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First Order Fractional Fourier Transform Moment Based on Ambiguity Function

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***Abstract*—The fractional Fourier transform can be considered as a rotated standard Fourier transform in general and its benefit in signal processing is growing to be known more. In this paper, we have computed the first order moment of fractional Fourier transform according to the ambiguity function exactly. In addition we have derived relations between time and spectral moments with those obtained in fractional domain. We will prove that the first moment in fractional Fourier transform can be considered as a rotated the time and frequency gravity in general. For more satisfaction, we choose three different types signals and obtain analytically their fractional Fourier transform as well.**

***Index Terms*—Fractional Fourier transform, first order moment**

I. INTRODUCTION

In the mathematics literature, the concept of fractional order Fourier transform (FT) was proposed some years ago [1]-[3]. The ordinary FT being a transform of order 1, and the signal in time is of order zero. The fractional FT depends on a parameter α and can be interpreted as a rotation by an angle α in the time- frequency plane. The relationship between fractional FT order and angle is given by $\alpha = a\frac{\pi}{2}$. The fractional FT of function $x(t)$ can be written in the form:

$$X_\alpha(u) = \int_{-\infty}^{+\infty} x(t) K_\alpha(t,u) dt \tag{1}$$

where the kernel is given by $K_\alpha(t,u) = \sqrt{\frac{1 - j\cot\alpha}{2\pi}} e^{j\frac{t^2+u^2}{2}\cot\alpha - jut\csc\alpha}$, and "j" represents the imaginary unit ($\sqrt{-1}$). The parameter α is continuous and interpreted as a rotation angle in the phase plane. When α increases from 0 to $\frac{\pi}{2}$, the fractional FT produce a continuous transformation of a signal to its Fourier image. If α or $\alpha + \pi$ is a multiple of 2π, the kernel reduces to $\delta(t-u)$ or $\delta(t+u)$ respectively. We also note that for $\alpha = \frac{\pi}{2}$, the kernel coincide with the kernel of the ordinary FT. In summary, the fractional FT is a linear transform, and continuous in the angle α, which satisfies the basic conditions for being interpretable as a rotation in the time-frequency plane [4]. Fractional FT is the energy-preserving transform [4], it means:

$$\int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \int_{-\infty}^{+\infty} |X_\alpha(u)|^2 du \tag{2}$$

Due to the energy-preserving property of the FT, the squared magnitude of the FT of a signal $|X(\omega)|^2$ is often called the energy spectrum of the signal and is interpreted as the distribution of the signal's energy among the different frequencies. As the fractional FT is also energy conservative, $|X_\alpha(u)|^2$ is named as the fractional energy spectrum of the signal x(t), with angle α.

In time-frequency representations, one normally uses a plane with two orthogonal axes corresponding to time and frequency respectively, (Fig. 1).

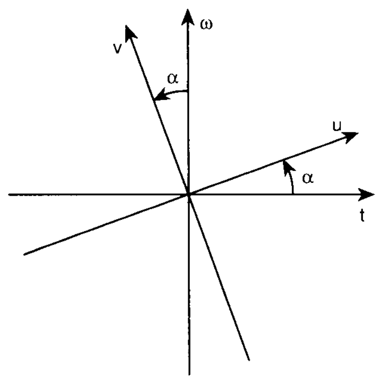

Fig. 1: Time- frequency plane and a set of coordinates (u, v) rotated by an angle α relative to the original coordinates (t, ω).

A signal represented along the frequency axis is the FT of the signal representation $x(t)$ along the time axis. It can also be represented along an axis making some angle α with the time axis. Along this axis, we define the fractional FT of $x(t)$ at angle α defined as the linear integral transform, equation (1). It is easy to prove that pairs (t, ω) and (u, v) corresponding to an axis rotation by:

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$$\begin{pmatrix} t \\ \omega \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{3}$$

Although many properties were known for fractional FT, it is convenient to include in this preliminary section three results which will be useful later on. Now according to $x(t) \overset{FrFT}{\Longleftrightarrow} X_\alpha(u)$, we denote these properties, they are named shift, modulation, and multiplication as follows:

$$x(t-\tau) \overset{FrFT}{\Longleftrightarrow} e^{j\frac{\tau^2}{2}\sin\alpha\cos\alpha - ju\tau\sin\alpha} X_\alpha(u-\tau\cos\alpha) \tag{4}$$

$$x(t)e^{-j\theta t} \overset{FrFT}{\Longleftrightarrow} e^{-j\frac{\theta^2}{2}\sin\alpha\cos\alpha - ju\theta\cos\alpha} X_\alpha(u+\theta\sin\alpha) \tag{5}$$

$$tx(t) \overset{FrFT}{\Longleftrightarrow} u\cos\alpha\, X_\alpha(u) + j\sin\alpha\, X'_\alpha(u) \tag{6}$$

