On Riemann-Liouville and Caputo Impulsive Fractional Calculus

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***Abstract*- This paper gives formulas for Riemann-Liouville impulsive fractional integral calculus and for Riemann- Liouville and Caputo impulsive fractional derivatives.**

***Keywords*- Rimann-Liouville fractional calculus, Caputo fractional derivative, Dirac delta, Distributional derivatives, High- order distributional derivatives.**

I. INTRODUCTION

Fractional calculus has been used in a set of applications, mainly, to deal with modelling errors in differential equations and dynamic systems. There are also applications in Signal Processing and sampling and hold algorithms, [1-3]. Fractional integrals and derivatives can be of non-integer orders and even of complex order. The related fractional calculus facilitates the description of some problems which are not easily described by ordinary calculus due to modelling errors, [1-5]. There are several approaches for the integral fractional calculus, the most popular ones being the Riemann-Liouville fractional integral. There is also a fractional Riemann- Liouville derivative. However, the well- known Caputo fractional derivative are less involved since the associated integral operator manipulates the derivatives of the primitive function under the integral symbol. This paper extends the basic fractional differ-integral calculus to impulsive functions described through the use of Dirac distributions and Dirac distributional derivatives, [5], of real fractional orders. In the general case, it is admitted a presence of infinitely many impulsive terms at certain isolated point of the relevant function domains.

II. GENERALIZED RIEMANN-LIOUVILLE FRACTIONAL INTEGRAL

Let us denote the set of positive real numbers by $\boldsymbol{R}_+ = \{ r \in \boldsymbol{R} : r > 0 \}$ and left-sided and right-sided Lebesgue integrals, respectively, as:

$\int_0^x g(\tau) d\tau := \lim_{t \to x \equiv x^-} \int_0^t g(\tau) d\tau$ (the identification $x \equiv x^-$ is used for all x in order to simplify the notation), and

$$\int_0^{x^+} g(\tau) d\tau := \lim_{t \to x^+} \int_0^t g(\tau) d\tau$$

Now, consider real functions $f, \bar{f} : \boldsymbol{R}_+ \to \boldsymbol{R}$,such that $\int_0^x (x-t)^{\mu-1} \bar{f}(t) dt$ exists, $\forall x \in \boldsymbol{R}_+$, fulfilling:

$$f(x) = \bar{f}(x) + \sum_{x_i \in IMP} K_i \delta(x - x_i) = \bar{f}(x) + \sum_{i \in I(\infty)} K_i \delta(x - x_i)$$

$\delta(x)$ denotes the Dirac delta distribution, $K_i \delta(0) = f(x_i^+) - f(x_i)$ with $K_i \in \mathbf{R}$; $\forall i \in I(\infty) \subset \boldsymbol{Z}_+$, [5], and $IMP := \bigcup_{x \in \boldsymbol{R}_+} IMP(x) = \bigcup_{x \in \boldsymbol{R}_+} IMP(x^+)$ of indexing set $I(\infty)$ is the whole impulsive set defined via empty or non-empty) partial impulsive strictly ordered denumerable sets:

$$IMP(x) := \{ x_i \in \mathbf{R}_+ : f(x_i^+) - f(x_i) = K_i \delta(0), x_i < x \} \quad (1)$$

of indexing set $I(x) := \{ i \in \boldsymbol{Z}_{0+} : x_i \in IMP(x) \} \subset I(x^+) \subset \boldsymbol{Z}_+$, for each $x \in \boldsymbol{R}_+$; and

$$IMP(x) \subset IMP(x^+)_+ := \{ x_i \in \boldsymbol{R}_+ : f(x_i^+) - f(x_i) = K_i \delta(0), x_i \le x^+ \} \subset \boldsymbol{R} \quad (2)$$

of indexing set $I(x) \subset I(x^+) := \{ i \in \boldsymbol{Z}_{0+} : x_i \in IMP(x^+) \} \subset \boldsymbol{Z}_+$, for each $x \in \boldsymbol{R}_+$ with the indexing set of IMP being $I(\infty) = \bigcup_{x \in IMP(x)} I(x) = \bigcup_{x \in IMP(x^+)} I(x^+)$. If we are interested in studying the fractional derivative of the impulsive function $f : \boldsymbol{R}_+ \to \boldsymbol{R}$ then $\bar{f} : \boldsymbol{R}_+ \to \boldsymbol{R}$ is non- uniquely defined as $\bar{f}(x) = f(x)$ for $x \in \boldsymbol{R}_+ \setminus IMP$, and $f(x_i) = \bar{f}(x_i)$, $f(x_i^+) = f(x_i) + K_i \delta(0) = \bar{f}(x_i) + K_i \delta(0)$, for $x_i \in IMP$ with $\bar{f}(x^+) \in \boldsymbol{R}$ (non-uniquely) defined being bounded arbitrary (for instance, being zero or $\bar{f}(x^+) = f(x)$) if $x \in IMP$. Note that IMP and $I(\infty)$ have infinite cardinals if there are infinitely many impulsive values of the function f(t).

