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Improved Modified Nodal Analysis of Nonlinear Analog Circuits in the Time Domain

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Abstract—**The advantages of nodal analyses of nonlinear analog circuits in the time domain are widely recognized. However, it requires some restrictions related to redundant variables or circuit topology. The paper proposes an improvement that allows treating nonlinear analog circuits of any topology, including floating capacitors, magnetically coupled inductors, excess elements and controlled sources. The method keeps the advantage of the simplicity in formulating of MNA-based mathematical models. The procedure has been implemented in a dedicated program that systematically builds the symbolic differential algebraic equation (DAE) system and solves it numerically.**

Index Terms—**Modified nodal approach, floating capacitor, nonlinear analog circuit, symbolic mathematical model.**

I. Introduction

The transient analysis of analog nonlinear circuits requires a numerical integration that is commonly performed through associated discrete circuit models. In this manner, resistive circuits are solved sequentially at each time step [1,2]. Different strategies involve the construction of the state or semistate mathematical model, as differential or differential-algebraic equation system [3-5]. It is solved by specific numerical methods without engaging equivalent circuit models. In this manner, the problem of circuit analysis is transferred to a pure mathematical one. The latter strategy was extended during the last decades, taking advantage of the information technology development [6-11].

The paper is focused on a semistate equations-based method, associated to the modified nodal approach. The proposed method avoids singular matrices in the equation system and overcomes the restriction related to floating capacitors. Thus, the numerical integration algorithm is reliable.

The semistate mathematical model corresponding to the modified nodal approach (MNA) can be expressed as

$$\begin{cases} \boldsymbol{M}(\boldsymbol{x},t)\cdot\dot{\boldsymbol{x}}(t)+\boldsymbol{N}(\boldsymbol{x},t)\cdot\boldsymbol{x}(t)=\boldsymbol{f}(\boldsymbol{x},t), \\ \boldsymbol{x}(t_0)=\boldsymbol{x}_0. \end{cases} \tag{1}$$

The vector of circuit variables $\boldsymbol{x}(t)$, with the initial value $\boldsymbol{x}_0$, contains the node voltages vector $\boldsymbol{v}_{n-1}$ and the branch currents vector $\boldsymbol{i}_m$ that can not be expressed in terms or node voltages and/or theirs first-order derivatives:

$$\boldsymbol{x}(t)=\begin{bmatrix} \boldsymbol{v}_{n-1}(t) \\ \boldsymbol{i}_m(t) \end{bmatrix}. \tag{2}$$

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Therefore, the vector $\boldsymbol{i}_m$ contains the currents of independent and controlled voltage sources, the controlling currents of current controlled sources, the inductors currents and the currents of the current controlled nonlinear resistors [2,4,12].

$\boldsymbol{M}(\boldsymbol{x},t)$ and $\boldsymbol{N}(\boldsymbol{x},t)$ are square and generally state and time dependent matrices. The parameters of the nonlinear elements are stored here; the matrix $\boldsymbol{M}$ contains the dynamic inductances and capacitances of nonlinear energy storage circuit elements and $\boldsymbol{N}$ contains the dynamic resistances and conductances of nonlinear resistors. The matrix $\boldsymbol{M}$ is commonly singular, so that the mathematical model (1) requires a special treatment.

$\boldsymbol{f}(\boldsymbol{x},t)$ contains the circuits excitations and the parameters associated to the incremental sources used in the local linearization of the nonlinear resistors.

Unfortunately, although the building of the mathematical model is relatively simple, its compatibility requires some restrictions. However the proposed method is less restrictive than those presented in the known previous work.

The paper is organized as follows: the section II explains the problem of floating capacitors, the improved version of the MNA is described in section III and an example is treated in the section IV.

II. The Problem of Floating Capacitors

If the circuit capacitors subgraph is not connected, redundant variables appear in eq. (1) and the circuit response can not be computed, as follows.

The time-domain MNA requires that any linear or nonlinear capacitor is linked to the reference node through a path of capacitors. A capacitor that does not accomplish this requirement is a "floating capacitor", as in fig. 1.

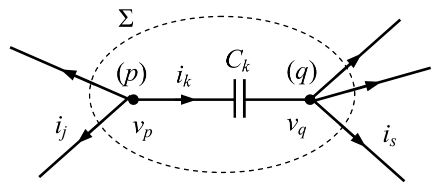

Fig. 1. Floating capacitor.

