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On Adjacency Matrix of One Type of Graph and Pell Numbers

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Abstract— **Recently there is huge interest on graph theory and intensive study on computing integer powers of matrices. As it is well-known, the (*i,j*)th entry of A^m (arbitrary positive integer power of *A*) is just the number of the different paths from vertex *i* to vertex *j*.**

In the present paper, we consider adjacency matrix of one type of graph, which is a block-diagonal matrix, and we investigate relations between the matrix and well-known Pell sequence.

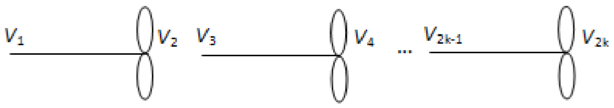
Index Pell number, matrix power, adjacency matrix

I. Introduction

There are many special types of matrices which have great importance in many scientific works. For example matrices of tridiagonal, pentadiagonal and others. These types of matrices frequently appear in interpolation, numerical analysis, solution of boundary value problems, high order harmonic spectral filtering theory and so on. In [3]-[5], authors investigated integer powers of some type of these matrices.

Among numerical sequences, the Pell sequence which is defined by the recurrence $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$ with initial conditions $P_1 = 1$ and $P_2 = 2$, has achieved a kind of celebrity status [1].

Let $G=(V,E)$ be a graph with set of vertices $V(G) = \{1,2,\dots,n\}$ and set of edges $E(G)=\{e_1,e_2,\dots,e_m\}$, as following:



The adjacency matrix of G is a block-diagonal matrix of even order whose diagonal blocks have the form $[0,1,1,2]$, i.e.:

$$A = \begin{cases} 1, \; a_{i,i+1} = a_{i+1,i} & \text{for } i = 1,3,4,\dots,n-1 \\ 2, \; a_{i,i} & \text{for } i = 2,4,6,\dots,n \\ 0, & \text{otherwise.} \end{cases} \qquad (1)$$

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In the present paper, we compute arbitrary integer powers of the block diagonal matrix exploiting some properties of the Pell sequence.

II. Main Results

Let A be n-square $(n = 2k$ and $k = 1,2,\dots)$ block-diagonal matrix as in (1).

One can observe that all integer powers of A are specified to the famous Pell numbers with positive and negative signs.

As it is well-known that the rth ($r \in \mathbb{N}$) power of a matrix is computed by using the known expression $A^r = TJ^rT^{-1}$ [2], here J is the Jordan form of the matrix A and T is the transforming matrix. The matrices J and T are obtained using eigenvalues and eigenvectors of the matrix A. The eigenvalues of A are the roots of the characteristic equation defined by $|A - \lambda I| = 0$ where I is the identity matrix of nth order.

Let $P_n(x)$ be the characteristic polynomial of the matrix A which is defined in (1). Then, we can write:

$$\begin{aligned} P_2(x) &= x?\ 2x - 1 \\ P_4(x) &= x^4 - 4x?+\ x?+\ x+1 \\ P_6(x) &= x^6 - 6x^5 + 9x^4 + 4x^3 - 9x^2 - 6x - 1 \qquad (2) \\ &\vdots \\ P_n(x) &= (x?\ 2x-1)^{\frac{n}{2}} \end{aligned}$$

Taking (2) into account

$$P_n(\lambda) = (\lambda?\ 2\lambda - 1)^{\frac{n}{2}}, \qquad n = 2k \text{ and } k = 1,2,\dots$$
$$= [(\lambda - \alpha)(\lambda - ß)]^{\frac{n}{2}}$$

where $\alpha = \left(1+\sqrt{2}\right)$ and $\beta = \left(1-\sqrt{2}\right)$. The eigenvalues of the matrix are multiple according to the order of the matrix. Then the Jordan's form of the matrix A is

$$J = J_k = diag(\underbrace{\alpha,\dots,\alpha}_{k}, \underbrace{\text{薐}..,\ }_{k}),\ k = 1,2,\dots \qquad (3)$$

Let us consider the relation $J = T^{-1}AT$ $(AT = TJ)$; here A is n-square matrix in (1) ($n=2k$, $k=1,2,\dots$), J is the jordan form of the matrix A and T is the transforming matrix.

We will find the transforming matrix T. Let denote the jth column of T by T_j. Then $T = (T_1, T_2, \dots, T_n)$ and

$$(AT_1,\dots,AT_n) = (\alpha T_1,\dots,\alpha T_k, \beta T_{k+1},\dots,\beta T_n).$$

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In other words,

$$\begin{aligned} AT_1 &= \alpha T_1 \\ AT_2 &= \alpha T_2 \\ &\vdots \\ AT_k &= \alpha T_k \\ AT_{k+1} &= \beta T_{k+1} \\ AT_{k+2} &= \beta T_{k+2} \\ &\vdots \\ AT_n &= \beta T_n . \end{aligned} \quad (4)$$

