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# Optimization of First Order Partial Differential Inclusions in Gradient Form

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***Abstract*—This paper is dedicated to optimization of so-called first order differential $(P_C)$ inclusions in gradient form on a square domain. As a supplementary problem, discrete-approximation problem is considered. In the Euler-Lagrange form, necessary and sufficient conditions are derived for partial differential inclusions $(P_C)$. The results obtained are based on a new concept of locally adjoint mappings.**

***Index Terms*—Locally adjoint mappings, discrete and differential inclusions, discrete approximation, necessary and sufficient conditions**

## I. Introduction

Multivalued mappings with ordinary and partial differential inclusions are important tools by which some problems from extensively developing fields of optimal control theory can be descibed [1], [2], [3], [4]. Consequently, a lot of problems in economic dynamics, classical optimal control theory, especially in hydrodynamical engineering, vibrations, chemical, heat, diffusion, i.e. processes can be reduced to such researches. Some duality relations regarding a variety of this kind of optimization problems with partial differential inclusions are given in papers of Mahmudov [1], [2], [3], [4], [5].

In this study feasible solutions of the considered differential inclusions are taken from the space of absolutely continuous functions with summable first partial derivatives. Obviously, different classes of solutions for partial differential inclusions like classical, generalized, almost everywhere can also be used for this purpose.

This paper is divided into three parts.

In Section II, optimization problems for discrete and then corresponding partial differential inclusions including the gradient vector of searched functions are posed. In terms of Hamiltonian function to multivalued mapping the locally adjoint mapping (LAM) is introduced and necessary and sufficient conditions of optimality are formulated for non-convex discrete inclusions. Moreover, as is shown in the paper, the use of convex upper approximation (CUA) [6] and local tents are very suitable to obtain the optimality conditions for stated problems. Further such concept of LAM and construction of convex and non-smooth analysis facilitate to have a new necessary and sufficient conditions in the Euler-Lagrange form.

In Section 3 the sufficient conditions, that are in the Euler-Lagrange form, for optimality of partial differential inclusions are formulated separately. As is seen from the considered examples, this form implies the Weierstrass-Pontryagin maximum condition. Note that the adjoint partial differential inclusion for continuous problem which involves gradient function is expressed in terms of div (divergent) operation.

Finally in the conclusion section, the work done in the paper is summarized.

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## II. Supplementary definitions and the Statement of the Problem

The basic definitions and concepts used in this section can be found in [3]. Let $R^n$ be the n-dimensional Euclidean space, $(u_1, u_2)$ is a pair of elements $u_1, u_2 \in R^n$ and $\langle u_1, u_2 \rangle$ is their inner product. We say that a multivalued mapping $F : R^n \to 2^{R^{2n}}$ is convex if its graph $\operatorname{gph} F = \{(u, \upsilon_1, \upsilon_2) : (\upsilon_1, \upsilon_2) \in F(u)\}$ is a convex subset of $R^{3n}$. It is convex valued if $F(u)$ is a convex set for each $u \in \operatorname{dom} F = \{u : F(u) \neq \emptyset\}$ . $F$ is closed if $\operatorname{gph} F$ is a closed set in $R^{3n}$.

Let us introduce notations:

$$M(u, \upsilon_1^*, \upsilon_2^*) = \sup_{\upsilon_1, \upsilon_2} \{\langle \upsilon_1, \upsilon_1^* \rangle + \langle \upsilon_2, \upsilon_2^* \rangle : (\upsilon_1, \upsilon_2) \in F(u)\}, \quad \upsilon_1^*, \upsilon_2^* \in R^n,$$

$$F(u, \upsilon_1^*, \upsilon_2^*) = \{(\upsilon_1, \upsilon_2) \in F(u) : \langle \upsilon_1, \upsilon_1^* \rangle + \langle \upsilon_2, \upsilon_2^* \rangle = M(u, \upsilon_1^*, \upsilon_2^*)\}.$$

These function and set are called Hamiltonian function and argmaximum set respectively. Assume that $\operatorname{ri} A$ is relative interior of a set $A \subset R^n$ , i.e., is a set of interior points of $A$ with respect to its carrier subspace $\operatorname{Lin} A$.

