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Numerical Solution of Sine-Gordon Equation by Reduced Differential Transform Method

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***Abstract*— Reduced differential transform method (RDTM), which does not need small parameter in the equation is implemented for solving the sine-Gordon equation. The approximate analytical solution of the equation is calculated in the form of a series with easily computable components. Comparing the methodology with some other known techniques shows that the present approach is effective and powerful. Three test modeling problems from mathematical physics, both nonlinear and coupled are discussed to illustrate the effectiveness and the performance of the proposed method.**

***Index Terms*— Reduced differential transform method, sine-Gordon equations Variational iteration method.**

I. Introduction

One of the most important of all partial differential equations occurring in applied mathematics is that associated with the name of sine-Gordon. The sine-Gordon equation plays an important role in the propagation of fluxons in Josephson junctions [1-3] between two superconductors, then in many scientific fields such as the motion of a rigid pendula attached to a stretched wire [4], solid state physics, nonlinear optics, stability of fluid motions.

We consider the sine-Gordon equation

$$u_{tt} - u_{xx} + \sin(u) = 0 \tag{1}$$

subject to initial conditions

$$u(x,0) = f(x),\ u_t(x,0) = g(x) \tag{2}$$

where *u* is a function of *x* and *t*, *f(x)* and *g*(*x*) are a known analytic function. Many numerical methods were developed for this type of nonlinear partial differential equations such as the Adomian Decomposition Method (ADM) [5-9], the EXP function method [10], the Homotopy Perturbation Method (HPM) [11-13], the Homotopy Analysis Method (HAM) [14], the variable separated ODE method [4,15] and Variational Iteration Method (VIM) [16-17].

In this paper, we solve some sine-Gordon equations by the reduced differential transform method [18-20] which is presented to overcome the demerit of complex calculation of differential transform method (DTM) [21]. The main advantage of the method is the fact that it provides its user with an analytical approximation, in many cases an exact solution, in a rapidly convergent sequence with elegantly computed terms.

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The structure of this paper is organized as follows. In section 2, we begin with some basic definitions and the use of the proposed method. In section 3, we apply the reduced differential transform method to solve three test examples in order to show its ability and efficiency.

II. Traditional Differential Transform Method

A. One Dimensional Differential Transform Method

The differential transform of the function $w(x)$ is defined as follows:

$$W(k) = \frac{1}{k!}\left[\frac{\partial^k}{\partial x^k} w(x)\right]_{x=0} \tag{3}$$

where $w(x)$ is the original function and $W(k)$ is the transformed function. Here $\frac{d^k}{dx^k}$ means the k the derivative with respect to x.

The differential inverse transform of $W(k)$ is defined as

$$w(x) = \sum_{k=0}^{\infty} W(k) x^k . \tag{4}$$

Combining (3) and (4) we obtain

$$w(x) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{d^k}{dx^k} w(x)\right]_{x=0} x^k . \tag{5}$$

From above definitions it is easy to see that the concept of differential transform is derived from Taylor series expansion. With the aid of (3) and (4) the basic mathematical operations are readily be obtained and given in Table 1.

Table 1 One-dimensional differential transformation

Functional Form	Transformed Form
$u(x) \pm v(x)$	$U(k) \pm V(k)$
$cu(x)$	$cU(k)$
$\frac{d^m u(x)}{dx^m}$	$\frac{(k+m)!}{k!} U(k+m)$
$w(x) = u(x)v(x)$	$W(k) = \sum_{r=0}^{k} U(r)V(k-r)$

B. Two Dimensional Differential Transform Method

Similarly, the two dimensional differential transform of the function $w(x,t)$ can be defined as follows:

$$W(k,h) = \frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial x^k \partial t^h} w(x,t)\right]_{(0,0)} \tag{6}$$

where $w(x,t)$ is the original function and $W(k,h)$ is the transformed function. The differential inverse transform of $W(k,h)$ is

$$w(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} W(k,h) x^k t^h . \tag{7}$$

Then combining equation (6) and (7) we write

$$w(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial x^k \partial t^h}w(x,t)\right]_{(0,0)}x^k t^h \ . \qquad (8)$$

Therefore we can obtain basic mathematical operations of two-dimensional differential transform as follows in Table 2.