Note that the existence of $\int_0^x (x-t)^{\mu-1} \bar{f}(t) dt$ implies that of $\int_0^x (x-t)^{\mu-1} f(t) dt = \int_0^x (x-t)^{\mu-1} \bar{f}(t) dt$ if $x \notin IMP(x)$, since $\int_0^x (x-t)^{\mu-1} \bar{f}(t) dt$ exists, and that of

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$$\int_0^{x_i^+}(x-t)^{\mu-1}f(t)dt=\int_0^{x_i}(x-t)^{\mu-1}\bar{f}(t)dt+(x-x_i)^{\mu-1}\left(f\left(x_i^+\right)-f(x_i)\right)$$

if $x_i\in IMP\left(x^+\right)$ (3)

Theorem 2.1. The extended fractional Riemann- Liouville integrals by considering impulsive functions are defined for any fixed order $\mu\in\boldsymbol{R}_+$ and all $x\in\boldsymbol{R}_+$ by

$$\left(J^{\mu}f\right)(x):=\frac{1}{\Gamma(\mu)}\int_0^x(x-t)^{\mu-1}f(t)dt$$

$$=\frac{1}{\Gamma(\mu)}\left(\int_0^x(x-t)^{\mu-1}\bar{f}(t)dt+\sum_{i\in I(x)}(x-x_i)^{\mu-1}\left(f\left(x_i^+\right)-f(x_i)\right)\right)$$

$$=\frac{1}{\Gamma(\mu)}\sum_{i\in I(x)\cup\{0\}}\int_{x_i^+}^{x_{i+1}}(x-t)^{\mu-1}f(t)dt$$

$$+\frac{1}{\Gamma(\mu)}\int_{x_{n(x)}^+}^{x}(x-t)^{\mu-1}f(t)dt+\sum_{i\in I(x)}(x-x_i)^{\mu-1}\left(f\left(x_i^+\right)-f(x_i)\right) \quad (4)$$

$$\left(J^{\mu}f\right)\left(x^+\right):=\frac{1}{\Gamma(\mu)}\int_0^{x^+}(x-t)^{\mu-1}f(t)dt$$

$$=\frac{1}{\Gamma(\mu)}\left(\int_0^x(x-t)^{\mu-1}\bar{f}(t)dt+\sum_{i\in I(x^+)}(x-x_i)^{\mu-1}\left(f\left(x_i^+\right)-f(x_i)\right)\right)$$

$$=\frac{1}{\Gamma(\mu)}\sum_{i\in I(x^+)\cup\{0\}}\int_{x_i^+}^{x_{i+1}}(x-t)^{\mu-1}f(t)dt$$

$$+\frac{1}{\Gamma(\mu)}\sum_{i\in I(x^+)}(x-x_i)^{\mu-1}\left(f\left(x_i^+\right)-f(x_i)\right) \quad (5)$$

$$\left(J^0f\right)\left(x^+\right)\equiv\left(J^0f\right)(x):=f(x)$$

where $\Gamma:\boldsymbol{R}_{0+}\to\boldsymbol{R}_+$ is the Γ - function , [1-5] and $n:IMP\to\boldsymbol{Z}_+$ is defined by $n(x)=card\,I(x)=card\,IMP(x)$. □

Note that if $x\in IMP$ then

$$\left(J^{\mu}f\right)\left(x^+\right)=\frac{1}{\Gamma(\mu)}\sum_{i\in I(x^+)\cup\{0\}}\int_{x_i^+}^{x_{i+1}}(x-t)^{\mu-1}f(t)dt$$

$$+\frac{1}{\Gamma(\mu)}\sum_{i\in I(x^+)}(x-x_i)^{\mu-1}\left(f\left(x_i^+\right)-f(x_i)\right)$$

$$=\left(J^{\mu}f\right)(x)+\left(x-x_{n(x)}\right)^{\mu-1}\left(f\left(x_{n(x)}^+\right)-f\left(x_{n(x)}\right)\right)$$

$$\neq\left(J^{\mu}f\right)(x)=\frac{1}{\Gamma(\mu)}\sum_{i\in I(x)\cup\{0\}}\int_{x_i^+}^{x_{i+1}}(x-t)^{\mu-1}f(t)dt$$

$$+\frac{1}{\Gamma(\mu)}\sum_{i\in I(x)}(x-x_i)^{\mu-1}\left(f\left(x_i^+\right)-f(x_i)\right) \quad (6)$$

and if $x\notin IMP$, since $I(x^+)=I(x)$, then $\left(J^{\mu}f\right)\left(x^+\right)=\left(J^{\mu}f\right)(x)$.