The time-domain nodal equations for such a structure are:

$$\begin{cases} C_k\cdot\dot{v}_p - C_k\cdot\dot{v}_q + \sum\limits_{j\in(p)} i_j = 0, \\ -C_k\cdot\dot{v}_p + C_k\cdot\dot{v}_q + \sum\limits_{s\in(q)} i_s = 0, \end{cases} \tag{3}$$

or

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$$\begin{bmatrix} C_k & -C_k \\ -C_k & C_k \end{bmatrix} \cdot \begin{bmatrix} \dot{v}_p \\ \dot{v}_q \end{bmatrix} + \begin{bmatrix} \sum\limits_{j\in(p)} i_j \\ \sum\limits_{s\in(q)} i_s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{4}$$

where the state matrix is obviously singular. Therefore, such a mathematical model is improper. If one of the equations (3) is replaced by the cutset current law expressed for the cutset Σ:

$$\begin{cases} C_k \cdot \dot{v}_p - C_k \cdot \dot{v}_q + \sum\limits_{j\in(p)} i_j = 0, \\ \sum\limits_{\substack{j\in(p) \\ j\neq k}} i_j + \sum\limits_{\substack{s\in(q) \\ s\neq k}} i_s = 0, \end{cases} \tag{5}$$

then the singular matrix is avoided. The second equation in (5) can be obtained simply by adding both nodal equations (3). Nevertheless, the equation system (5) contains a redundant variable (two dynamic variables are present and only one differential equation is written). Moreover, we obtained two state variables for only one capacitor.

If any subgraph of capacitors is floating, according to similar reason, then the number of variables exceeds the number of essential capacitors, one variable being redundant.

If the capacitor is grounded, see $v_q = 0$, the nodal equation associated to the node (*p*) is

$$C_k \cdot \dot{v}_p + \sum_{j\in(p)} i_j = 0 \tag{6}$$

and the problem of singular matrix or redundant variable does not appear.

III. MNA IMPROVEMENT

To overcome the problem of singular matrices and redundant variables introduced by the floating capacitors, our method requires following three main steps:

- build the modified nodal equations by ignoring the floating capacitor problem;
- identify all floating subgraphs of capacitors, i.e. the nodal equations related to their nodes [2]; for each such a subgraph, replace one of the nodal equations by the cutset current law expressed for the cutset surrounding the subgraph, as in eq. (5);
- perform a change of variables: the node voltages vector $\boldsymbol{v}_{n-1}$ are replaced with the tree-branch voltages vector $\boldsymbol{u}_t$, the vector $\boldsymbol{i}_m$ remaining unchanged.

As it is known, in general the MNA does not require finding a normal tree of the given circuit. Concerning our goal, a normal tree is required. We developed previously a simple and efficient method to build normal trees systematically [1,2,12], that requires only few preliminary adjustments in the circuit diagram, as: the controlling branches of voltage-controlled sources must be modeled by zero-independent current sources and the controlling branches of current-controlled sources must be modeled by zero-independent voltage sources. The magnetically coupled inductors need to be modeled through equivalent diagrams with controlled sources. Thus, the normal tree is necessary for identifying the excess capacitors and inductors.

Since a normal tree was found, the node-branch incidence matrix can be partitioned as:

$$\boldsymbol{A} = \left[\boldsymbol{A}_t \;\vdots\; \boldsymbol{A}_c\right], \tag{7}$$

where $\boldsymbol{A}_t$ corresponds to the tree branches and $\boldsymbol{A}_c$ corresponds to the cotree branches [2,13]. Next, the tree-branch voltages may be expressed in terms of nodes voltages [1,13] using the transpose of the square nonsingular matrix $\boldsymbol{A}_t$:

$$\boldsymbol{u}_t = \boldsymbol{A}_t^{\ \mathrm{t}} \cdot \boldsymbol{v}_{n-1}. \tag{8}$$

Since the existence of the normal tree guarantees that the matrix $\boldsymbol{A}_t$ is always square and nonsingular, the node voltages of (8) can be expressed in terms of the tree-branch voltages:

$$\boldsymbol{v}_{n-1} = \boldsymbol{A}' \cdot \boldsymbol{u}_t, \tag{9}$$

where $\boldsymbol{A}'$ signifies the inverse matrix of $\boldsymbol{A}_t^{\ \mathrm{t}}$.

