA) v_i ordered according to b_j, that is, according to the jth largest of the a_i.

As we can see, if $\beta = 1$, we get the OWA operator and if $\beta = 0$, the WA. The OWAWA operator accomplishes similar properties than the usual aggregation operators. Note that we can distinguish between descending and ascending orders, extend it by using mixture operators, and so on.

III. Decision Making with D-S Theory using the OWAWA Operator

A new method for decision making with D-S theory is possible by using the OWAWA operator. The main advantage of this approach is that we can use probabilities, WAs and OWAs in the same formulation. Thus, we are able to represent the decision problem in a more complete way because we can use objective and subjective information and the attitudinal character (degree of optimism) of the decision maker. The decision process can be summarized as follows.

Assume we have a decision problem in which we have a collection of alternatives $\{A_1, \ldots, A_q\}$ with states of nature $\{S_1, \ldots, S_n\}$. a_{ih} is the payoff if the decision maker selects alternative A_i and the state of nature is S_h. The knowledge of the state of nature is captured in terms of a belief structure m with focal elements $B_1, \ldots, B_r$ and associated with each of these focal elements is a weight $m(B_k)$. The objective of the problem is to select the alternative which gives the best result to the decision maker. In order to do so, we should follow the following steps:

Step 1: Calculate the results of the payoff matrix.

Step 2: Calculate the belief function m about the states of nature.

Step 3: Calculate the attitudinal character (or degree of orness) of the decision maker $\alpha(W)$ [6-7,13].

Step 4: Calculate the collection of weights, w, to be used in the OWAWA aggregation for each different cardinality of focal elements. Note that it is possible to use different methods depending on the interests of the decision maker [6,12,15]. Note that for the WA aggregation we have to

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calculate the weights according to a degree of importance (or subjective probability) of each state of nature. This can be carried out by using the opinion of a group of experts that has some information about the possibility that each state of nature will occur.

Step 5: Determine the results of the collection, M_{ik}, if we select alternative A_i and the focal element B_k occurs, for all the values of i and k. Hence $M_{ik} = \{a_{ih} \mid S_h \in B_k\}$.

Step 6: Calculate the aggregated results, V_{ik} = OWAWA(M_{ik}), using Eq. (4), for all the values of i and k.

Step 7: For each alternative, calculate the generalized expected value, C_i, where:

$$C_i = \sum_{r=1}^{r} V_{ik} m(B_k) \tag{4}$$

Step 8: Select the alternative with the largest C_i as the optimal. Note that in a minimization problem, the optimal choice is the lowest result.

From a generalized perspective of the reordering step, it is possible to distinguish between ascending and descending orders in the OWAWA aggregation.

IV. The BS-OWAWA operator

Analyzing the aggregation in *Steps* 6 and 7 of the previous subsection, it is possible to formulate in one equation the whole aggregation process. We will call this process the belief structure – OWAWA (BS-OWAWA) aggregation. It can be defined as follows.

Definition 5. A BS-OWAWA operator is defined by

$$C_i = \sum_{k=1}^{r} \sum_{j_k=1}^{q_k} m(B_k) \hat{v}_{j_k} b_{j_k} \tag{5}$$

where $\hat{v}_{j_k}$ is the weighting vector of the kth focal element such that $\sum_{j=1}^{n} \hat{v}_{j_k} = 1$ and $\hat{v}_{j_k} \in [0, 1]$, b_{j_k} is the j_kth largest of the a_{i_k}, each argument a_{i_k} has an associated weight (WA) v_{i_k} with $\sum_{i=1}^{n} v_{i_k} = 1$ and $v_{i_k} \in [0, 1]$, and a weight (OWA) w_{j_k} with $\sum_{j=1}^{n} w_{j_k} = 1$ and $w_{j_k} \in [0, 1]$, $\hat{v}_{j_k} = \beta w_{j_k} + (1-\beta) v_{j_k}$ with $\beta \in [0, 1]$ and v_j is the weight (WA) v_i ordered according to b_j, that is, according to the jth largest of the a_{i_k}, and $m(B_k)$ is the basic probability assignment.

Note that q_k refers to the cardinality of each focal element and r is the total number of focal elements.

The BS-OWAWA operator is monotonic, bounded and idempotent. By choosing a different manifestation in the weighting vector of the OWAWA operator, we are able to develop different families of BS-OWAWA operators [6]. As it can be seen in Definition 5, each focal element uses a different weighting vector in the aggregation step with the OWAWA operator. Therefore, the analysis needs to be done individually.

Remark 1. For example, it is possible to obtain the following cases:

- The maximum-WA is formed if $w_1 = 1$ and $w_j = 0$, for all $j \neq 1$.
- The minimum-WA is obtained if $w_n = 1$ and $w_j = 0$, for all $j \neq n$.
- The average is found when $w_j = 1/n$ and $v_i = 1/n$, for all a_i.
- The step-OWAWA operator is found when $w_k = 1$ and $w_j = 0$, for all $j \neq k$.
- The arithmetic-WA is obtained when $w_j = 1/n$ for all j.
- Note that if $v_i = 1/n$, for all i, we get the arithmetic-OWA (A-OWA).
- The olympic-OWAWA is generated when $w_1 = w_n = 0$, and for all others $w_{j*} = 1/(n - 2)$.
- Note that it is possible to develop a general form of the olympic-OWAWA by considering that $w_j = 0$ for $j = 1, 2, \ldots, k, n, n-1, \ldots, n-k+1$, and for all others $w_{j*} = 1/(n - 2k)$, where $k < n/2$.