III. GENERALIZED RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVE

Assume that $f\in C^{m-1}\left(\boldsymbol{R}_+,\boldsymbol{R}\right)$ and its $m-th$ derivative exists everywhere in $\boldsymbol{R}_+$. Then, the Caputo fractional derivative of order $\mu\geq 0$ with $m-1\leq\mu\left(\in\mathbf{R}_+\right)\leq m$, $m\in\boldsymbol{Z}_+$ is for any $x\in\boldsymbol{R}_+$:

$$\left(D^{\mu}f\right)(x):=\left(\frac{d}{dx}\right)^m\left(J^{m-\mu}f\right)(x)$$

$$=\frac{1}{\Gamma(m-\mu)}\left(\frac{d}{dx}\right)^m\left(\int_0^x(x-t)^{m-\mu-1}f(t)dt\right) \quad (7)$$

The following particular cases follow from this formula for $\mu=m-1$:

(a) $\mu=-1;m=0$ yields $\left(D^{-1}f\right)(x)=\int_0^x f(t)dt$ which is the standard integral of the function f . This case does not verify the "derivative constraint " $0\leq m-1\leq\mu\left(\in\boldsymbol{R}_+\right)<m$ leading to an integral result.

(b)) $\mu=0;m=1$ yields $\left(D^0f\right)(x)=f(x)$ which so that D^0f is the identity operator

(c) $\mu=1;m=2$ yields $\left(D^1f\right)(x)=f^{(1)}(x)$

(d) $\mu=2;m=3$ yields $\left(D^2f\right)(x)=f^{(2)}(x)$ which is the standard first- derivative of the function f.

Compared to the parallel cases with the Caputo fractional derivative, note that the Riemann- Liouville fractional derivative, compared to the Caputo corresponding one, does not depend on the conditions at zero of the function and its derivatives. Define the Kronecker delta $\delta(a,b)$ of any pair of real numbers (a,b) as $\delta(a,b)=1$ if $a=b$ and $\delta(a,b)=0$ if $a\neq b$ and then evaluate recursively the Riemann – Liouville fractional derivative of order $\mu\geq 0$ from the above formula by using Leibniz´s differentiation rule by noting that, since $\mu\neq m-j;\forall\, j\left(\in\boldsymbol{Z}_+\right)>1$, only the differential part corresponding to the differentiation of the integrand is non zero for $j>m-\mu$. This yields the following result:

Theorem 3.1. Assume that $f\in C^{m-2}\left(\boldsymbol{R}_+,\boldsymbol{R}\right)$ and $f^{(m-1)}$ exists everywhere in $\boldsymbol{R}_+$ and that $f(t)$ is integrable on $\mathbf{R}_+$, then:

$$\left(D^{\mu}f\right)(x)=\frac{1}{\Gamma(m-\mu)}\left(\frac{d}{dx}\right)^m\left(\int_0^x(x-t)^{m-\mu-1}f(t)dt\right)$$

$$=\frac{1}{\Gamma(m-\mu)}\left(\frac{d}{dx}\right)^{m-1}$$

$$\left[\int_0^x(m-\mu-1)(x-t)^{m-\mu-2}f(t)dt+f(x)\delta(\mu,m-1)\right]$$

$$=\frac{1}{\Gamma(m-\mu)}f^{(m-1)}(x)\delta(\mu,m-1)$$

$$+\frac{1}{\Gamma(m-\mu)}\left(\frac{d}{dx}\right)^{m-1}\left(\int_0^x(m-\mu-1)(x-t)^{m-\mu-2}f(t)dt\right)$$

$$=\frac{1}{\Gamma(m-\mu)}f^{(m-1)}(x)\delta(\mu,m-1)$$

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$$+\frac{1}{\Gamma(m-\mu)}\left(\sum_{i=1}^{m-2}\prod_{j=i+1}^{m-1}[j-\mu]\right)f^{(i)}(x)\delta(\mu,m-i)$$

$$+\frac{1}{\Gamma(m-\mu)}\left[\prod_{j=0}^{m-1}[j-\mu]\right]\left(\int_0^x (x-t)^{-(\mu+1)}f(t)dt\right) \tag{8}$$

