Optimization of First Order Partial Differential Inclusions in Gradient Form

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Abstract—This paper is dedicated to optimization of so-called first order differential (PC₆) inclusions in gradient form on a square domain. As a supplementary problem, discrete-approximation problem is considered. In the Euler-Lagrange form, necessary and sufficient conditions are derived for partial differential inclusions (PC₆). The results obtained are based on a new concept of locally adjoint mappings.

Index Terms—Locally adjoint mappings, discrete and differential inclusions, discrete approximation, necessary and sufficient conditions

I. INTRODUCTION

MULTIVALUED mappings with ordinary and partial differential inclusions are important tools by which some problems from extensively developing fields of optimal control theory can be described [1], [2], [3], [4]. Consequently, a lot of problems in economic dynamics, classical optimal control theory, especially in hydrodynamical engineering, vibrations, chemical, heat, diffusion, i.e. processes can be reduced to such researches. Some duality relations regarding a variety of this kind of optimization problems with partial differential inclusions are given in papers of Mahmudov [1], [2], [3], [4], [5].

In this study feasible solutions of the considered differential inclusions are taken from the space of absolutely continuous functions with summable first partial derivatives. Obviously, different classes of solutions for partial differential inclusions like classical, generalized, almost everywhere considered examples, this form implies the Weierstrass-Pontryagin maximum condition. Note that the adjoint partial differential inclusion for continuous problem which involves gradient function is expressed in terms of div (divergent) operation.

Finally in the conclusion section, the work done in the paper is summarized.

II. SUPPLEMENTARY DEFINITIONS AND THE STATEMENT OF THE PROBLEM

The basic definitions and concepts used in this section can be found in [3]. Let $R^n$ be the n-dimensional Euclidean space, $(u_1, u_2)$ is a pair of elements $u_1, u_2 \in R^n$ and $(u_1, u_2)$ is their inner product. We say that a multivalued mapping $F : R^n \rightarrow 2^{R^2n}$ is convex if its graph $gph F = \{(u, v_1, v_2) : (v_1, v_2) \in F(u)\}$ is a convex subset of $R^{3n}$. It is convex valued if $F(u)$ is a convex set for each $u \in dom F = \{u : F(u) \neq \emptyset\}$. $F$ is closed if $gph F$ is a closed set in $R^{3n}$.

Let us introduce notations:

$$M(u, v_1^*, v_2^*) = \sup_{v_1, v_2} \{(v_1, v_1^*) + (v_2, v_2^*) : (v_1, v_2) \in F(u)\}.$$  

$$F(u, v_1^*, v_2^*) = \{(v_1, v_2) \in F(u) : (v_1, v_1^*) + (v_2, v_2^*) \leq M(u, v_1^*, v_2^*)\}.$$  

These function and set are called Hamiltonian function and argmaximum set respectively. Assume that $ri A$ is relative interior of a set $A \subset R^n$, i.e., is a set of interior points of $A$ with respect to its carrier subspace Lin $A$.

**Definition 1:** A multivalued mapping $F^*$ from $R^{2n}$ into $R^n$ that is defined by

$$F^*(v_1^*, v_2^*, (u_0, v_1^0, v_2^0)) = \{u^* : (u^*, -v_1^*, -v_2^*) \in K_F(u_0, v_1^0, v_2^0)\},$$

is called locally adjoint mapping (LAM) to the convex mapping $F$ at the point $(u_0, v_1^0, v_2^0)$, where $K_F(u_0, v_1^0, v_2^0)$ is the cone dual to the cone $K_F(u_0, v_1^0, v_2^0)$.

**Definition 2:** A multivalued mapping $F^*$ defined by

$$F^*(v_1^*, v_2^*, (u, v_1, v_2)) = \{u^* : M(u, v_1, v_2) - M(\tilde{u}, v_1^*, v_2^*) \leq (u^*, u - \tilde{u}) \}$$

is called the LAM to non-convex mapping $F$ at a point $(\tilde{u}, v_1, v_2) \in gph F$.

