Application of the General Variational Iteration Method to a Nonlinear System

A.Lotfavar, H.Rafieipour, H.Latifizadeh

Abstract : In this paper, a new method called general variational iteration method (GVIM) has been presented for deriving accurate/approximate analytical solution to strong nonlinear oscillators. Oscillator problems are frequently encountered in many major fields of science and engineering. Only one iteration leads to high accuracy of the solutions and the relative error for the approximate period is less than 1.6% for amplitudes as high as 130◦. Furthermore, it is shown that a large class of linear or nonlinear differential equations can be solved without the tangible restriction of sensitivity to the degree of the nonlinear term, adding that the method is quite convenient due to reduction in size of calculations. Results obtained by General variational iteration method (GVIM) are compared with Homotopy Perturbation Method (HPM) and Energy Balance Method (EBM) and it is shown that, simply one term is enough to obtain a highly accurate result.
The results are valid not only for weakly nonlinear systems, but also for strongly nonlinear ones. We believe that the present study may be a suitable and fruitful exercise for teaching and better understanding of analytical techniques in advanced undergraduate courses on classical mechanics.

Keywords **: General variational iteration method, strongly nonlinear systems, nonlinear pendulum**

I. INTRODUCTION

There are several methods used to find approximate solutions to nonlinear problems, such as perturbation techniques [4,9–12] or harmonic balance based methods [13–15]. A review of some asymptotic methods for strongly nonlinear equations can be found in detail in [16]. And it is well known that the perturbation method is one of the commonly used quantitative methods for analyzing nonlinear problems. Nayfeh [1] has presented an account of various perturbation techniques, pointing out their similarities, differences and advantages, as well as their limitations. The perturbation method is valid in principle only for problems containing small (or large) parameters.
Its basic idea is to transform, by means of small parameters, a nonlinear problem into an infinite number of linear sub-problems, or a complicated linear problem into an infinite number of simpler ones. Therefore, the small parameter plays a very important role in the perturbation method. It determines not only the accuracy of the perturbation approximations but also the validity of the perturbation method itself. Therefore, it is the small parameter that greatly restricts the applications of the perturbation method. However, in science and engineering, there exist many nonlinear problems which do not contain any small parameters, especially those with strong nonlinearity. Thus, it is necessary to develop and improve some nonlinear analytical techniques which are independent of small parameters.[5,6]
In this paper, for removing the demerits of the perturbation method we propose Generalization of the variational iteration technique to obtain an approximate expression for the period of the nonlinear pendulum. While in HPM (as one of best method for solving nonlinear problems), which requires neither a small parameter nor a linear term in the differential equation, an artificial perturbation equation is constructed by embedding an artificial parameter $\varepsilon \in [0,1]$, which is used as an expanding parameter [13]. This technique yields a very rapid convergence of the solution series; in most cases only one iteration leads to high accuracy of the solution. An important advantage of this method is that it can be applied to nonlinear oscillatory problems for which the nonlinear terms are not 'small', i.e., no perturbation parameter needs to exist [13]. For this type of oscillators—the nonlinear pendulum is one of them—the traditional perturbation methods such as the Lindstedt–Poincar´e method cannot be applied because a linear term and a perturbation parameter are not present [7, 10,13]. The use of General variational iteration method (GVIM) is of great interest for students, since this technique can be applied to solve differential equations that could not be solved using standard perturbation methods. In our opinion, students of physics must at least once analyze a nonlinear differential equation by using perturbation techniques, since these ones represent a relatively easy way of solving this type of equations without the use of complex mathematics.

II. DESCRIPTION OF THE NOVEL METHOD

To illustrate the basic ideas of the proposed method in [1,2,8], the following differential equation is considered

$$\mathrm{L}u(t) + \mathrm{N}[u(t)] = f(t) \qquad (1)$$

Where L is a linear operator, N a nonlinear operator and $f(t)$ an inhomogeneous term. Eq. (2) which can be written as operator from as follows

$$\chi[u] = f(t) \rightarrow \psi[u(t)] = h(u(t)) \qquad (2)$$

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where ψ is a nonlinear operator that embraces the nonlinear source and the rest of linear operator of the Eq. (1); therefore, considering initial conditions to be zero with regard to the independent variable and by taking Laplace transform of both sides of Eq. (2) in the usual manner (since u_0 is chosen so that $Lu_0 = 0$) and also by taking inverse Laplace transform, the following iteration formulation can be obtained:

$$u_{n+1}(t) = u_0(t) + \int_0^t h(u_n(\mu))b(t-\mu)d\mu \qquad (3)$$

provided that u_0 is an initial solution with or without unknown parameters, $P(s)$ is a polynomial with the degree of the highest derivative in Eq. (1) and $L[h(u(t))] = H$,

$$B(s) = 1/P(s), L[b(t)] = B(s), \qquad (4)$$

In this method, the problems are initially approximated with possible unknowns and, till here, there has been no dependence on small parameters; therefore, it can be applied in non-linear problems without linearization or small perturbation. The approximate solutions obtained by the proposed method rapidly converge to its exact solution.

III. NONLINEAR PENDULUM

In this section, We consider the simple mathematical nonlinear pendulum which can be written in the form [3]:

$$\frac{d^2\theta}{dt^2} + \Omega^2 \sin\theta = 0, \qquad (5)$$

where θ is the angular displacement, t is the time, $\Omega^2 = g/L$ is the natural frequency of the small oscillations of the pendulum, L is the length of the pendulum and g is the acceleration due to gravity. The oscillations of the pendulum are subjected to the initial conditions $\theta(0) = A$, and $\dot{\theta}(0) = 0$, , A being the amplitude of the oscillations.

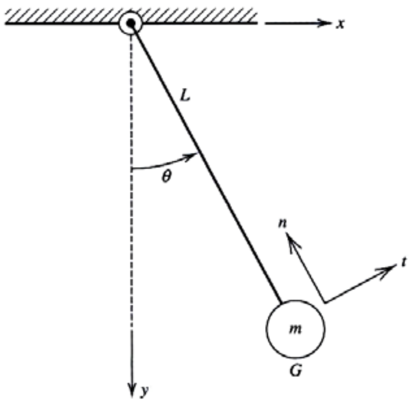

Fig. 1. The simple pendulum

The periodic solution $\theta(t)$ of equation (1) and the period depend on the amplitude A. Equation (1), although straightforward in appearance, is in fact difficult to solve because of the nonlinearity of the trigonometric function.
To solve Eq. (2) by the GVIM method, it can be rewritten as

$$\ddot{\theta}(t) + \omega^2\theta(t) = F(\theta(t)), \qquad (6)$$

where

$$F(\theta(t)) = \omega^2\theta(t) - \Omega^2 \sin\theta, \qquad (7)$$

By applying the GVIM, the following recursive iteration will be constructed.
Now the GVIM is applied to solve the Eq. (6) and the following recursive iteration is constructed

$$\theta_{n+1}(t) = \theta_0 + \frac{1}{\omega}\int_0^t \sin(\omega(t-\mu))F(\theta_n(\mu))d\mu, \qquad (8)$$

The trial function was used for determining the angular frequency ω, i.e., the first approximation to Eq. (5) was assumed to be

$$\theta_0(t) = A\cos(\omega t), \qquad (9)$$

To obtain and eliminate secular terms in equation (8) we need to obtain contributions proportional to $\cos(\omega t)$ due to $\sin(A\cos(\omega t))$. To do this, we can consider the following Taylor series expansion:

$$\sin(A\cos\omega t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1}\cos^{2n+1}\omega t, \qquad (10)$$

The formula that allows us to obtain the odd power series of the cosine is

$$\cos^{2n+1}\omega t = \frac{1}{2^{2n}}\sum_{k=0}^{n}\binom{2n+1}{n-k}\cos\left[(2k+1)\omega t\right] \qquad (11)$$

Substituting equation (11) into equation (10) gives

$$\sin(A\cos\omega t) = \sum_{n=0}^{\infty}\frac{(-1)^n A^{2n+1}}{2^{2n}(2n+1)!}\sum_{k=0}^{n}\binom{2n+1}{n-k}\cos\left[(2k+1)\omega t\right], \qquad (12)$$