This paper is organized as follows. In section II, we derive the first fractional FT moment according to ambiguity function (AF) and find some relations among time-frequency and fractional moment respectively. In section III, we obtain the fractional FT of some signals those can be used as an additive noise model. Finally section IV concludes the paper.

II. FRACTIONAL MOMENT BASED ON AMBIGUITY FUNCTION

In this paper, based on connection between the AF and the fractional FT, we derive the fractional moment. The AF of a signal $x(t)$ is defined as [5]:

$$\begin{aligned} AF_x(\theta,\tau) &= \int_{-\infty}^{+\infty} x(t+\frac{\tau}{2})x^*(t-\frac{\tau}{2})e^{-j\theta t}dt \\ &= \int_{-\infty}^{+\infty} X(\omega+\frac{\theta}{2})X^*(\omega-\frac{\theta}{2})e^{j\tau\omega}d\omega \end{aligned} \tag{7}$$

Where asterisk '*' refers to the complex conjugate operation. It is easy to show that:

$$<t> = \frac{1}{(-j)} \cdot \left.\frac{\partial AF_x(\theta,0)}{\partial\theta}\right|_{\theta=0} \tag{8}$$

$$AF_x(\theta,0) = \int_{-\infty}^{+\infty} x(t)x^*(t)e^{-j\theta t}dt$$

$$<\omega> = \frac{1}{(j)} \cdot \left.\frac{\partial AF_x(0,\tau)}{\partial\tau}\right|_{\tau=0} \tag{9}$$

$$AF_x(0,\tau) = \int_{-\infty}^{+\infty} X(\omega)X^*(\omega)e^{j\tau\omega}d\omega$$

Before starting to derive the first order moment in fractional FT based on the first order moment in time and frequency, we recall that as fractional FT is a linear transform and energy conservative, so in general the fractional first order moment can be considered as:

$$<u> = \int_{-\infty}^{+\infty} u\,|X_\alpha(u)|^2du \tag{10}$$

Now according to the fractional FT definition, equation (1), and shift and modulation properties (equation (4) and equation (5)), we rewrite the AF in equation (11):

$$\begin{aligned} AF_x(\theta,\tau) = e^{j\frac{\theta\tau}{2}} e^{j\frac{u^2}{2\sin\alpha\cos\alpha}} & \int_{-\infty}^{+\infty} e^{-j\frac{\sin\alpha\cos\alpha}{2}(\tau-\frac{u}{\cos\alpha})^2} X^*_\alpha(u-\tau\cos\alpha) \\ & \times e^{-j\frac{\sin\alpha\cos\alpha}{2}(\theta+\frac{u}{\sin\alpha})^2} X_\alpha(u+\theta\sin\alpha)du \end{aligned} \tag{11}$$

A. Time Moment

Although it takes really long analytic computation, we try to obtain the first order moment belong to time according to equation (8) and by using equation (11) as follows:

$$\begin{aligned} AF_x(\theta,0) = e^{j\frac{1}{2\tan\alpha}u^2} \\ \times \int_{-\infty}^{+\infty} X^*_\alpha(u)\cdot e^{-j\frac{\sin\alpha\cos\alpha}{2}(\theta+\frac{u}{\sin\alpha})^2} X_\alpha(u+\theta\sin\alpha)du \end{aligned} \tag{12}$$