A function $g$ is said to be proper if it does not take the value $-\infty$ and is not identically equal to $+\infty$. 

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Lemma 1: Let $F : R^n \rightarrow R^{2n}$ be a convex multivalued mapping. Then

$$F^*(v_1^*, v_2^*, (u, v_1, v_2)) = \begin{cases} \partial_u M(u, v_1^*, v_2^*), & (v_1, v_2) \in F(u, v_1^*, v_2^*), \\ \emptyset, & (u, v_1, v_2) \notin F^*(u, v_1^*, v_2^*) \end{cases}$$

At first we consider the following optimization problem $(PD)$ for discrete inclusions with distributed parameters:

minimize $\sum_{x=0 \cdots L-1} g_{t,x}(u_{t,x})$ subject to $u_{t+1,x}, u_{t,x+1} \in F_{t,x}(u_{t,x})$, and $u_{t,L} = \alpha_{t,L}, t \in H_1, u_{0,x} = \beta_{0,x}, x \in L_0$

where $H_1 = 0, \ldots, T-1, \ L_0 = 0, \ldots, L, g_{t,x} : R^n \rightarrow R^1 \cup \{\pm \infty\}$ are functions taking values on the extended line, $F_{t,x}$ is multivalued mapping $F_{t,x} : R^n \rightarrow 2^{R^n}$, and $\alpha_{t,L}, \beta_{0,x}$ are fixed vectors. A set of points $\{u_{t,x}(t) \in \mathbb{H} \times L_0 = \{u_{t,x} : (t, x) \in H \times L, (t, x) \neq (T, L)\}$, $H = \{0, \ldots, T\}$ is called a feasible solution for the problem $(1)$-(3) if it satisfies the inclusion (2) and boundary conditions (3). It is easy to see that, for fixed natural numbers $T$ and $L$, the conditions (3) enable us to choose some feasible solution, and the number of points to be determined coincides with the number of discrete inclusions (2). The following condition is assumed below for the functions $g_{t,x}, t = 1, \ldots, T, x \in L_1, L_1 = \{0, \ldots, L - 1\}$ and the mapping $F_{t,x}$.

Hypothesis 1: (H1) Suppose that in the problem $(PD)$, the mapping $F_{t,x}$ is such that the cone $K_{\partial t,x}(u_{t,x}, u_{t,x+1}, u_{t+1,x+1})$ of tangent directions is a local tent [3], [4], [5], where $u_{t+1,x}, u_{t,x+1}$ are the points of the optimal solution $(\tilde{u}_{t,x}(t) \in \mathbb{H} \times L_0$), Suppose, moreover, that the functions $g_{t,x}$ admit a CUA [1], [2], [3], [4], $h_{t,x}(u, \tilde{u}_{t,x})$, at the points $\tilde{u}_{t,x}$ that is continuous with respect to $\tilde{u}$. The latter means that the subdifferentials $\partial h_{t,x}(\tilde{u}_{t,x}) = \partial h_{t,x}(0, \tilde{u}_{t,x})$ are defined.

The problem $(PD)$ is said to be convex if the mapping $F_{t,x}$ is convex and the $g_{t,x}$ are convex proper functions.

Hypothesis 2: (H2) Assume that in convex problem $(PD)$, for some feasible solution $\{u^0_{t,x}(t,x) \in \mathbb{H} \times L_0 \}$ of the following conditions is fulfilled:

(a) $(u^0_{t,x}, u^0_{t+1,x}, u^0_{t,x+1}) \in \text{ri} gph F_{t,x}, (t, x) \in H \times L_1, u^0_{t,x} \in \text{ri} dom g_{t,x}, (t, x) \in H \times L_0$

(b) $(u^0_{0,x}, u^0_{1,x}, u^0_{t,x+1}) \in \text{int} gph F_{t,x}, (t, x) \in H \times L_1, (t, x) \neq (t_0,x_0)$ is the fixed pair and $g_{t,x}$ are continuous at the points $u^0_{t,x}$.