*Definition 1:* A multivalued mapping $F^*$ from $R^{2n}$ into $R^n$ that is defined by

$$F^*(\upsilon_1^*, \upsilon_2^*, (u^0, \upsilon_1^0, \upsilon_2^0)) = \{u^* : (u^*, -\upsilon_1^*, -\upsilon_2^*) \in K_F^*(u^0, \upsilon_1^0, \upsilon_2^0)\}$$

is called locally adjoint mapping (LAM) to the convex mapping F at the point $(u^0, \upsilon_1^0, \upsilon_2^0)$, where $K_F^*(u^0, \upsilon_1^0, \upsilon_2^0)$ is the cone dual to the cone $K_F(u^0, \upsilon_1^0, \upsilon_2^0)$.

*Definition 2:* A multivalued mapping $F^*$ defined by

$$F^*(\upsilon_1^*, \upsilon_2^*, (\tilde{u}, \tilde{\upsilon}_1, \tilde{\upsilon}_2)) = \{u^* : M(u, \upsilon_1^*, \upsilon_2^*) - M(\tilde{u}, \upsilon_1^*, \upsilon_2^*) \le \langle u^*, u - \tilde{u} \rangle, \forall (u, \upsilon_1, \upsilon_2) \in R^{3n}\}, (\tilde{\upsilon}_1, \tilde{\upsilon}_2) \in F(\tilde{u}, \upsilon_1^*, \upsilon_2^*)$$

is called the LAM to non-convex mapping $F$ at a point $(\tilde{u}, \tilde{\upsilon}_1, \tilde{\upsilon}_2) \in \operatorname{gph} F$ .

A function $g$ is said to be proper if it does not take the value $-\infty$ and is not identically equal to $+\infty$.

*Lemma 1:* Let $F : R^n \to R^{2n}$ be a convex multivalued mapping. Then

$$F^*(\upsilon_1^*, \upsilon_2^*, (u, \upsilon_1, \upsilon_2)) = \begin{cases} \partial_u M(u, \upsilon_1^*, \upsilon_2^*), & (\upsilon_1, \upsilon_2) \in F(u, \upsilon_1^*, \upsilon_2^*), \\ \emptyset, & (\upsilon_1, \upsilon_2) \notin F(u, \upsilon_1^*, \upsilon_2^*). \end{cases}$$

At first we consider the following optimization problem $(P_D)$ for discrete inclusions with distributed parameters:

$$\text{minimize} \quad \sum_{\substack{t=1,\ldots,T \\ x=0,\ldots,L-1}} g_{t,x}(u_{t,x}) \tag{1}$$

$$\text{subject to} \quad (u_{t+1,x}, u_{t,x+1}) \in F_{t,x}(u_{t,x}), \tag{2}$$

$$\text{and} \quad u_{t,L} = \alpha_{tL},\ t \in H_1,\ u_{0,x} = \beta_{0x},\ x \in L_0 \quad (\alpha_{0L} = \beta_{0L}) \tag{3}$$

where $H_1 = 0, \ldots, T-1,\ L_0 = 0, \ldots, L,\ g_{t,x} : R^n \to R^1 \cup \{\pm\infty\}$ are functions taking values on the extended line, $F_{t,x}$ is multivalued mapping $F_{t,x} : R^n \to 2^{R^{2n}}$, and $\alpha_{tL}, \beta_{0x}$ are fixed vectors. A set of points $\{u_{t,x}\}_{(t,x)\in H\times L_0} = \{u_{t,x} : (t,x) \in H \times L_0,\ (t,x) \neq (T,L)\},\ H = \{0, \ldots, T\}$ is called a feasible solution for the problem (1) - (3) if it satisfies the inclusion (2) and boundary conditions (3). It is easy to see that, for fixed natural numbers $T$ and $L$, the conditions (3) enable us to choose some feasible solution, and the number of points to be determined coincides with the number of discrete inclusions (2). The following condition is assumed below for the functions $g_{t,x},\ t = 1, \ldots, T,\ x \in L_1,\ L_1 = \{0, \ldots, L-1\}$ and the mapping $F_{t,x}$.