Using (9) to substitute the vector $\boldsymbol{v}_{n-1}$ in (1), the mathematical model becomes

$$\boldsymbol{M} \cdot \begin{bmatrix} \boldsymbol{A}' & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{1} \end{bmatrix} \cdot \begin{bmatrix} \dot{\boldsymbol{u}}_t \\ \dot{\boldsymbol{i}}_m \end{bmatrix} + \boldsymbol{N} \cdot \begin{bmatrix} \boldsymbol{A}' & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{1} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{u}_t \\ \boldsymbol{i}_m \end{bmatrix} = \boldsymbol{f} \tag{10}$$

or

$$\begin{cases} \boldsymbol{P}(\boldsymbol{x}',t) \cdot \dot{\boldsymbol{x}}'(t) + \boldsymbol{Q}(\boldsymbol{x}',t) \cdot \boldsymbol{x}'(t) = \boldsymbol{f}(\boldsymbol{x}',t) \\ \boldsymbol{x}'(t_0) = \boldsymbol{x}'_0, \end{cases} \tag{11}$$

where obvious notations were used. The new vector of variables is $\boldsymbol{x}'(t)$. We extract from $\boldsymbol{x}'$ the essential capacitor voltages and the essential inductor currents, as elements of the state vector of length *s* (the subscript *s* comes from "state"):

$$\boldsymbol{x}_s = \begin{bmatrix} \boldsymbol{u}_C \\ \boldsymbol{i}_L \end{bmatrix}. \tag{12}$$

The remained elements of $\boldsymbol{x}'$ are grouped in the vector $\boldsymbol{x}_a$. The vector of variables organized as above involves splitting the equation system (11) as

$$\begin{bmatrix} \boldsymbol{P}_{ss} & \boldsymbol{0}_{sa} \\ \boldsymbol{0}_{as} & \boldsymbol{0}_{aa} \end{bmatrix} \cdot \begin{bmatrix} \dot{\boldsymbol{x}}_s \\ \dot{\boldsymbol{x}}_a \end{bmatrix} + \begin{bmatrix} \boldsymbol{Q}_{ss} & \boldsymbol{Q}_{sa} \\ \boldsymbol{Q}_{as} & \boldsymbol{Q}_{aa} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{x}_s \\ \boldsymbol{x}_a \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_s \\ \boldsymbol{f}_a \end{bmatrix}. \tag{13}$$

We remark that only the partition $\boldsymbol{P}_{ss}$ of size $s \times s$ of the matrix $\boldsymbol{P}$ is nonsingular, the other elements being zeros.

A differential-algebraic equation system has been emphasized

$$\begin{cases} \boldsymbol{P}_{ss} \cdot \dot{\boldsymbol{x}}_s(t) + \boldsymbol{Q}_{ss} \cdot \boldsymbol{x}_s(t) + \boldsymbol{Q}_{sa} \cdot \boldsymbol{x}_a(t) = \boldsymbol{f}_s(\boldsymbol{x}_s, \boldsymbol{x}_a, t), \\ \boldsymbol{Q}_{as} \cdot \boldsymbol{x}_s(t) + \boldsymbol{Q}_{aa} \cdot \boldsymbol{x}_a(t) = \boldsymbol{f}_a(\boldsymbol{x}_s, \boldsymbol{x}_a, t), \end{cases} \tag{14}$$

with the initial condition

$$\boldsymbol{x}_s(t_0) = \begin{bmatrix} \boldsymbol{u}_C(t_0) \\ \boldsymbol{i}_L(t_0) \end{bmatrix}. \tag{15}$$

Therefore, the vector $\boldsymbol{x}_s$ contains the variables of the differential equation system, while $\boldsymbol{x}_a$ combines the variables of the algebraic equation system (the subscript a comes from "algebraic").

In order to find the time-domain solution, many numerical techniques suitable for DAE can be used. In principle, the computation procedure requires the discretization of the analysis time and running the following steps:

- Solve the algebraic equation from (14), assigning to the state variables the initial values:

$$\boldsymbol{Q}_{as} \cdot \boldsymbol{x}_s(t_0) + \boldsymbol{Q}_{aa} \cdot \boldsymbol{x}_a = \boldsymbol{f}_a \tag{16}$$

in order to find the solution $\boldsymbol{x}_a(t_0)$.

- Perform a numerical integration of the differential equation from (14), for the first discrete time interval (t_0, t_1), assigning the value $\boldsymbol{x}_a(t_0)$ to the vector $\boldsymbol{x}_a$ and considering (15) as initial condition:

$$\begin{cases} \boldsymbol{P}_{ss} \cdot \dot{\boldsymbol{x}}_s + \boldsymbol{Q}_{ss} \cdot \boldsymbol{x}_s + \boldsymbol{Q}_{sa} \cdot \boldsymbol{x}_a(t_0) = \boldsymbol{f}_s, \\ \boldsymbol{x}_s(t_0) = \boldsymbol{x}_{s0}. \end{cases} \tag{17}$$

The solution $\boldsymbol{x}_s(t_1)$ is obtained.