In the Section 3 we study the convex problem for differential inclusions with distributed parameters:

minimize $J(u(\cdot, \cdot)) = \int_0^T g(u(t,x), t,x) dt dx + \int_0^1 g_0(u(1,x), x) dx$

subject to $\nabla u(t, x) \in F(u(t,x), t,x)$, $0 \leq t \leq 1, 0 \leq x \leq 1$, and $u(t,1) = \alpha(t), u(0,x) = \beta(x), \alpha(0) = \beta(1), R = [0,1] \times [0,1], (6)$

where $\nabla u = \begin{pmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial u}{\partial x} \end{pmatrix}$.

Here $F(\cdot, t,x) : R^n \rightarrow 2^{R^n}$ is a convex multivalued mapping, $g(\cdot, t,x)$ and $g_0(\cdot,x)$ are continuous functions; $g : R^n \times R \rightarrow R^n, g_0 : R^n \times [0,1] \rightarrow R^n$ and $\alpha(t)$ and $\beta(x)$ are absolutely continuous functions, $\alpha : [0,1] \rightarrow R^n, \beta : [0,1] \rightarrow R^n$. We label this continuous problem as $(PC_c)$.

The problem is to find a solution $\tilde{u}(t,x)$ of the boundary value problem (5), (6) that minimizes (4). Here an admissible solution is understood to be an absolutely continuous function satisfying almost everywhere (a.e.) (5) with summable first partial derivatives. Note that if $u(\cdot, \cdot) \in L_1(R)$ has generalized derivatives belonging to $L_1(R)$, then $u(\cdot, \cdot)$ and $u(t, \cdot)$ are absolutely continuous functions for almost every $x$ and $t$, respectively.

At first we consider the convex problem $(PD_c)$.

**Theorem 1:** Assume that $F_{t,x}$ is a convex multivalued mapping and $g_{t,x}$ are convex proper functions that are continuous at the points of some feasible solution $\{u^*_t(x) \in \mathbb{H} \times L_0 \}$. Then for the solution $\{u^*_t(x) \in \mathbb{H} \times L_0 \}$ to be an optimal solution of the problem $(PD_c)$, it is necessary that there exist a number $\lambda = 0$ or 1 and vectors $\{u^*_t(x)\}$ and $\{g^*_t(x)\}$ simultaneously not all zero such that:

(i) $u^*_t(x) + g^*_t(x) \in F_{t,x}(u^*_t(x), u^*_t(x+1)) - \lambda \partial g_{t,x}(u^*_t(x), \partial g_{t,x}(u^*_t(x))) \equiv 0$.

(ii) $-u^*_t(x) \in \lambda \partial g_{t,x}(u^*_t(x), \partial g_{t,x}(u^*_t(x)))$. Under the Hypothesis 2 (H2) the conditions (1) and (2) are also sufficient for the optimality of $\{u^*_t(x) \in \mathbb{H} \times L_0 \}$.

**Theorem 2:** Assume the Hypothesis 1 (H1) for the problem $(PD_c)$. Then for $\{u^*_t(x) \in \mathbb{H} \times L \}$ to be a solution of this non-convex problem it is necessary that there exist a number $\lambda = 0$ or 1 and vectors $\{u^*_t(x)\}$, $\{g^*_t(x)\}$ simultaneously not all zero, satisfying the conditions (1) and (2) of Theorem 1.

**III. SUFFICIENT CONDITIONS IN THE CONTINUOUS PROBLEM $(PC_c)$**

In this section, we formulate a sufficient condition for optimality for the continuous problem $(PC_c)$.