Table 2. Two dimensional differential transformation

Functional Form	Transformed Form
$u(x,t)\pm v(x,t)$	$U(k,h)\pm V(k,h)$
$cu(x,t)$	$cU(k,h)$
$\frac{\partial u(x,t)}{\partial x}$	$(k+1)U(k+1,h)$
$\frac{\partial u(x,t)}{\partial t}$	$(h+1)U(k,h+1)$
$\frac{\partial u^{r+s}(x,t)}{\partial^r x \partial t^s}$	$\frac{(k+r)!}{k!}\frac{(h+s)!}{s!}U(k+r,h+s)$
$u(x,t)v(x,t)$	$\sum_{r=0}^{k}\sum_{s=0}^{h}U(r,h-s)V(k-r,s)$

Now we can state our main results in the next section.

III. Analysis of the Reduced Differential Transform Method

The basic definitions of reduced differential transform method are introduced as follows:

Definition 1.

If function $u(x,t)$ is analytic and differentiated continuously with respect to time t and space x in the domain of interest, then let

$$U_k(x)=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0} \qquad (9)$$

where the t-dimensional spectrum function $U_k(x)$ is the transformed function. In this paper, the lowercase $u(x,t)$ represent the original function while the uppercase $U_k(x)$ stand for the transformed function.

The differential inverse transform of $U_k(x)$ is defined as follows:

$$u(x,t)=\sum_{k=0}^{\infty}U_k(x)t^k \ . \qquad (10)$$

Then combining equation (9) and (10) we write

$$u(x,t)=\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0}t^k \ . \qquad (11)$$

From the above definitions, it can be found that the concept of the reduced differential transform is derived from the power series expansion.

For the purpose of illustration of the methodology to the proposed method, we write the gas dynamic equation in the standard operator form

$$L(u(x,t))-R(u(x,t))+N(u(x,t))=0 \qquad (12)$$

with initial condition

$$u(x,0)=f(x) \qquad (13)$$

where $L(u(x,t))=u_{tt}(x,t)$ is a linear operator which has partial derivatives, $R(u(x,t))=u_{xx}(x,t)$, $N(u(x,t))=\sin(u(x,t))$ is a nonlinear term.

According to the RDTM and Table 3, we can construct the following iteration formula:

$$\frac{(k+2)!}{k!}U_{k+2}(x)=\frac{\partial^2}{\partial x^2}U_k(x)-N(U_k(x)) \qquad (14)$$

where $N(U_k(x))$ is the transformations of the functions $N(u(x,t))$ respectively.

Table 3. Reduced differential transformation [18-20]

Functional Form	Transformed Form
$u(x,t)$	$U_k(x)=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0}$
$u(x,t)\pm v(x,t)$	$U_k(x)\pm V_k(x)$
$\alpha u(x,t)$	$\alpha U_k(x)$ (α is a constant)
$x^m t^n$	$x^m\delta(k-n)$
$x^m t^n u(x,t)$	$x^m U(k-n)$
$u(x,t)v(x,t)$	$\sum_{r=0}^{k}U_r(x)V_{k-r}(x)$
$\frac{\partial^r}{\partial t^r}u(x,t)$	$\frac{(k+r)!}{k!}U_{k+r}(x)$
$\frac{\partial}{\partial x}u(x,t)$	$\frac{\partial}{\partial x}U_k(x)$
$Nu(x,t)$	Maple Code for Nonlinear Function restart; NF:=Nu(x,t):#Nonlinear Function m:=5: # Order u[t]:=sum(u[b]*t^b,b=0..m): NF[t]:=subs(Nu(x,t)=u[t],NF): s:=expand(NF[t],t): dt:=unapply(s,t): for i from 0 to m do n[i]:=((D@@i)(dt)(0)/i!): print(N[i],n[i]); #Transform Function od:

For the easy to follow of the reader, we can give the first few nonlinear term are

$$N_0=\sin(U_0(x))$$

$$N_1=\cos(U_0(x))U_1(x)$$

$$N_2=\cos(U_0(x))U_2(x)-\frac{1}{2}\sin(U_0(x))U_1^2(x)$$

From initial condition (2), we write

$$U_0(x)=f(x) \qquad (15)$$

Substituting (15) into (14) and by a straight forward iterative calculations, we get the following $U_k(x)$ values. Then the inverse transformation of the set of values $\{U_k(x)\}_{k=0}^{n}$ gives approximation solution as,

$$\tilde{u}_n(x,t)=\sum_{k=0}^{n}U_k(x)t^k \qquad (16)$$

where *n* is order of approximation solution.

Therefore, the exact solution of problem is given by

$$u(x,t)=\lim_{n\to\infty}\tilde{u}_n(x,t) \ . \qquad (17)$$

IV. Applications

To show the efficiency of the new method described in the previous part, we present some examples.