*Hypothesis 1:* (H1) Suppose that in the problem $(P_D)$, the mapping $F_{t,x}$ is such that the cone $K_{F_{t,x}}(\tilde{u}_{t,x},\ \tilde{u}_{t+1,x},\ \tilde{u}_{t,x+1})$ of tangent directions is a local tent [3], [4], [5], where $\tilde{u}_{t,x}$ are the points of the optimal solution $\{\tilde{u}_{t,x}\}_{(t,x)\in H\times L_0}$. Suppose, moreover, that the functions $g_{t,x}$ admit a CUA [1], [2], [3], [4], $h_{t,x}(\bar{u}, \tilde{u}_{t,x})$, at the points $\tilde{u}_{t,x}$ that is continuous with respect to $\bar{u}$. The latter means that the subdifferentials $\partial g_{t,x}(\tilde{u}_{t,x}) = \partial h_{t,x}(0, \tilde{u}_{t,x})$ are defined.

The problem $(P_D)$ is said to be convex if the mapping $F_{t,x}$ is convex and the $g_{t,x}$ are convex proper functions.

*Hypothesis 2:* (H2) Assume that in convex problem $(P_D)$, for some feasible solution $\{u_{t,x}^0\}_{(t,x)\in H\times L_0}$ one of the following conditions is fulfilled:

(a) $(u_{t,x}^0, u_{t+1,x}^0, u_{t,x+1}^0) \in \operatorname{ri} \operatorname{gph} F_{t,x},\ (t,x) \in H_1 \times L_1,\ u_{t,x}^0 \in \operatorname{ri} \operatorname{dom} g_{t,x},$ $(t,x) \in H_1 \times L_0$

(b) $(u_{t,x}^0, u_{t,x+1}^0, u_{t+1,x}^0) \in \operatorname{int} \operatorname{gph} F_{t,x},\ (t,x) \in H_1 \times L_1,\ (t,x) \neq (t_0, x_0)$ ($(t_0, x_0)$ is the fixed pair) and $g_{t,x}$ are continuous at the points $u_{t,x}^0$.

In the Section 3 we study the convex problem for differential inclusions with distributed parameters:

$$\text{minimize} \quad J(u(\cdot,\cdot)) = \iint_R g(u(t,x),t,x)\, \mathrm{d}t\, \mathrm{d}x + \int_0^1 g_0(u(1,x),x) \mathrm{d}x \tag{4}$$

$$\text{subject to} \quad \nabla u(t,x) \in F(u(t,x),t,x), \quad 0 < t \leq 1,\ 0 \leq x < 1, \tag{5}$$

$$\text{and} \quad u(t,1) = \alpha(t),\ u(0,x) = \beta(x),\ \alpha(0) = \beta(1), \quad R = [0,1] \times [0,1], \tag{6}$$

where $\nabla u = \operatorname{grad} u = \left(\dfrac{\partial u}{\partial t}, \dfrac{\partial u}{\partial x}\right).$

Here $F(\cdot,t,x) : R^n \to 2^{R^{2n}}$ is a convex multivalued mapping, $g(\cdot,t,x)$ and $g_0(\cdot,x)$ are continuous functions; $g : R^n \times R \to R^1,\ g_0 : R^n \times [0,1] \to R^1$ and $\alpha(t)$ and $\beta(x)$ are absolutely continuous functions, $\alpha : [0,1] \to R^n,\ \beta : [0,1] \to R^n$. We label this continuous problem as $(P_C)$. The problem is to find a solution $\tilde{u}(t,x)$ of the boundary value problem (5), (6) that minimizes (4). Here an admissible solution is understood to be an absolutely continuous function satisfying almost everywhere (a.e) (5) with summable first partial derivatives. Note that if $u(\cdot,\cdot) \in L_1(R)$ has generalized derivatives belonging to $L_1(R)$, then $u(\cdot,x)$ and $u(t,\cdot)$ are absolutely continuous functions for almost every $x$ and $t$, respectively.

At first we consider the convex problem $(P_D)$.