**Theorem 3:** Assume that the functions $g(u(t,x), \cdot)$ and $g_0(u(x))$ are continuous and convex with respect to $u$, and $F^*(\cdot, t,x)$ is a convex mapping for all fixed $(t, x)$. Then for the optimality of the solution $\tilde{u}(t,x)$ it is sufficient that there exist a solution $\varphi^*(t,x), u^*(t,x)$ such that the conditions (i)-(iii) hold:

(i) $-\nabla \psi(t,x) \in F^*(\psi(t,x), \tilde{u}(t,x), \nabla \tilde{u}(t,x), t,x)$

(ii) $\varphi^*(t,x) = 0, -u^*(1,x) \in \partial g_0(\tilde{u}(1,x), x)$.

**Theorem 4:** Let $\tilde{u}(\cdot, \cdot)$ be some feasible solutions of a non-convex problem $(PC_c)$ and $\{u^*(\cdot, \cdot), \varphi^*(\cdot, \cdot)\}$ be a pair of a feasible solutions satisfying the conditions (i)-(iii):
Thus, we have obtained the following result.

Theorem 5: The solution \( \tilde{u}(t,x) \) corresponding to the control \( \tilde{w}(t,x) \) minimizes \( J(u(\cdot,\cdot)) \) in the Problem (7), if there exists a solution, satisfying the conditions (8), (10), (11).

Now, let us consider the following example:

\begin{equation}
\text{ minimize } J(u(t,x)),
\end{equation}

subject to \[
\frac{\partial u(t,x)}{\partial t} = A_1 u(t,x) + B_1 w(t,x),
\]
\[
\frac{\partial u(t,x)}{\partial x} = A_2 u(t,x) + B_2 w(t,x),
\]
\[w(t,x) \in V\]
\[u(t,1) = \alpha(t), \quad u(0,x) = \beta(x)\]

where \( A_1 \) and \( A_2 \) are \( n \times n \) matrices, \( B_1, B_2 \) are \( n \times r \) matrices, \( V \subset \mathbb{R}^r \) is a convex closed set, and \( g \) and \( g_0 \) are continuously differentiable functions on \( u \). It is required to find a controlling parameter \( v(t,x) \in V \) such that the solution \( \tilde{u}(t,x) \) corresponding to it minimizes \( J(u(\cdot,\cdot)) \). In this case

\[F(u) = \{A_1 u + B_1 V, A_2 u + B_2 V\}\]

By elementary computations we find that

\[F^*(v_1^*, v_2^*, (u, v_1, v_2)) = \begin{cases}
(A_1^* u^* + A_2^* v_2), & -B_1^* u^* - B_2^* v_2 \in K_\gamma^*(w),
\emptyset, & -B_1^* u^* - B_2^* v_2 \notin K_\gamma^*(w)
\end{cases}\]

where \( v_1 = A_1 u + B_1 v, v_2 = A_2 u + B_2 v \). Then, using Theorem 3, we get the relations

\[-\text{div} \psi(t,x) = A_1^* u^*(t,x) + A_2^* \varphi^*(t,x),
-\varphi^*(t,0) = 0\]

\[\langle w - \tilde{w}(t,x), -B_1^* u^*(t,x) - B_2^* \varphi^*(t,x) \rangle \geq 0, \quad w \in V\]

\[-u^*(1,x) = g_0^*(\tilde{u}(1,x), x), \quad \varphi^*(t,0) = 0\]

Obviously (9) can be written in the form

\[\langle B \tilde{u}(t,x), \psi(t,x) \rangle = \sup_{w \in V} \langle Bw, \psi(t,x) \rangle,\]

where \( B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \)

Thus, we have obtained the following result.

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Continuous problems one of which is a linear type and the other of which is a constant convex type are stated and the necessary and sufficient conditions to the respective problems are formulated in order to have a better understanding of the conditions.

As a future work the duality relations about this problem can be considered. It is expected that the dual problem exhibits similar properties with other dual problems, i.e. they should have strong relations with the optimality conditions.

REFERENCES