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A. *Example 1*

We first consider the homogeneous sine-Gordon equation [4,6,10,17]

$$u_{tt} - u_{xx} + \sin(u) = 0 \tag{18}$$

with initial conditions:

$$u(x,0) = 0,\; u_t(x,0) = 4\operatorname{sech}(x) \tag{19}$$

where $u = u(x,t)$ is a function of the variables x and t.

Now if we use the VIM, based on the correction functional given [17,22]

$$u_{n+1} = u_n + \int_0^t (\tau - t)\left(\frac{\partial^2 u_n}{\partial \tau^2} - \frac{\partial^2 u_n}{\partial x^2} + \sin(u_n)\right) d\tau$$

then we will find it too difficult to evaluate the solution components because we should evaluate the integral $\int_0^t (\tau - t)\sin(u_n) d\tau$, which is not easily computed. So, the RDTM will be more efficient for this example.

Then, by using the basic properties of the reduced differential transformation, we can find the transformed form of equation (18) as

$$\frac{(k+2)!}{k!} U_{k+2}(x) = \frac{\partial^2}{\partial x^2} U_k(x) - N_k(x)\,. \tag{20}$$

Using the initial conditions (19), we have

$$U_0(x) = 0,\, U_1(x) = 4\operatorname{sech}(x) \tag{21}$$

Now, substituting (21) into (20), we obtain the following $U_k(x)$ values successively

$$U_k(x) = 0,\, U_3(x) = -\frac{4}{3}\frac{1}{\cosh^3(x)},\, U_4(x) = 0,$$

$$U_5(x) = \frac{4}{5}\frac{1}{\cosh^5(x)},\, U_6(x) = 0, U_7(x) = -\frac{4}{5}\frac{1}{\cosh^7(x)}$$

$$U_k(x) = \begin{cases} (-1)^{k-1/2}\dfrac{4}{k}\dfrac{1}{\cosh^k(x)}, & \text{for k is odd} \\ 0, & \text{for k is even} \end{cases}$$

Finally the differential inverse transform of $U_k(x)$ gives

$$u(x,t) = \sum_{k=1,3,\ldots}^{\infty} U_k(x) t^k = 4\sum_{k=1,3,\ldots}^{\infty} \frac{(-1)^{k-1/2}}{k\cosh^k(x)} t^k \tag{22}$$

Hence the closed form of (22) is

$$u(x,t) = 4\arctan\left(t\sec(x)\right)$$

which is the exact solutions of (18)–(19) (see Figure 1 and 2).

Figure 1. The numerical results for $u_7(x,y)$: (a) in comparison with the analytical solutions $u(x,t) = 4\arctan\left(t\sec(x)\right)$, (b) for the solitary wave solution with the initial condition of Example 1.

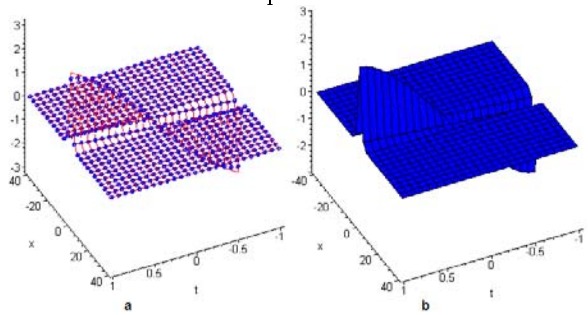
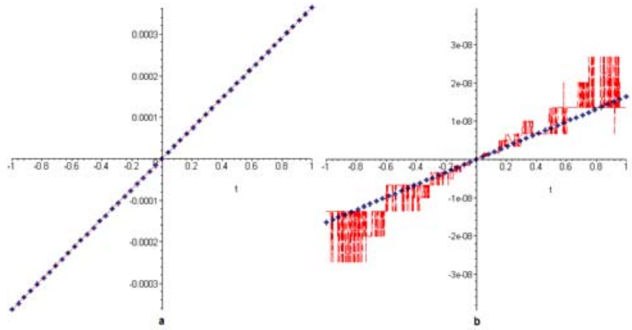


Figure 2. The numerical results for $u_7(x,y)$ (show in +) (a) in comparison with the analytical solutions $u(x,t) = 4\arctan\left(\sec(x)t\right)$ (show in □) and VIM solution (show in --) for $x = 10$, (b) $x = 20$.



B. *Example 2*

Now, we will find the approximate analytical solution of the sine-Gordon equation

$$u_{tt} - u_{xx} + \sin(u) = 0 \tag{23}$$

with the initial conditions

$$u(x,0) = \pi + \alpha\cos(\beta x),\; u_t(x,0) = 0 \tag{24}$$

where $\beta = \frac{\sqrt{2}}{2}$ and α is a constant.