*Theorem 1:* Assume that $F_{t,x}$ is a convex multivalued mapping and $g_{t,x}$ are convex proper functions that are continuous at the points of some feasible solution $\{u_{t,x}^0\}_{(t,x)\in H\times L_0}$. Then for the solution $\{\tilde{u}_{t,x}\}_{(t,x)\in H\times L_0}$ to be an optimal solution of the problem $(P_D)$, it is necessary that there exist a number $\lambda = 0$ or 1 and vectors $\{u_{t,x}{}^*\}$ and $\{\varphi_{t,x}^*\}$, simultaneously not all zero such that:

(1) $u_{t,x}^* + \varphi_{t,x}^* \in F_{t,x}^*\left(u_{t+1,x}^* \varphi_{t,x+1}^*, (\tilde{u}_{t,x}, \tilde{u}_{t+1,x}, \tilde{u}_{t,x+1})\right) - \lambda \partial g_{t,x}(\tilde{u}_{t,x}), \partial g_{0,x}(\tilde{u}_{0,x}) \equiv 0, \quad (t,x) \in H_1 \times L_1,$

(2) $-u_{T,x}^* \in \lambda \partial g_{T,x}(\tilde{u}_{T,x}), \quad \varphi_{t,0}^* = 0.$

Under the Hypothesis 2 (H2) the conditions (1) and (2) are also sufficient for the optimality of $\{\tilde{u}_{t,x}\}_{(t,x)\in H\times L_0}$

*Theorem 2:* Assume the Hypothesis 1 (H1) for the problem $(P_D)$. Then for $\{\tilde{u}_{t,x}\}_{(t,x)\in H\times L}$ to be a solution of this non-convex problem it is necessary that there exist a number $\lambda = 0$ or 1 and vectors $\{u_{t,x}^*\}, \{\varphi_{t,x}^*\}$ simultaneously not all zero, satisfying the conditions (1) and (2) of Theorem 1.

## III. Sufficient Conditions in the Continuous Problem $(P_C)$

In this section, we formulate a sufficient condition for optimality for the continuous problem $(P_C)$.

*Theorem 3:* Assume that the functions $g(u,t,x)$ and $g_0(u,x)$ are continuous and convex with respect to $u$, and $F(\cdot,(t,x))$ is a convex mapping for all fixed $(t,x)$. Then for the optimality of the solution $\tilde{u}(t,x)$ it is sufficient that there exist a solution $\varphi^*(t,x), u^*(t,x)$ such that the conditions (i)-(iii) hold:

(i)

$$-\operatorname{div} \psi(t,x) \in F^*\left(\psi(t,x), \tilde{u}(t,x), \nabla\tilde{u}(t,x), t, x\right) - \partial g(\tilde{u}(t,x),t,x), \text{a.e.}$$

$$\psi(t,x) = (u^*(t,x), \varphi^*(t,x)),$$

$$\operatorname{div} \psi(t,x) = \frac{\partial u^*(t,x)}{\partial t} + \frac{\partial \varphi^*(t,x)}{\partial x}$$

(ii) $\varphi^*(t,0) = 0,\ -u^*(1,x) \in \partial g_0(\tilde{u}(1,x),x)$. Let us formulate the condition ensuring that the LAM $F^*$ is nonempty (see Lemma 1):

(iii) $\nabla\tilde{u}\,(t,x) \in F\left(\tilde{u}(t,x), \psi(t,x), t, x\right)$ a.e.

*Theorem 4:* Let $\tilde{u}(\cdot,\cdot)$ be some feasible solutions of a non-convex problem $(P_C)$ and $\{u^*(\cdot,\cdot), \varphi^*(\cdot,\cdot)\}$ be a pair of a feasible solutions satisfying the conditions (i)-(iii):

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(i)

$$
\begin{aligned}
&-\operatorname{div}\psi(t,x)+u^*(t,x)\in\\
&\quad F^*(\psi(t,x),(\tilde{u}(t,x),\nabla\tilde{u}(t,x)),t,x) \text{ a.e.}
\end{aligned}
$$

(ii)

$$
\begin{aligned}
g(u,t,x)-g(\tilde{u}(t,x),t,x)&\geq\langle u^*(t,x),u-\tilde{u}(t,x)\rangle,\\
&\forall u,\\
g_0(u,x)-g_0(\tilde{u}(1,x)&\geq\langle -u^*(1,x),u-\tilde{u}(1,x)\rangle,\\
&\varphi^*(t,0)=0,\forall u
\end{aligned}
$$

(iii) $\langle\psi(t,x),\nabla\tilde{u}(t,x)\rangle=M(\tilde{u}(t,x),\psi(t,x),t,x)$

Then the solution $\tilde{u}(t,x)$ is optimal.