By using the Eq. (7) and consider three term of Eq. (12) we have

$$\begin{aligned} F(\theta_0(t)) = &\left(\frac{1}{8}\Omega^2A^3 - \Omega^2A - \frac{1}{192}\Omega^2A^5 + \omega^2A\right)\cos(\omega t) \\ &-\frac{1}{384}\Omega^2A^5\cos(3\omega t) + \frac{1}{24}\Omega^2A^3\cos(3\omega t) \\ &-\frac{1}{1920}\Omega^2A^5\cos(5\omega t) \end{aligned} \qquad (13)$$

Since no secular terms should be present in Eq.(13), the

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coefficient of $\cos(\omega t)$ was set equal to zero which yielded the corresponding approximate period of the oscillation as:

$$\omega_0 = \frac{1}{24}\Omega\sqrt{-72A^2+576+3A^4} \qquad (14)$$

therefore the first approximation to the periodic solution of the nonlinear oscillator was given by the following equation

$$\theta_1(t) = A\cos(\omega t) + \frac{\Omega^2 A^3}{46080\omega^2}\begin{pmatrix} -16\cos(\omega t)A^2 + 240\cos(\omega t) \\ +15\cos(3\omega t)A^2 - 240\cos(3\omega t) \\ +A^2\cos(5\omega t) \end{pmatrix} \qquad (15)$$

and so on the second order approximation of frequency of oscillator is obtained by setting the coefficient of $\cos(\omega t)$ in $F(\theta_1)$ in Eq.(16) equal to zero.

$$F(\theta_1(t)) = -\Omega^2\left(\alpha - \frac{1}{6}\alpha^3 + \frac{1}{120}\alpha^5 + \omega^2\alpha\right) \qquad (16)$$

Where

$$\alpha = A\cos(\omega t) + \frac{1}{46080}\frac{\Omega^2 A^3}{\omega^2}\begin{pmatrix} -16\cos(\omega t)A^2 + 240\cos(\omega t) \\ +15\cos(3\omega t)A^2 - 240\cos(3\omega t) \\ +A^2\cos(5\omega t) \end{pmatrix} \qquad (17)$$

Then, after getting the first approximation to the frequency one can reach to the second approximation to the periodic solution of the nonlinear oscillator (5) by using the iteration formula in Eq. (8). Obtained numerical results are illustrated in Figs. 2, 3 and Table (1). In order to have a comparison between the approximate and exact frequency expressions of the oscillation amplitude, the relative error is analytically obtained for GVIM and EBM and HPM in this section.

IV. Conclusion

Analytical approach was successfully applied to strong nonlinear equations such as pendulum problems to obtain an approximate expression for the period of the pendulum. In addition, in Table (1) the accuracy of the method was investigated by a comparison which was made between HPM and energy balance method. It was shown in Table (1) that the accuracy of our method by using three terms in Taylor's series solution of sine function and only one iteration is higher than HPM. The method is useful to obtain analytical solution for all oscillators and vibration problems, such as in the fields of civil structures, fluid mechanics, electromagnetics and waves, etc. These provide a simple and pedagogical example for introducing this technique by means of an example. On the other hand, undergraduate students can easily see that the method considered in this paper is extremely simple in its principle, quite easy to use, and gives a very good numerical accuracy to the periodic solutions.

Results in Tables 1 and Figs. 2, 3 reveal that this method can be considered as a viable alternative for conventional methods which can solve highly nonlinear oscillatory systems.

Table 1. Comparison between different methods to obtain the frequency ($\Omega = 1$)