If $f\in PC^k(\boldsymbol{R}_+,\boldsymbol{R})$ with $f^{(k)}(x)$ being discontinuous of first class then $f^{m-1}(x)=\delta^{(j(x))}(x)$ with $j(x)=m-1-k(x)$, one uses to obtain the right value of (8) the perhaps high-order distributional derivatives formula:

$$\left|f^{(m-1)}(x^+)-f^{(m-1)}(x)\right|=$$
$$\frac{(-1)^k k!}{x^k}\left|f^{(m-1-k)}(x^+)-f^{(m-1-k)}(x)\right|\delta(0)=\infty \tag{9}$$

to yield

$$(D^\mu f)(x^+)=\frac{1}{\Gamma(m-\mu)}$$
$$\left[\frac{(-1)^{k(x)}k(x)!}{x^{k(x)}}\left|f^{(m-1-k(x))}(x^+)-f^{(m-1-k(x))}(x)\right|\delta(0)\delta(\mu,m-1)\right.$$
$$+\left(\sum_{i=1}^{m-2}\prod_{j=i+1}^{m-1}[j-\mu]\right)f^{(i)}(x)\delta(\mu,m-i)$$
$$\left.+\left[\prod_{j=0}^{m-1}[j-\mu]\right]\left(\int_0^x (x-t)^{-(\mu+1)}f(t)dt\right)\right] \tag{10}$$

If $\mu=m-1$ then

$$(D^{m-1}f)(x)$$
$$=f^{(m-1)}(x)+\left[\prod_{j=0}^{m-1}[j-\mu]\right]\left(\int_0^x (x-t)^{-m}f(t)dt\right)$$

(11)

provided that $\left(\int_0^x (x-t)^{-(\mu+1)}f(t)dt\right)$ exists for $x\in\mathbf{R}_+$ (which is guaranteed if $f(t)$ is Lebesgue-integrable on $\mathbf{R}_+$), $f\in C^{m-2}(\boldsymbol{R}_+,\boldsymbol{R})$ and f^{m-1} exists everywhere in $\mathbf{R}_+$. The correction (10) applies when the derivative does not exist. □

If $\mu\neq m-1$ with $m-1\le\mu(\in\mathbf{R}_+)\le m$ then after defining the impulsive sets, its associated indexing sets and the function $\bar{f}:\mathbf{R}_+\to\mathbf{R}$ as for the extended Riemann-Liouville fractional integral, one gets:

$$(D^\mu f)(x)$$
$$=\frac{1}{\Gamma(m-\mu)}\left[\prod_{j=0}^{m-1}[j-\mu]\right]\left(\int_0^x (x-t)^{-(\mu+1)}f(t)dt\right)$$
$$=$$
$$=\frac{1}{\Gamma(m-\mu)}\left[\prod_{j=0}^{m-1}[j-\mu]\right]\sum_{i\in I(x)\cup\{0\}}\int_{x_i^+}^{x_{i+1}}(x-t)^{-(\mu+1)}f(t)dt$$
$$+\frac{1}{\Gamma(m-\mu)}\left[\prod_{j=0}^{m-1}[j-\mu]\right]\int_{x_{n(x)}^+}^{x}(x-t)^{-(\mu+1)}f(t)dt$$
$$+\frac{1}{\Gamma(m-\mu)}\left[\prod_{j=0}^{m-1}[j-\mu]\right]\sum_{i\in I(x)}(x-x_i)^{-(\mu+1)}\left(f(x_i^+)-f(x_i)\right) \tag{12}$$

$$(D^\mu f)(x^+)=\frac{1}{\Gamma(m-\mu)}\left[\prod_{j=0}^{m-1}[j-\mu]\right]$$
$$\times\left(\int_0^x (x-t)^{-(\mu+1)}\bar{f}(t)dt+\sum_{i\in I(x)}(x-x_i)^{-(\mu+1)}\left(f(x_i^+)-f(x_i)\right)\right) \tag{13}$$

IV. GENERALIZED CAPUTO FRACTIONAL DERIVATIVE

Assume that $f\in C^{m-1}(\boldsymbol{R}_+,\boldsymbol{R})$ and its $m-th$ derivative exists everywhere in $\boldsymbol{R}_+$. Then, the Caputo fractional derivative of order $\mu\ge 0$ with $m-1\le\mu(\in\boldsymbol{R}_+)<m$, $m\in\boldsymbol{Z}_+$ is for any $x\in\boldsymbol{R}_+$:

$$(D_*^\mu f)(x):=\left(J^{m-\mu}f^{(m)}\right)(x)$$
$$=\frac{1}{\Gamma(m-\mu)}\int_0^x (x-t)^{m-\mu-1}f^{(m)}(t)dt \tag{14}$$

; $m-1\le\mu<m$, $m\in\boldsymbol{Z}_+$, $x\in\boldsymbol{R}_+$

The following particular cases occur with $\mu=m-1$ leading to

$$(D_*^{m-1}f)(x)=\int_0^x f^{(m)}(t)dt=f^{(m-1)}(x)-f^{(m-1)}(0^+) \tag{15}$$

(a) $\mu=-1;\ m=0$ yields $(D_*^{-1}f)(x)=f^{(-1)}(x)-f^{(-1)}(0^+)$ which is an integral result f . Note that this case does not verifies the "derivative constraint" $0\le\mu(\in\boldsymbol{R}_+)<m$ leading to an integral result.