In the conclusion of this section, we consider an example:

$$
\begin{aligned}
&\text{minimize} \quad J(u(t,x)),\\
&\text{subject to} \quad \frac{\partial u(t,x)}{\partial t}=A_1u(t,x)+B_1w(t,x)\\
&\qquad\qquad \frac{\partial u(t,x)}{\partial x}=A_2u(t,x)+B_2w(t,x),\\
&\qquad\qquad w(t,x)\in V\\
&\qquad\qquad u(t,1)=\alpha(t),\quad u(0,x)=\beta(x)
\end{aligned} \tag{7}
$$

where $A_1$ and $A_2$ are $n\times n$ matrices, $B_1$, $B_2$ are rectangular $n\times r$ matrices, $V\subset R^r$ is a convex closed set, and $g$ and $g_0$ are continuously differentiable functions on $u$. It is required to find a controlling parameter $\tilde{w}(t,x)\in V$ such that the solution $\tilde{u}(\cdot,\cdot)$ corresponding to it minimizes $J(u(\cdot,\cdot))$. In this case

$$F(u)=\{A_1u+B_1V,A_2u+B_2V\}$$

By elementary computations we find that

$$
\begin{aligned}
&F^*(v_1^*,v_2^*,(u,v_1,v_2))=\\
&\quad\begin{cases}(A_1^*v_1^*+A_2^*v_2^*), & -B_1^*v_1^*-B_2^*v_2^*\in K_V^*(w),\\ \emptyset, & -B_1^*v_1^*-B_2^*v_2^*\notin K_V^*(w)\end{cases}
\end{aligned}
$$

where $v_1=A_1u+B_1w, v_2=A_2u+B_2w$. Then, using Theorem 3, we get the relations

$$
\begin{aligned}
-\operatorname{div}\psi(t,x)=&A_1^*u^*(t,x)+A_2^*\varphi^*(t,x)\\
&-g'(\tilde{u}(t,x),t,x),
\end{aligned} \tag{8}
$$

$$\langle w-\tilde{w}(t,x),-B_1^*u^*(t,x)-B_2^*\varphi^*(t,x)\rangle\geq 0,\ w\in V \tag{9}$$

$$-u^*(1,x)=g_0'(\tilde{u}(1,x),x),\quad \varphi^*(t,0)=0 \tag{10}$$

Obviously (9) can be written in the form

$$\langle B\tilde{w}(t,x),\psi(t,x)\rangle=\sup_{w\in V}\langle Bw,\psi(t,x)\rangle, \tag{11}$$

$$\text{where } B=\begin{pmatrix}B_1\\B_2\end{pmatrix}$$

Thus, we have obtained the following result.

*Theorem 5:* The solution $\tilde{u}(t,x)$ corresponding to the control $\tilde{w}(t,x)$ minimizes $J(u(\cdot,\cdot))$ in the Problem (7), if there exists a solution, satisfying the conditions (8), (10), (11).

Now, let us consider the following example:

$$
\begin{aligned}
&\text{minimize} \quad J(u(t,x))\\
&\text{subject to} \quad \frac{\partial u(t,x)}{\partial t}\in\Omega,\quad \Omega\subset R^n,\\
&\qquad\qquad \frac{\partial u(t,x)}{\partial x}=Cu(t,x)+w_0(t,x),\\
&\qquad\qquad\qquad w_0(t,x)\in P\in R^n,\\
&\qquad\qquad u(t,1)=\alpha_0(t),\\
&\qquad\qquad u(0,x)=\beta_0(x),
\end{aligned} \tag{12}
$$

where $\Omega$ and $P$ are convex closed sets, $C$ is $n\times n$ matrix, $g$ and $g_0$ are continuously differentiable functions on $u$. Our aim is to find a controlling parameter $\tilde{w}_0(t,x)\in P$ such that the associated solution minimizes $J(u(t,x))$.