A	ω				
	Exact[4]	**EBM[17]**	**HPM[18]**	**GVIM(0)**	**GVIM(1)**
0.01	**0.99999375**	**0.99999375**	**0.99999375**	**0.99999375**	**0.99999375**
0.1	**0.999375033**	**0.999374926**	**0.999375065**	**0.999375065**	**0.999375033**
0.2	**0.997500519**	**0.997498817**	**0.997501044**	**0.997501044**	**0.997500522**
0.3	**0.994377614**	**0.99436899**	**0.994380303**	**0.994380303**	**0.994377651**
0.4	**0.990008204**	**0.98998092**	**0.990016835**	**0.990016835**	**0.990008415**
0.5	**0.984394851**	**0.984328151**	**0.984416335**	**0.984416335**	**0.984395658**
0.6	**0.977540699**	**0.977402169**	**0.977586313**	**0.977586313**	**0.97754313**
0.7	**0.969449373**	**0.969192229**	**0.96953624**	**0.96953624**	**0.969455558**
0.8	**0.960124819**	**0.959685134**	**0.960277738**	**0.960277738**	**0.960138745**
0.9	**0.94957112**	**0.948864947**	**0.949824819**	**0.949824819**	**0.949599694**
1.0	**0.937792258**	**0.936712632**	**0.938194187**	**0.938194188**	**0.937846766**
1.1	**0.924791819**	**0.923205598**	**0.925405598**	**0.925405598**	**0.924889888**
1.2	**0.910572619**	**0.908317125**	**0.911482309**	**0.911482309**	**0.910740803**
1.3	**0.895136237**	**0.892015644**	**0.896451628**	**0.896451628**	**0.895413412**
1.4	**0.878482424**	**0.874263817**	**0.880345576**	**0.880345576**	**0.878924193**
1.5	**0.860608347**	**0.855017361**	**0.863201707**	**0.863201708**	**0.861292771**

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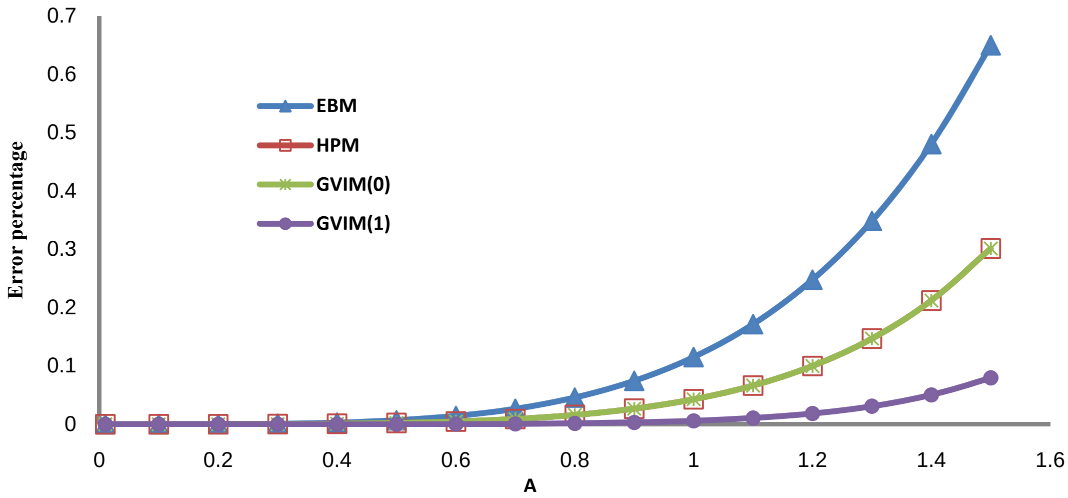

Fig. 2. Comparison between error percentages of different methods to obtain the frequency

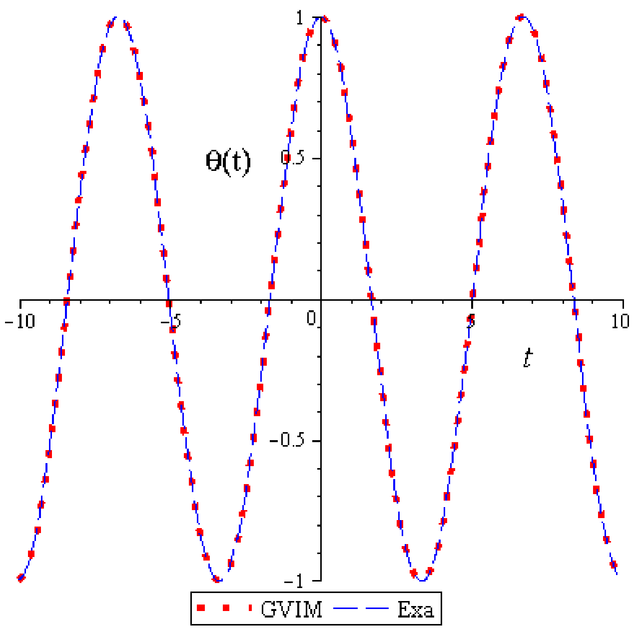

Fig. 3. Comparison of the GVIM and the exact solution for A=1

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