(b)) $\mu=0;\ m=1$ yields
$(D_*^0 f)(x)=f^{(0)}(x)-f^{(0)}(0^+)=f(x)-f(0^+)$

(c) $\mu=1;\ m=2$ yields $(D_*^1 f)(x)=f^{(1)}(x)-f^{(1)}(0^+)$

(d) $\mu=2;\ m=3$ yields $(D_*^2 f)(x)=f^{(2)}(x)-f^{(2)}(0^+)$

We can extend the above formula to real functions with impulsive $m-th$ derivative as follows. Assume that $f\in C^{m-2}(\mathbf{R}_+,\mathbf{R})$ with bounded piecewise $(m-1)-th$ derivative existing everywhere in $\boldsymbol{R}_+$ and $f^{(m)}(x)\equiv\frac{d^m f(x)}{dx^m}$ being impulsive with

$$f^{(m)}(x_i)=K_i\,\delta(0)=\left(f^{(m-1)}(x_i^+)-f^{(m-1)}(x_i)\right)\delta(0)$$

; $\forall x_i\in IMP$, equivalently , $\forall i\in I(\infty)$, at the eventual discontinuity points $x_i>0$ at the impulsive set $IMP:=\bigcup_{x\in\boldsymbol{R}_+}IMP(x)$, where the partial impulsive sets are re-defined as follows:

$$IMP(x):=\left\{x_i\in\mathbf{R}_+ : f^{(m-1)}\left(x_i^+\right)-f^{(m-1)}(x_i)=K_i\,,\,x_i<x\right\}\subset IMP\left(x^+\right) \quad (16)$$

$$IMP\left(x^+\right):=\left\{x_i\in\mathbf{R}_+ : f^{(m-1)}\left(x_i^+\right)-f^{(m-1)}(x_i)=K_i\,,\,x_i\le x^+\right\}\subset IMP\left(x^+\right) \quad (17)$$

Now, consider $f\in C^{m-1}(0,\infty)$ with $f^{(m)}(x)\equiv\frac{d^m f(x)}{dx^m}$ being almost everywhere piecewise continuous in $\boldsymbol{R}_+$ except possibly on a non-empty discrete impulsive set IMP. Define a non-impulsive real function $\bar f:\boldsymbol{R}_+\to\boldsymbol{R}$ defined as $\bar f^{(m)}(x)=f^{(m)}(x)$ for $x\in\boldsymbol{R}_+\setminus IMP$, and $f^{(m)}(x_i)=\bar f^{(m)}(x_i)$, $f^{(m)}(x_i)=\bar f^{(m)}(x_i)+K_i\,\delta(0)$ for $x_i\in IMP$ with $\bar f^{(m)}\left(x^+\right)=f^{(m)}(x)$; $x\in IMP$ (defined being bounded arbitrary (for instance, zero) if $x\in IMP$. Through a similar reasoning as that used for Riemann-Liouville fractional integral by replacing the function $f:\boldsymbol{R}_+\to\boldsymbol{R}$ by its m-th derivative, one obtains the following result:

Theorem 4.1. The Caputo fractional derivative of order $\mu\in\boldsymbol{R}_+$ satisfying $m-1<\mu\le m$; $m\in\boldsymbol{Z}_+$ and all $x\in\boldsymbol{R}_+$ is given below:

$$\left(D_*^{\mu} f\right)(x):=\frac{1}{\Gamma(m-\mu)}\int_0^x (x-t)^{m-\mu-1} f^{(m)}(t)dt$$
$$=\frac{1}{\Gamma(m-\mu)}\int_0^x (x-t)^{m-\mu-1}\,\bar f^{(m)}(t)\,dt$$
$$+\frac{1}{\Gamma(m-\mu)}\sum_{i\in I(x)}(x-x_i)^{m-\mu-1}\left(f^{(m-1)}\left(x_i^+\right)-f^{(m-1)}(x_i)\right)\delta(x-x_i)$$

$$=\frac{1}{\Gamma(m-\mu)}\sum_{i\in I(x)\cup\{0\}}\int_{x_i^+}^{x_{i+1}}(x-t)^{m-\mu-1}\,\bar f^{(m)}(t)dt$$
$$+\frac{1}{\Gamma(m-\mu)}\int_{x_{n(x)}^+}^{x}(x-t)^{m-\mu-1}\,\bar f^{(m)}(t)dt$$
$$+\frac{1}{\Gamma(m-\mu)}\sum_{i\in I(x)}(x-x_i)^{m-\mu-1}\left(f^{(m-1)}\left(x_i^+\right)-f^{(m-1)}(x_i)\right)\delta(x-x_i) \quad (18)$$

$$\left(D_*^{\mu} f\right)\left(x^+\right):=\frac{1}{\Gamma(m-\mu)}\int_0^{x^+}(x-t)^{m-\mu-1} f^{(m)}(t)dt$$
$$=\frac{1}{\Gamma(m-\mu)}\int_0^x (x-t)^{m-\mu-1}\,\bar f^{(m)}(t)dt$$
$$+\frac{1}{\Gamma(m-\mu)}\sum_{i\in I(x^+)}(x-x_i)^{m-\mu-1}\left(f^{(m-1)}\left(x_i^+\right)-f^{(m-1)}(x_i)\right)\delta(x-x_i) \quad (19)$$

where $n: IMP\to\boldsymbol{Z}_+$ is a discrete function defined by $n(x)=card\,I(x)=card\,IMP(x)$. □