Taking differential transform of (23) and the initial conditions (24) respectively, we obtain

$$\frac{(k+2)!}{k!} U_{k+2}(x) = \frac{\partial^2}{\partial x^2} U_k(x) - N_k(x) \tag{25}$$

where $N_k(x)$ is transformed form of $\sin\left(u(x,t)\right)$.

The transformed initial conditions

$$U_0(x) = \pi + \alpha\cos(\beta x),\; U_1(x) = 0 \tag{26}$$

Substituting (26) into (25), we obtain the following $U_k(x)$ values successively

$$U_2(x) = -\frac{1}{2}\alpha\beta^2\cos(\beta x) + \frac{1}{2}\sin\left(\alpha\cos(\beta x)\right),\, U_3(x) = 0$$

$$U_4(x) = \frac{1}{24}\alpha\beta^4\cos(\beta x) - \frac{1}{24}\alpha^2\beta^2\sin\left(\alpha\cos(\beta x)\right)$$
$$+\frac{1}{24}\alpha^2\beta^2\cos^2(\beta x)\sin\left(\alpha\cos(\beta x)\right) - \frac{1}{12}\alpha\beta^2\cos(\beta x)\cos\left(\alpha\cos(\beta x)\right)$$
$$+\frac{1}{24}\sin\left(\alpha\cos(\beta x)\right)\cos\left(\alpha\cos(\beta x)\right)$$

$$U_5(x) = 0$$

Then, the inverse transformation of the set of values $\{U_k(x)\}_{k=0}^{5}$ gives five-term approximation solution as

$$\tilde{u}_5(x,y) = \pi + \alpha\cos(\beta x) + \left(\frac{1}{2}\sin\left(\alpha\cos(\beta x)\right) - \frac{1}{2}\alpha\beta^2\cos(\beta x)\right)t^2$$
$$+\left(\frac{1}{24}\alpha\beta^4\cos(\beta x) - \frac{1}{24}\alpha^2\beta^2\sin\left(\alpha\cos(\beta x)\right) + \frac{1}{24}\alpha^2\beta^2\cos^2(\beta x)\sin\left(\alpha\cos(\beta x)\right)\right.$$
$$\left.-\frac{1}{12}\alpha\beta^2\cos(\beta x)\cos\left(\alpha\cos(\beta x)\right) + \frac{1}{24}\sin\left(\alpha\cos(\beta x)\right)\cos\left(\alpha\cos(\beta x)\right)\right)t^4$$

(27)

To demonstrate the numerical stability of the RDTM, we take four γ values (γ=0.001, γ=0.05, γ=0.1 and γ=1.0), these values have previously been used by Kaya [6], and some of γ values (γ=0.05, 0.1) have been used by Ablowitz et al. [1]. In the present numerical experiment, (3.10) has been used to draw the graphs as shown in Figure 3-4. The numerical solutions of example 2 have been shown in Figure 3-4 using VIM. In the present numerical computation we have assumed γ=0.001, γ=0.05, γ=0.1 and γ=1 respectively.

Figure 3. The comparison of the RDTM (line) approximation and the VIM solution (circle) for (a) γ=0.001, (b) γ=0.05

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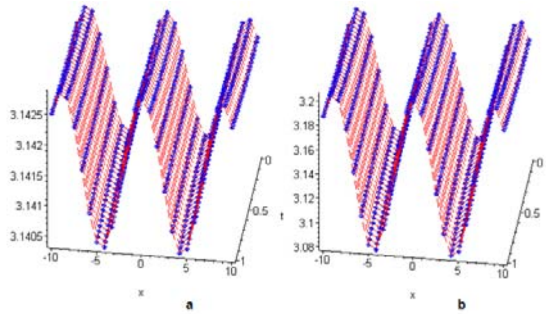
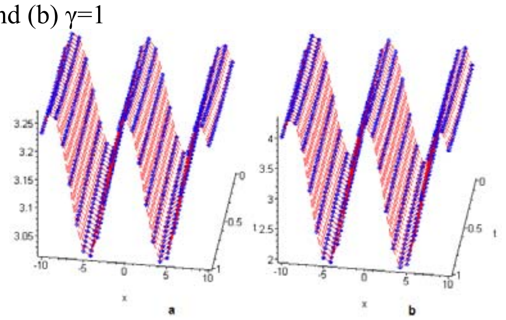


Figure 4. The comparison of the RDTM (line) approximation and the VIM solution (circle) for (a) γ=0.1 and (b) γ=1