V. Numerical example

In the following, we are going to develop a numerical example about the use of the OWAWA in a decision making problem with D-S theory. We focus on the selection of monetary policies.

Assume a government that it is planning his monetary policy for the next year and considers five possible alternatives.

- A_1 = Develop a strong expansive monetary policy.
- A_2 = Develop an expansive monetary policy.
- A_3 = Do not make any change.
- A_4 = Develop a contractive monetary policy.
- A_5 = Develop a strong contractive monetary policy.

In order to evaluate these monetary policies, the group of experts of the government considers that the key factor is the economic situation of the world for the next year. After careful analysis, the experts have considered five possible situations that could happen in the future.

- S_1 = Very bad economic situation.
- S_2 = Bad economic situation.
- S_3 = Regular economic situation.
- S_4 = Good economic situation.
- S_5 = Very good economic situation.

Depending on the situation that could happen in the future, the experts establish the payoff matrix. The results are shown in Table 1.

Table 1: Payoff matrix.

	S_1	S_2	S_3	S_4	S_5
A_1	70	60	80	40	50
A_2	30	60	80	50	70
A_3	50	40	50	70	80
A_4	40	60	90	70	40
A_5	50	50	40	70	70

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After careful analysis of the information, the experts have obtained some probabilistic information about which state of nature will happen in the future. This information is represented by the following belief structure about the states of nature.

Focal element
$B_1 = \{S_1, S_2, S_3\} = 0.3$
$B_2 = \{S_1, S_3, S_5\} = 0.3$
$B_3 = \{S_3, S_4, S_5\} = 0.4$

The attitudinal character of the enterprise is very complex because it involves the opinion of different members of the board of directors. After careful evaluation, the experts establish the following weighting vectors for both the WA and the OWA operator: $W = (0.2, 0.4, 0.4)$ and $V = (0.3, 0.3, 0.4)$. Note that they assume that the OWA has a degree of importance of 30% and the WA a degree of 70%. With this information, we can obtain the aggregated results. They are shown in Table 2.

Table 2: Aggregated results.

	AM	WA	OWA	OWAWA
V_{11}	70	71	68	70.1
V_{12}	66.6	65	64	64.7
V_{13}	56.6	56	52	54.8
V_{21}	56.6	59	52	56.9
V_{22}	60	61	56	59.5
V_{23}	66.6	67	64	66.1
V_{31}	46.6	47	46	46.7
V_{32}	60	62	56	60.2
V_{33}	66.6	68	64	66.8
V_{41}	63.3	66	58	63.6
V_{42}	56.6	55	50	53.5
V_{43}	66.6	64	62	63.4
V_{51}	46.6	46	46	46
V_{52}	53.3	55	50	53.5
V_{53}	60	61	58	60.1

Once we have the aggregated results, we have to calculate the generalized expected value. The results are shown in Table 3.

Table 3: Generalized expected value.

	AM	WA	OWA	OWAWA
A_1	63.62	63.2	60.4	62.36
A_2	61.62	62.8	58	61.36
A_3	58.62	60	56.2	58.79
A_4	62.61	61.9	57.2	60.49
A_5	54	54.7	52	53.89

As we can see, depending on the aggregation operator used, the results and decisions may be different. Note that in this case, our optimal choice is the same for all the aggregation operators but in other situations we may find different decisions between each aggregation operator.

A further interesting issue is to establish an ordering of the policies. Note that this is very useful when the decision maker wants to consider more than one alternative. The results are shown in Table 4.

Table 4: Ordering of the policies.

	Ordering		Ordering
AM	$A_1 \succ A_4 \succ A_2 \succ A_3 \succ A_5$	OWA	$A_1 \succ A_2 \succ A_4 \succ A_3 \succ A_5$
WA	$A_1 \succ A_2 \succ A_4 \succ A_3 \succ A_5$	OWAWA	$A_1 \succ A_2 \succ A_4 \succ A_3 \succ A_5$

As we can see, depending on the aggregation operator used, the ordering of the monetary policies may be different. Note that in this example the optimal choice is clearly A_1.

VI. Conclusion

We have presented a new decision making approach with D-S belief structure by using the OWAWA operator. The main advantage of this approach is that it deals with probabilities, WAs and OWAs in the same framework. Therefore, we are able to consider subjective and objective information and the attitudinal character of the decision maker. For doing so, we have developed the BS-OWAWA operator. It is a new aggregation operator that uses belief structures with the OWAWA operator. We have studied some families of BS-OWAWA operators and we have seen that it contains the OWA and the WA aggregation as particular cases. Moreover, by using the OWAWA we can consider a wide range of inter medium results giving different degrees of importance to the WA and the OWA.

We have also developed a numerical example of the new approach. We have focused on a decision making problem about selection of monetary policies. The main advantage of this approach is that it provides a more complete representation of the decision process because the decision maker can consider many different scenarios depending on his interests.

In future research, we expect to develop further extensions of this approach by considering more complex aggregation operators such as those ones that use uncertain information or order-inducing variables.

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