Note that if $x\in IMP$ then

$$\left(D_*^{\mu} f\right)\left(x^+\right)=$$
$$\frac{1}{\Gamma(m-\mu)}\sum_{i\in I(x^+)\cup\{0\}}\int_{x_i^+}^{x_{i+1}}(x-t)^{m-\mu-1}\,\bar f^{(m)}(t)dt$$
$$+\frac{1}{\Gamma(m-\mu)}\sum_{i\in I(x^+)}(x-x_i)^{m-\mu-1}\left(f^{(m-1)}\left(x_i^+\right)-f^{(m-1)}(x_i)\right)\delta(x-x_i)$$
$$=\left(D_*^{\mu} f\right)(x)+\left(x-x_{n(x)}\right)^{m-\mu-1}\left(f^{(m-1)}\left(x_{n(x)}^+\right)-f^{(m-1)}\left(x_{n(x)}\right)\right)\delta(0)$$
$$\ne\left(D_*^{\mu} f\right)(x)=\frac{1}{\Gamma(m-\mu)}\sum_{i\in I(x)\cup\{0\}}\int_{x_i^+}^{x_{i+1}}(x-t)^{m-\mu-1}\,\bar f^{(m)}(t)dt$$
$$+\frac{1}{\Gamma(m-\mu)}\sum_{i\in I(x)}(x-x_i)^{m-\mu-1}\left(f^{(m-1)}\left(x_i^+\right)-f^{(m-1)}(x_i)\right)\delta(x-x_i)$$

and if $x\notin IMP$, since $I(x^+)=I(x)$, then $\left(D_*^{\mu} f\right)\left(x^+\right)=\left(D_*^{\mu} f\right)(x)$. The above formalism applies when $f^{(m-1)}:\boldsymbol{R}_+\to\boldsymbol{R}$ is piecewise continuous with isolated first- class discontinuity points, that is $f\in PC^{m-1}(\boldsymbol{R}_+,\boldsymbol{R})$ implying that $f\in C^{m-2}(\boldsymbol{R}_+,\boldsymbol{R})$. A more general situation arises when the discontinuities can point-wise arise for points of the function itself of for any successive derivative up- till order m. This would lead to a more general description than that given as follows. Define partial sets of positive integers as $\bar k:=\{1,2,....,k\}$

Assume that $f\in PC^{j}(\boldsymbol{R}_+,\boldsymbol{R})$ and x is a discontinuity point of first class of $f^{(j)}(x)$ for some $j\in\overline{m-1}\cup\{0\}$. Then , $f^{(j+\ell)}(x)$ are impulsive for $\ell\in\overline{m-j}$ of high order being increasing with ℓ. Define the (j+1) – th impulsive sets of the function f on $(0,x)\subset\boldsymbol{R}$ as:

$$IMP_{j+1}(x):=\left\{z\in\boldsymbol{R}_+ : z<x,\ 0<\left|f^{(j)}\left(z^+\right)-f^{(j)}(z)\right|<\infty\right\}\ ;$$
$$j\in\overline{m-1}\cup\{0\},\ x\in\boldsymbol{R}_+ \quad (20)$$

This leads directly the definition of the following impulsive sets:

$$IMP_{j+1}:=\left\{x\in\boldsymbol{R}_+ :\ 0<\left|f^{(j)}\left(x^+\right)-f^{(j)}(x)\right|<\infty\right\}$$
$$\equiv\bigcup_{x\in\boldsymbol{R}_+} IMP_{j+1}(x) \quad (21)$$
$$IMP:=\left\{x\in\boldsymbol{R}_+ :\ 0<\left|f^{(j)}\left(x^+\right)-f^{(j)}(x)\right|<\infty,\ some\ j\in\overline{m-1}\cup\{0\}\right\}$$
$$\equiv\bigcup_{x\in\boldsymbol{R}_+}\left(\bigcup_{j\in\overline{m-1}\cup\{0\}} IMP_{j+1}(x)\right) \quad (22)$$

which can be empty . Thus , if $z\in IMP_{j+1}$ then $f^{(j-1)}\left(x^+\right)=f^{(j-1)}(x)$ exists with identical left and right limits, $f^{(j)}\left(x^+\right)-f^{(j)}(x)=K=K(x)\ne 0$ and $f^{(j)}(x)=K\delta(0)$ with successive higher-order derivatives represented by higher- order Dirac distributional derivatives

The above definitions yield directly the following simple results:

Assertion 5.2. $x\in IMP\Rightarrow x\in IMP_j$ for a unique $j=j(x)\in\overline{m}$.