Solving the set of equations system, we obtain the eigenvectors of the matrix A:

$$T = \begin{pmatrix} 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & \alpha & 0 & 0 & \cdots & 0 & \beta \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \beta & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & \alpha & \cdots & 0 & 0 & \beta & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ \underbrace{\alpha \quad 0 \quad \cdots \quad 0}_{k} & & & & \underbrace{\beta \quad 0 \quad \cdots \quad 0 \quad 0}_{k} \end{pmatrix} \quad (5)$$

We shall find the inverse matrix T^{-1} denoting the ith row of the inverse matrix T^{-1} by $T^{-1} = (t_1, t_2, \ldots, t_n)$ and implementing the necessary transformations, we obtain:

$$T^{-1} = \frac{1}{\alpha-\beta}\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\beta & 1 \\ 0 & 0 & 0 & 0 & \cdots & -\beta & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & -\beta & 1 & \cdots & 0 & 0 & \alpha & -1 \\ -\beta & 1 & 0 & 0 & \cdots & \alpha & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \alpha & -1 & \cdots & 0 & 0 & 0 & 0 \\ \alpha & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix} \quad (6)$$

Using the equalities (3), (5) and (6); we derive the expression for the rth power of the matrix A:

$$A = TJT^{-1} \Rightarrow A^r = TJ^rT^{-1} = \left[a_{(i,j)}(r)\right]$$

That is,

$$A^r = \begin{cases} a_{i-1,i-1} = \dfrac{1}{(\alpha-\beta)}\left[\beta^r\alpha - \beta\alpha^r\right] \\ a_{i,i} = \dfrac{1}{(\alpha-\beta)}\left[\alpha^{r+1} - \beta^{r+1}\right] \\ a_{i-1,i} = \dfrac{1}{(\alpha-\beta)}\left[\alpha^r - \beta^r\right] \\ a_{i,i-1} = \dfrac{1}{(\alpha-\beta)}\left[-\beta\alpha^{r+1} + \beta^{r+1}\alpha\right] \\ 0, \ otherwise \end{cases}$$

where $i = 2, 4, 6, \ldots, n$.

Lemma: Let A be as in (1). Then

$$\det A = (-1)^k$$

where $n = 2k, k = 1, 2, \ldots, \frac{n}{2}$.

Proof: Using Laplace expansion, the determinant can be obtained.

Theorem: Let $B = A^{-1}$ be a matrix as below:

$$B = \begin{cases} 1, \ a_{i,i-1} = a_{i-1,i} & \text{for } i = 2,4,6,\ldots,n \\ -2, \ a_{i,i} & \text{for } i = 1,3,5,\ldots,n-1 \\ 0, & \text{otherwise.} \end{cases} \quad (7)$$

Then;

$$B^r = A^{-r} = \begin{cases} b_{i-1,i-1} = \dfrac{1}{(\alpha-\beta)}\left[-\beta(-\beta)^r + \alpha(-\alpha)^r\right] \\ b_{i,i} = \dfrac{1}{(\alpha-\beta)}\left[\alpha(-\beta)^r - \beta(-\alpha)^r\right] \\ b_{i-1,i} = \dfrac{1}{(\alpha-\beta)}\left[-\alpha\beta(-\beta)^r + \alpha\beta(-\alpha)^r\right] \\ b_{i,i-1} = \dfrac{1}{(\alpha-\beta)}\left[(-\beta)^r - (-\alpha)^r\right] \\ 0, \ \text{otherwise} \end{cases}$$

where $i = 2, 4, 6, \ldots, n$ and $r = 1, 2, \ldots$.

Proof. We will compute arbitrary negative integer power of A using the well-known equality $A^{-r} = TJ^{-r}T^{-1}$, here T is the transforming matrix and J is the Jordan matrix.

Let $Q_n(x)$ be the characteristic polynomial of the matrix A^{-1} which is defined in (7). Then we can write:

$$\begin{aligned} &Q_2(x) = x?+ \ x - 1 \\ &Q_4(x) = x^4 + 4x^3 + 2x^2 - 4x + 1 \\ &Q_6(x) = x^6 + 6x^5 + 9x^4 - 4x^3 - 9x^2 + 6x - 1 \\ &\quad\vdots \\ &Q_n(x) = (x?+ \ x - 1)^{\frac{n}{2}} \end{aligned} \quad (8)$$

By (8), we can write

$$\begin{aligned} Q_n(\lambda) &= (\lambda?+ \ \lambda - 1)^{\frac{n}{2}}, \quad n = 2k \text{ and } k = 1, 2, \ldots \\ &= [(\lambda - \alpha)(\lambda + ß)]^{\frac{n}{2}} \end{aligned}$$

here $\alpha = \left(1+\sqrt{2}\right)$ and $\beta = \left(1-\sqrt{2}\right)$. The eigenvalues of the matrix are multiple according to the order of the matrix. Then the Jordan's form of the matrix A^{-1} is

$$J^{-r} = diag\left[\underbrace{(-\beta)^r, \ldots, (-\beta)^r}_{k}, \underbrace{(-\alpha)^r, \ldots, (-\alpha)^r}_{k}\right]? \quad (9)$$

where $k = 1, 3, \ldots$.