C. *Example 3*

We now consider a system of coupled sine-Gordon equations [8,13,16]

$$\begin{aligned} u_{tt}(x,t) - u_{xx}(x,t) &= -\alpha^2 \sin(u(x,t) - v(x,t)) \\ v_{tt}(x,t) - c^2 v_{xx}(x,t) &= \sin(u(x,t) - v(x,t)) \end{aligned} \tag{28}$$

with initial conditions

$$\begin{aligned} &u(x,0) = A\cos(kx),\ u_t(x,0) = 0 \\ &v(x,0) = 0,\ v_t(x,0) = 0 \end{aligned} \tag{29}$$

Taking the differential transform of (28), it can be obtained that

$$\begin{aligned} \frac{(k+2)!}{k!}U_{k+2}(x) &= \frac{\partial^2}{\partial x^2}U_k(x) - \alpha^2 N_k(x) \\ \frac{(k+2)!}{k!}V_{k+2}(x) &= c^2\frac{\partial^2}{\partial x^2}V_k(x) + N_k(x) \end{aligned} \tag{30}$$

where the t-dimensional spectrum function $U_k(x)$ and $V_k(x)$ are the transformed function and $N_k(x)$ is transformed form of $\sin\left(u(x,t) - v(x,t)\right)$.

From the initial condition (4.12) we write

$$\begin{aligned} &U_0(x) = A\cos(kx),\ U_1(x) = 0 \\ &V_0(x) = 0,\ V_1(x) = 0 \end{aligned} \tag{31}$$

Substituting (31) into (30), we obtain the following $U_k(x)$ and $V_k(x)$ values successively. Then, the inverse transformation of the set of values $\{U_k(x)\}_{k=0}^{5}$ and $\{V_k(x)\}_{k=0}^{5}$ gives five term approximation solution as

$$\tilde{u}_5(x,y) = A\cos(kx) + \left(-\frac{A\cos(kx)k^2}{2} - \frac{\sin(A\cos(kx))\alpha^2}{2}\right)t^2 + \left(\frac{A\cos(kx)k^4}{24} + \frac{\alpha^2\sin(A\cos(kx))A^2k^2}{24} - \frac{\alpha^2\sin(A\cos(kx))A^2k^2\cos^2(kx)}{24} + \frac{\alpha^2\cos(A\cos(kx))Ak^2\cos(kx)}{12} + \frac{\alpha^4\cos(A\cos(kx))\sin(A\cos(kx))}{24} + \frac{\alpha^2\cos(A\cos(kx))\sin(A\cos(kx))}{24}\right)t^4$$

$$\tilde{v}_5(x,y) = \frac{\sin(A\cos(kx))}{2}t^2 + \left(\frac{c^2\sin(A\cos(kx))A^2k^2}{24} + \frac{c^2\sin(A\cos(kx))\cos^2(kx)A^2k^2}{24} - \frac{c^2\cos(A\cos(kx))Ak^2\cos(kx)}{24} - \frac{\cos(A\cos(kx))Ak^2\cos(kx)}{24} - \frac{\alpha^2\cos(A\cos(kx))\sin(A\cos(kx))}{24} - \frac{\cos(A\cos(kx))\sin(A\cos(kx))}{24}\right)t^4$$

This solution is convergent to the adomian decomposition method solution [8,13] and the same as approximate solution of the variational iteration method [16]. (see Figure 5)

Figure 5. The comparison of the RDTM (line) approximation and the VIM solution (point).

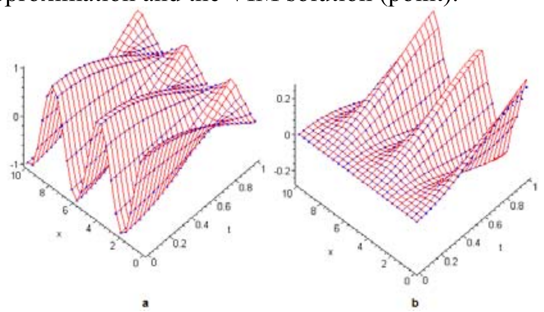


Figure 5 (a-b) shows the comparison of the RDTM approximation solution of order five and the VIM solution $u(x,t)$ (Figure a) and $v(x,t)$ (Figure b), the solid line represents the solution by the RDTM (shown in red), while the circle represents the VIM (shown in blue).

V. CONCLUSIONS

The sine-Gordon equations have been analyzed using the reduced differential transform method. All the examples show that the reduced differential transform method is a powerful mathematical tool to solving sine-Gordon equation. It is also a promising method to solve other nonlinear equations. This method solves the problem without any need to discretization of the variables, therefore, it is not affected by computation round off errors and one does not face the need of large computer memory and time. In our work, we made use of the Maple Package to calculate the series obtained from the reduced differential transform method.

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