- At the time step k, the algebraic equation is solved, assigning to the state variables the values $\boldsymbol{x}_s(t_k)$ calculated previously, during the numerical integration on the time interval (t_{k-1}, t_k):

$$\boldsymbol{Q}_{as} \cdot \boldsymbol{x}_s(t_k) + \boldsymbol{Q}_{aa} \cdot \boldsymbol{x}_a = \boldsymbol{f}_a. \tag{18}$$

The solution $\boldsymbol{x}_a(t_k)$ is found.

- Perform a numerical integration of the differential equation, for the next discrete time interval (t_k, t_{k+1}), assigning the previously computed value $\boldsymbol{x}_a(t_k)$ to the vector $\boldsymbol{x}_a$, and considering as initial condition the values $\boldsymbol{x}_s(t_k)$:

$$\begin{cases} \boldsymbol{P}_{ss} \cdot \dot{\boldsymbol{x}}_s + \boldsymbol{Q}_{ss} \cdot \boldsymbol{x}_s + \boldsymbol{Q}_{sa} \cdot \boldsymbol{x}_a(t_k) = \boldsymbol{f}_s, \\ \boldsymbol{x}_s(t_k) = \boldsymbol{x}_{sk}. \end{cases} \tag{19}$$

The solution $\boldsymbol{x}_s(t_{k+1})$ is obtained.

The last two steps are followed until the final moment of the analysis time is reached.

It is noticeable that the efficiency of the iterative algorithms used for nonlinear algebraic equation solving is significantly increased if $\boldsymbol{x}_a(t_{k-1})$ is considered as start point.

The above described method has been implemented in a computation program under the high performance computing environment MATLAB. It recognizes the input data stored in a SPICE-compatible netlist, performs a topological analysis in order to build a normal tree and incidence matrices, identifies the excess elements and floating capacitors, builds the symbolic mathematical model as in expression (14), solves it numerically and represents the solution graphically.

IV. EXAMPLE

Let us study the transient behavior of a brushed permanent magnet DC motor supplied by a chopper, whose equivalent diagram built according to the transient model is shown in fig. 2. There is not our goal to explain here the correspondence between the electromechanical system and the circuit diagram.

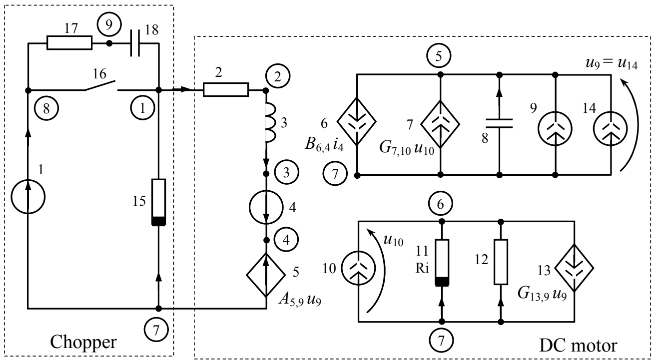

Fig. 2. Circuit example.

The diagram contains two floating capacitors and two nonlinear resistors (theirs characteristics are shown in fig. 3). The independent zero-current sources 9, 10 and 14 modelize the controlling branches of the voltage controlled sources 5, 13 and 7 respectively, while the independent zero-voltage source 4 modelizes the controlling branch of the current controlled current source of the branch 6. The switch is modeled by a time-variable linear resistance (fig. 4).

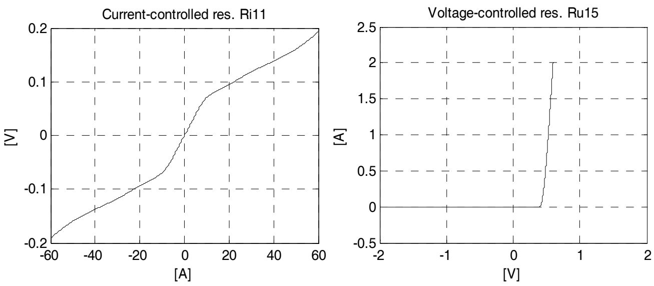

Fig. 3. Nonlinear resistors characteristics.