Easy using equations (12), and (7) show the fractional FT is energy conservative or unique signal has unique fractional FT and so the transform is reversible ($E_x = AF_x(0,0) = \int_{-\infty}^{+\infty} |x(t)|^2\,dt = \int_{-\infty}^{+\infty} |X_\alpha(u)|^2du$). We consider the signal energy is 1 ($E_x = 1$). Now we determine the first derivative in order to determine the first order moment in time domain, equation (13):

$$\begin{aligned} \left.\frac{\partial AF_x(\theta,0)}{\partial\theta}\right|_{\theta=0} &= \int_{-\infty}^{+\infty} X^*_\alpha(u)\cdot \left.\frac{\partial X_\alpha(u+\theta\sin\alpha)}{\partial\theta}\right|_{\theta=0} du \\ &+ \int_{-\infty}^{+\infty} (-j\cos\alpha)u\,|X_\alpha(u)|^2du \end{aligned} \tag{13}$$

and then we have:

$$<t> = j\int_{-\infty}^{+\infty} X^*_\alpha(u)\cdot \left.\frac{\partial X_\alpha(u+\theta\sin\alpha)}{\partial\theta}\right|_{\theta=0} du + \cos\alpha <u> \tag{14}$$

Now we should simplify the derived equations for the first order moment in time domain. Using equations (1) and (11) for fractional FT definition, it is not too hard to prove the following relationship:

$$\left.\frac{\partial X_\alpha(u+\theta\sin\alpha)}{\partial\theta}\right|_{\theta=0} = \sin\alpha\frac{\partial X_\alpha(u)}{\partial u} \tag{15}$$

Thereby, we rewrite the first order moment:

$$<t> = j\sin\alpha\int_{-\infty}^{+\infty} X^*_\alpha(u)\cdot\frac{\partial X_\alpha(u)}{\partial u}du + \cos\alpha <u> \tag{16}$$

As u and v are orthogonal axes (Fig. 1), we can obtain the first order moment in v domain by using signal in u domain as:

$$<v> = \int_{-\infty}^{+\infty} X^*_\alpha(u)\frac{1}{j}\frac{\partial X_\alpha(u)}{\partial u}du \tag{17}$$

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Now the first order moment for time domain is obtained according to the fractional moment as follows:

$$<t>=-\sin\alpha<v>+\cos\alpha<u> \tag{18}$$

B. Frequency Moment

Exactly the same algebra is used in order to obtain frequency moment. By notifying equation (9) and using equation (11), we write equations (19) and (20):

$$AF_x(0,\tau)=e^{j\frac{\tan\alpha}{2}u^2}\int_{-\infty}^{+\infty}X_\alpha(u)e^{-j\frac{\sin\alpha\cos\alpha}{2}(\tau-\frac{u}{\cos\alpha})^2} \times X_\alpha^*(u-\tau\cos\alpha)du \tag{19}$$

$$\left.\frac{\partial AF_x(0,\tau)}{\partial\tau}\right|_{\tau=0}=\int_{-\infty}^{+\infty}X_\alpha(u)\left.\frac{\partial X_\alpha^*(u-\tau\cos\alpha)}{\partial\tau}\right|_{\tau=0}du + \int_{-\infty}^{+\infty}(j\sin\alpha)u\,|X_\alpha(u)|^2du \tag{20}$$

and as following we have:

$$<\omega>=-j\int_{-\infty}^{+\infty}X_\alpha(u)\cdot\left.\frac{\partial X_\alpha^*(u-\tau\cos\alpha)}{\partial\tau}\right|_{\tau=0}du+\sin\alpha<u> \tag{21}$$

In order to simplify the derived equation, the following relation by employing equations (1) and (11) are determined:

$$\left.\frac{\partial X_\alpha^*(u-\tau\cos\alpha)}{\partial\tau}\right|_{\tau=0}=-\cos\alpha\frac{\partial X_\alpha^*(u)}{\partial u} \tag{22}$$

And now, the first spectral moment is written as follows:

$$<\omega>=\cos\alpha<v>+\sin\alpha<u> \tag{23}$$

In order to make the derived equations more readable, we define the first order moment according to their corresponding plane. They are $m_0=<t>$; $m_{\frac{\pi}{2}}=<\omega>$; $m_\alpha=<u>$; $m_{\alpha+\frac{\pi}{2}}=<v>$. Therefore, we rewite the derived equations (18) and (23) as the following matrix form:

$$\begin{pmatrix}m_0\\ m_{\frac{\pi}{2}}\end{pmatrix}=\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}m_\alpha\\ m_{\alpha+\frac{\pi}{2}}\end{pmatrix} \tag{24}$$

As rotation is true for pairs (t,ω) and (u,v), equation (3), obviously it is also true for the first moment in original plane. This result emphasize on why fractional FT is considered as a rotation operator. The first order moment, $m_\alpha=<u>$, in a fractional domain defined by an arbitrary angle α can be calculated from the relationship $m_\alpha=\cos\alpha\; m_0+\sin\alpha\; m_{\frac{\pi}{2}}$.