Proof: Proceed by contradiction. Assume that $x\in\left(IMP_{i+1}\cap IMP_{j+1}\right)$ for $i,j(\ne i)\in\overline{m-1}\cup\{0\}$.Then:

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$0<\left|f^{(i)}\left(x^{+}\right)-f^{(i)}(x)\right|<\infty$;

$0<\left|f^{(j)}\left(x^{+}\right)-f^{(j)}(x)\right|<\infty$

Assume with no loss of generality that $j=i+k>i$ for some $k(\leq m-i-1)\in\boldsymbol{Z}_{+}$. Then,

$$\left|f^{(j)}\left(x^{+}\right)-f^{(j)}(x)\right|=\left|f^{(i+k)}\left(x^{+}\right)-f^{(i+k)}(x)\right|$$
$$=\frac{(-1)^{k}k!}{x^{k}}\left|f^{(i)}\left(x^{+}\right)-f^{(i)}(x)\right|\delta(0)=\infty$$

with $x\in\boldsymbol{R}_{+}$. If $\left|f^{(i)}\left(x^{+}\right)-f^{(i)}(x)\right|\neq 0$ which contradicts $0<\left|f^{(i)}\left(x^{+}\right)-f^{(i)}(x)\right|<\infty$ so that $i=j$. □

Assertion 5.3. $x\in IMP\Rightarrow$

$$\left(x\in IMP_{j}\Leftrightarrow\exists\text{ a unique } j=j(x)=\max_{i\in\overline{m}}\left|f^{(i-1)}\left(x^{+}\right)-f^{(i-1)}(x)\right|<\infty\right)$$

Furthermore, such a unique *j=j(x)* satisfies $\left|f^{(j-1)}\left(x^{+}\right)-f^{(j-1)}(x)\right|>0$.

Proof: The existence is direct by contradiction. If $\neg\exists j=j(x)\in\overline{m-1}\cup\{0\}$ such that $\left|f^{(j)}\left(x^{+}\right)-f^{(j)}(x)\right|<\infty$ then $x\notin IMP$. Now, assume there exist two nonnegative integers $i=i(x)=\left|f^{(i-1)}\left(x^{+}\right)-f^{(i-1)}(x)\right|<\infty$ and $j=j(x)=i+k=\left|f^{(i+k-1)}\left(x^{+}\right)-f^{(i+k-1)}(x)\right|<\infty$; for some $k\in\overline{m-i}$. But for $x>0$,

$$\infty=\frac{(-1)^{k}k!}{x^{k}}\left|f^{(i-1)}\left(x^{+}\right)-f^{(i-1)}(x)\right|\delta(0)$$
$$=\left|f^{(i+k-1)}\left(x^{+}\right)-f^{(i+k-1)}(x)\right|<\infty$$

which is a contradiction. Then,

$$x\in IMP_{j}\Rightarrow\exists\ j=j(x)=\max_{i\in\overline{m}}\left|f^{(i-1)}\left(x^{+}\right)-f^{(i-1)}(x)\right|<\infty$$

which is unique. Also, from the definition of the impulsive sets $IMP_{i}(x)$,

$$\left|f^{(j-1)}\left(x^{+}\right)-f^{(j-1)}(x)\right|<\infty\Rightarrow x\in\bigcup_{i\in\bar{j}\cup\{0\}}IMP_{i}(x)$$

Now, assume that $x\in\bigcup_{i\in\overline{j-1}\cup\{0\}}IMP_{i}(x)$. Thus, $0<\left|f^{(j-1)}\left(x^{+}\right)-f^{(j-1)}(x)\right|<\infty\Rightarrow\left|f^{(j)}\left(x^{+}\right)-f^{(j)}(x)\right|=\infty$ from the definition of the impulsive sets. Thus, $x\in IMP_{j}(x)$. The contrary logic implication

$$j=j(x)=\max_{i\in\overline{m}}\left|f^{(i-1)}\left(x^{+}\right)-f^{(i-1)}(x)\right|<\infty\Rightarrow x\in IMP_{j}$$

is proved. Then, it has been fully proved that $x\in IMP\Rightarrow$

$$\left(x\in IMP_{j}\Leftrightarrow\exists\text{ a unique } j=j(x)=\max_{i\in\overline{m}}\left|f^{(i-1)}\left(x^{+}\right)-f^{(i-1)}(x)\right|<\infty\right)$$