Let us consider the relation $J = T^{-1}BT$ $(BT = TJ)$; here B is nth order matrix (7) (n=2k, k=1,2,…) J^{-1} is the jordan form of the matrix $B = A^{-1}, T$ is the transforming matrix. We will find the transforming matrix T. Let denote the j-th column of T by T_j. Then $T = (T_1, T_2, \ldots, T_n)$ and

$$(BT_1, \ldots, BT_n) = (\alpha T_1, \ldots, \alpha T_k, \beta T_{k+1}, \ldots, \beta T_n)$$

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Solving the set of equations system, we obtain the eigenvectors of the matrix B :

$$T=\begin{pmatrix} 0 & 0 & \cdots & -\beta & 0 & 0 & \cdots -\alpha \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots 1 \\ \vdots & \vdots & \iddots & \vdots & \vdots & \vdots & \iddots \vdots \\ 0 & -\beta & \cdots & 0 & 0 & -\alpha & \cdots 0 \\ 0 & 1 & \cdots & 0 & 0 & 1 & \cdots 0 \\ -\beta & 0 & \cdots & 0 & -\alpha & 0 & \cdots 0 \\ 1 & 0 & \cdots & 0 & 1 & 0 & \cdots 0 \end{pmatrix} \quad (10)$$

We shall find the inverse matrix T^{-1} denoting the ith row of the inverse matrix T^{-1} by $T^{-1}=(t_1,t_2,\ldots,t_n)$ and implementing the necessary transformations, we obtain:

$$T^{-1}=\frac{1}{(\alpha-\beta)}\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \alpha \\ 0 & 0 & 0 & 0 & \cdots & 1 & \alpha & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & \alpha & \cdots & 0 & 0 & 1 & -\beta \\ 1 & \alpha & 0 & 0 & \cdots & 1 & -\beta & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & -1 & -\beta & \cdots & 0 & 0 & 0 & 0 \\ -1 & -\beta & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix} \quad (11)$$

Using the equalities (9), (10) and (11); we derive the expression for the rth power of the matrix B:

$$B=TJT^{-1} \Rightarrow B^r = A^{-r} = TJ^{-r}T^{-1}=\left[b_{(i,j)}(r)\right]$$

By matrix multiplication, proof can be seen easily.

III. EXAMPLES

Taking into account the derived expressions, arbitrary integer powers of the matrix can be computed. Let k=3;

If $r=4$;

$$a_{11}=a_{33}=a_{55}=\frac{1}{(\alpha-\beta)}(\beta^4\alpha-\beta\alpha^4)=5$$

$$a_{22}=a_{44}=a_{66}=\frac{1}{(\alpha-\beta)}(\alpha^5-\beta^5)=29$$

$$a_{12}=a_{34}=a_{56}=\frac{1}{(\alpha-\beta)}(\alpha^4-\beta^4)=12$$

$$a_{21}=a_{43}=a_{65}=\frac{1}{(\alpha-\beta)}(-\beta\alpha^5+\beta^5\alpha)=12$$

otherwise, 0

If $r=5$;

$$a_{11}=a_{33}=a_{55}=\frac{1}{(\alpha-\beta)}(\beta^5\alpha-\beta\alpha^5)=12$$

$$a_{22}=a_{44}=a_{66}=\frac{1}{(\alpha-\beta)}(\alpha^6-\beta^6)=70$$

$$a_{12}=a_{34}=a_{56}=\frac{1}{(\alpha-\beta)}(\alpha^5-\beta^5)=29$$

$$a_{21}=a_{43}=a_{65}=\frac{1}{(\alpha-\beta)}(-\beta\alpha^6+\beta^6\alpha)=29$$

otherwise, 0

If $r=-4$;

$$b_{11}=b_{33}=b_{55}=\frac{1}{(\alpha-\beta)}[-\beta(-\beta)^4+\alpha(-\alpha)^4]=29$$

$$b_{22}=b_{44}=b_{66}=\frac{1}{(\alpha-\beta)}[\alpha(-\beta)^4-\beta(-\alpha)^4]=5$$

$$b_{12}=b_{34}=b_{56}=\frac{1}{(\alpha-\beta)}[-\alpha\beta(-\beta)^4+\alpha\beta(-\alpha)^4]=-12$$

$$b_{21}=b_{43}=b_{65}=\frac{1}{(\alpha-\beta)}[(-\beta)^4-(-\alpha)^4]=-12$$

otherwise, 0

If $r=-5$;

$$b_{11}=b_{33}=b_{55}=\frac{1}{(\alpha-\beta)}[-\beta(-\beta)^5+\alpha(-\alpha)^5]=-70$$

$$b_{22}=b_{44}=b_{66}=\frac{1}{(\alpha-\beta)}[\alpha(-\beta)^5-\beta(-\alpha)^5]=-12$$

$$b_{12}=b_{34}=b_{56}=\frac{1}{(\alpha-\beta)}[-\alpha\beta(-\beta)^5+\alpha\beta(-\alpha)^5]=29$$

$$b_{21}=b_{43}=b_{65}=\frac{1}{(\alpha-\beta)}[(-\beta)^5-(-\alpha)^5]=29$$

otherwise, 0.

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