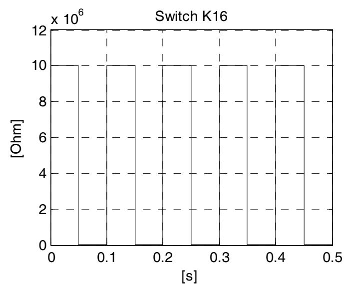

Fig. 4. Time-variable resistor as model of the switch.

If the node 9 is grounded, the topological analysis performed by our computing program gets the results:

The circuit does not contain excess inductors

Normal tree branches: 1 4 5 8 18 15 2 12

MNA-incompatible branches: 1 3 4 5 11

```
Floating capacitor subgraph 1:
  reference_node: 7
     other_nodes: 5
```

Therefore, the semistate variables are: $\boldsymbol{x}_s = [u_8, u_{18}, i_3]^{\mathrm{t}}$, and the variables of the algebraic equation system are:

$$\boldsymbol{x}_a = [u_1, u_4, u_5, u_{15}, u_2, u_{12}, i_1, i_4, i_5, i_{11}]^{\mathrm{t}}.$$

The computing program gets the mathematical model in the symbolic form, slightly modified as compared to eq. (14):

- The differential equation system:

$$\begin{cases} -C18^{*}Du18 = -GK16^{*}u1\text{-}(-Gd15\text{-}GK16)^{*}u15\text{-}G2^{*}u2 + J0R15 \\ -C8^{*}Du8 = G7\_10^{*}u12\text{-}B6\_4^{*}i4 + J9 + J14 \\ -L3^{*}Di3 = u4\text{-}u5 + u15 + u2 \end{cases}$$

- The algebraic equation system:

$$\begin{cases} i3\text{-}G2^{*}u2 = 0 \\ -i3 + i4 = 0 \\ -i4\text{-}i5 = 0 \\ -G13\_14^{*}u8 + G12^{*}u12 + i11 + J10 = 0 \\ G13\_14^{*}u8 + Gd15^{*}u15\text{-}G12^{*}u12 + i1 + i5\text{-}i11\text{-}J10 + J0R15 = 0 \\ -G17^{*}u18 + (-GK16\text{-}G17)^{*}u1 + (GK16 + G17)^{*}u15\text{-}i1 = 0 \\ u1 + E1 = 0 \\ u4 + E4 = 0 \\ A5\_9^{*}u8 + u5 = 0 \\ u12\text{-}Rd11^{*}i11\text{-}E0R11 = 0 \end{cases}$$

Since the mathematical model above is given by the computing program automatically, some unobvious notations are used (e.g. the first derivative of a state variable – $Du18$; the conductance of the time-variable resistance of the branch 16 – $GK16$; the conductance of the nonlinear resistance of the branch 15 – $Gd15$; the incremental current source used in the local linearization of a nonlinear voltage-controlled nonlinear resistance of the branch 15 – $J0R15$; the voltage gain of the voltage-controlled voltage source of the branch 5 controlled by the branch 9 – $A5\_9$).

Assuming zero-initial conditions, the solving algorithm gets the result as time-domain functions, an example being given in fig. 5. We remark that we obtained the same results with a witness SPICE simulation, using the version ICAP/4 from Intusoft [14].

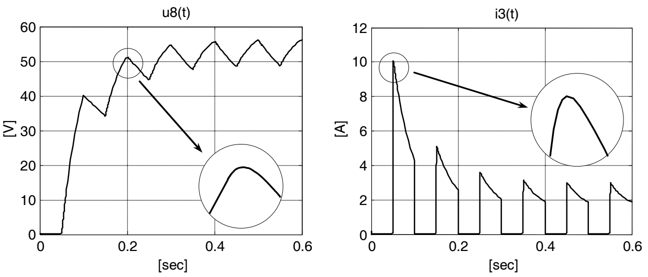

Fig. 5. Example of analysis results.

V. Conclusion

An efficient and totally feasible algorithm intended to the time-domain analysis of nonlinear lumped analog circuits was developed and implemented in a computation program. It overcomes some restrictions of the modified nodal approaches, having practically an unlimited degree of generality for RLCM circuits.

The algorithm benefits by the simplicity of the MNA and the numerical methods for solving the mathematical model are flexible and can be optimized without requiring any companion diagrams (as the SPICE-like algorithms). In this manner, the computation time and the computer requirements can be reduced as compared to other methods. Our contribution is proven by an example, the chosen circuit containing nonlinear resistors, floating capacitor, controlled sources and time-variable elements.

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