In addition, taking into account equations (18), and (23), we conclude the following relationship:

$$m_0^2+m_{\frac{\pi}{2}}^2=m_\alpha^2+m_{\alpha+\frac{\pi}{2}}^2 \tag{25}$$

According to what is derived, we can say that the first order moment is rotate invariant.

III. DIFFERENT SIGNALS

Fractional FT of a number of common signals such as $\exp(-t^2/2)$, $\delta(t)$, and e^{jkt} were computed before [1]. It was proved that fractional FT also exist for certain functions which are not square integrable (for example: $1,t,t^2$,etc) [1] (as in Z transform using r causes having this feature, here being α causes this effect). Fractional FT has attracted a great attention. Some researchers try to discover its features more [6], and some try to use it in application. Conventionally, the filtering systems are based on the FT, though the frequency of the noise and that of the signal usually overlap with each other, so it is very difficult to filter the noise completely. So it may conclude that filtering in the optimal fractional domain is significantly better than filtering in the conventional frequency domain. For further application of the fractional FT analysis, it is important to study its effects on different types of signals. It means that obtaining the central moment and explore their behavior are important topic for design an optimum filter in rotated domain or fractional FT. In this section, we will obtain the fractional FT for three different type functions which can be considered as a model for additive noise. We consider Gaussian function, One sided Gaussian function, and Rayleigh function and their fractional FT are written in Tabel. I. Obviously, it is easy to show that $X_\alpha(u)\big|_{\alpha=\frac{\pi}{2}}=X(j\omega)$, this result prove the computed procedure has done correctly.

IV. CONCLUSIONS

The fractional Fourier transform moment may be helpful in the search for the most appropriate fractional domain to perform a filtering operation. In this paper we have derived the relation between central moment in time, frequency, and fractional domain by employing the ambiguity function. In addition, we have obtained the fractional Fourier transform for different signals directly. Implementation will be our future work.

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Table I: Analytically computed the fractional FT.

Time Signal	Fractional Fourier Transform	Fourier Transform
$x(t)=e^{-\frac{t^2}{2\sigma^2}}$	$X_\alpha(u)=\sqrt{\dfrac{1-j\cot\alpha}{\frac{1}{\sigma^2}-j\cot\alpha}}\cdot\exp(\dfrac{-u^2}{2}\cdot\dfrac{1-\frac{j}{\sigma^2}\cot\alpha}{\frac{1}{\sigma^2}-j\cot\alpha})$	$X(j\omega)=\sigma\cdot e^{-\frac{\sigma^2}{2}\omega^2}$
$x(t)=e^{-\frac{t^2}{2\sigma^2}}u(t)$	$X_\alpha(u)=\dfrac{1}{2}\sqrt{\dfrac{1-j\cot\alpha}{\frac{1}{\sigma^2}-j\cot\alpha}}\cdot\exp(\dfrac{-u^2}{2}\cdot\dfrac{1-\frac{j}{\sigma^2}\cot\alpha}{\frac{1}{\sigma^2}-j\cot\alpha})$	$X(j\omega)=\dfrac{\sigma}{2}e^{-\frac{\sigma^2}{2}\omega^2}$
$x(t)=te^{-\frac{t^2}{2\sigma^2}}\mathrm{u}(t)$	$X_\alpha(u)=\sqrt{\dfrac{1-j\cot\alpha}{2\pi}}\cdot\dfrac{1}{\frac{1}{\sigma^2}-j\cot\alpha}\exp(\dfrac{u^2}{2}j\cot\alpha)$ $+\dfrac{1}{2}\dfrac{u}{j\sin\alpha}\cdot\dfrac{1}{\frac{1}{\sigma^2}-j\cot\alpha}\sqrt{\dfrac{1-j\cot\alpha}{\frac{1}{\sigma^2}-j\cot\alpha}}\cdot\exp(\dfrac{-u^2}{2}\cdot\dfrac{1-\frac{j}{\sigma^2}\cot\alpha}{\frac{1}{\sigma^2}-j\cot\alpha})$	$X(j\omega)=\dfrac{\sigma^2}{\sqrt{2\pi}}-\dfrac{\sigma^3}{2}\cdot j\omega\, e^{-\frac{\sigma^2}{2}\omega^2}$

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