In this case according to the Problem (4)-(6) $F=F_1\times F_2$, where $F_1(u)=\{\Omega: u\in R^n\}$ is a constant map and, $F_2(u)=Cu+P$, $P\subset R^n$. It is easy to calculate that

$$M(u,v^*)=M_{F_1}(u,v_1^*)+M_{F_2}(u,v_2^*),\ v^*=(v_1^*,v_2^*) \tag{13}$$

Here by $M_{F_i}$ $(i=1,2)$ we denote the support functions of the sets $F_i$. Now, on the Lemma 1 and Moreau-Rockafellar theorem [6], it follows from (13) that

$$
\begin{aligned}
&F^*(v^*,(u,v))=F_1^*(v_1^*,(u,v_1))+F_2^*(v_2^*,(u,v_2)),\\
&\qquad v=(v_1,v_2)
\end{aligned}
$$

i.e. the LAM to the Cartesian product of multifunctions is a sum of the corresponding locally adjoint mappings. But

$$F_1^*(v_1^*,(u,v_1))=\begin{cases}0, & -v_1^*\in K_\Omega^*(v_1),\\ \emptyset, & -v_1^*\notin K_\Omega^*(v_1)\end{cases}$$

$$F_2^*(v_2^*,(u,v_2))=\begin{cases}C^*v_2^*, & -v_2^*\in K_P^*(w_0),\\ \emptyset, & -v_2^*\notin K_P^*(w_0)\end{cases}$$

Consequently, we have

$$F^*(v^*,(u,v))=C^*v_2^*;\quad v_1^*\in K_\Omega^*(v_1),\quad -v_2^*\in K_P^*(w_0)$$

Then the conditions (i)-(iii) of Theorem 3 that suffices for optimality give us the following conditions for this example.

$$
\begin{aligned}
-\operatorname{div}\psi(t,x)&=C^*\varphi^*(t,x)-g'(\tilde{u}(t,x),t,x),\\
\langle\tilde{w}_0(t,x),\psi(t,x)\rangle&=\sup_{w_0\in P}\langle w_0,\psi(t,x)\rangle,\\
-u^*(1,x)&=g_0'(\tilde{u}(1,x),x),\quad \varphi^*(t,0)=0.
\end{aligned}
$$

## IV. Conclusion

At first the basic definitions and concepts, i.e. Hamiltonian function, argmaximum set, LAM to both convex and nonconvex multivalued mappings, and the Lemma (1) which can be considered as the core lemma for this paper are given. Then the optimal control problem for discrete inclusions with distributed parameters, and hypothesis required to establish the sufficient conditions are given. The continous problem of the same type is also expressed in gradent form. Then the theorem which states the necessary conditions for discrete problem in terms of some adjoint functions and inclusions containing the LAM and states under some hypothesis these necessary conditions are also sufficient are formulated. Then the analogous conditions for the continous problem is formulated in terms of an divergant operator. These conditions are given in two theorems one for convex the other for nonconvex problem. Finally two examples of

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continous problems one of which is a linear type and the other of which is a constant convex type are stated and the necessary and sufficient conditions to the respective problems are formulated in order to have a better understanding of the conditions.

As a future work the duality relations about this problem can ben considered. It is expected that the dual problem exhibits similar properties with other dual problems, i.e. they should have strong relations with the optimality conditions.

## References

[1] E. N. Mahmudov, "On duality in problems of optimal control described by convex differential inclusions of Goursat-Darboux type," *Journal of Mathematical Analysis and Applications*, vol. 307, no. 2, pp. 628 – 640, 2005.

[2] ——, "The optimality principle for discrete and first order partial differential inclusions," *Journal of Mathematical Analysis and Applications*, vol. 308, no. 2, pp. 605 – 619, 2005.

[3] ——, "Necessary and sufficient conditions for discrete and differential inclusions of elliptic type," *Journal of Mathematical Analysis and Applications*, vol. 323, no. 2, pp. 768 – 789, 2006.

[4] ——, "Locally adjoint mappings and optimization of the first boundary value problem for hyperbolic type discrete and differential inclusions," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 67, no. 10, pp. 2966 – 2981, 2007.

[5] ——, "Sufficient conditions for optimality for differential inclusions of parabolic type and duality," *Journal of Global Optimization*, vol. 41, pp. 31–42, 2008.

[6] B. N. Pshenichnyi, *Convex analysis and extremal problems*. Nauka, 1980.

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