Now, establish again a contradiction by assuming that

$$j=j(x)=\left|f^{(k-1)}\left(x^{+}\right)-f^{(k-1)}(x)\right|=\max$$

$$\left|f^{(i-1)}\left(x^{+}\right)-f^{(i-1)}(x)\right|=0<\infty;\ \forall k\in\overline{m}$$

what contradicts $x\in IMP$. This proves that the unique *j=j(x)* implying and being implied by $x\in IMP_{j}$ satisfies $\left|f^{(j-1)}\left(x^{+}\right)-f^{(j-1)}(x)\right|>0$. □

Using the necessary – high order distributional derivatives, one gets that

$$x\in IMP\Rightarrow f^{(m)}(x)=\frac{(-1)^{m-j}(m-j)!}{x^{m-j}}\left(f^{(j)}\left(x^{+}\right)-f^{(j)}(x)\right)\delta(0)$$

; with $j\in\overline{m-1}\cup\{0\}$ being uniquely defined so that $0<\left|f^{(j)}\left(x^{+}\right)-f^{(j)}(x)\right|<\infty$. Thus, the m-th distributional derivative of $f:\boldsymbol{R}_{+}\to\boldsymbol{R}$ can be represented as:

$$f^{(m)}(x)=\bar{f}^{(m)}(x)$$
$$+\sum_{x_{i}\in IMP_{j_{i+1}}}\frac{(-1)^{j_{i}}(m-j_{i})!}{x_{i}^{m-j_{i}}}\left(f^{(j_{i})}\left(x_{i}^{+}\right)-f^{(j_{i})}(x_{i})\right)\delta(x-x_{i})$$

, $x\in\boldsymbol{R}_{+}$

with $j_{i}=j_{i}(x_{i})$ being uniquely defied for each $x_{i}\in IMP$ so that $x_{i}\in IMP_{j_{i}}$, where $\bar{f}\in C^{m-1}(\boldsymbol{R}_{+},\boldsymbol{R})$ with everywhere continuous first-derivative defined as $\bar{f}^{(j)}(x)=f^{(j)}(x)$; $x\in\boldsymbol{R}_{+}$, $\bar{f}(0)=f(0)$. The above formula is applicable if $f\notin PC^{m}(\boldsymbol{R}_{+},\boldsymbol{R})$ but it is also applicable if $f\in PC^{m}(\boldsymbol{R}_{+},\boldsymbol{R})$ yielding:

$f^{(m)}\left(x^{+}\right)=f^{(m)}(x)=\bar{f}^{(m)}(x)$ if $x\notin IMP$

$$f^{(m)}(x)=\bar{f}^{(m)}(x)$$
$$f^{(m)}\left(x^{+}\right)=f^{(m)}(x)+\frac{(-1)^{m-j}(m-j)!}{x^{m-j}}\left(f^{(j)}\left(x^{+}\right)-f^{(j)}(x)\right)\delta(0)$$

if $x\in IMP$

$$f^{(m-1)}(x)=\bar{f}^{(m-1)}(x)$$
$$f^{(m-1)}\left(x^{+}\right)=f^{(m-1)}(x)+\frac{(-1)^{m-j}(m-1-j)!}{x^{m-1-j}}\left(f^{(j)}\left(x^{+}\right)-f^{(j)}(x)\right)\delta(0)$$

if $x\in IMP$ and $j<m-1$

$$f^{(m-1)}(x)=\bar{f}^{(m-1)}(x)$$

$f^{(m-1)}\left(x^{+}\right)=f^{(m-1)}(x)+\left(f^{(m-1)}\left(x^{+}\right)-f^{(m-1)}(x)\right)$ if $x\in IMP$ and $j=m-1$

for a unique $j=j(x)\in\overline{m-1}\cup\{0\}$ from Assertion 1. Denote further sets related to impulses as follows:

$IMP(x):=\{z\in IMP:z<x\}$; $IMP\left(x^{+}\right):=\{z\in IMP:z\leq x\}$; $\forall x\in\boldsymbol{R}_{+}$

being indexed by two subsets of integers of the same corresponding cardinals defined by:

$I(x)=\bar{j}=\overline{j(x)}$ indexing the members z_i of $IMP(x)$ in increasing order

$I(x^+)$, being either $I(x)$ or $I(x)+1$, indexing the members z_i of $IMP(x^+)$ in increasing order

The following result holds:

Theorem 5.4. The Caputo fractional derivative of $f:\boldsymbol{R}_+\to\boldsymbol{R}$ of order $\mu\in\boldsymbol{R}_+$ satisfying $m-1<\mu\le m$; $m\in\boldsymbol{Z}_+$ and all $x\in\boldsymbol{R}_+$ is after using distributional derivatives becomes in the most general case:

$$\left(D_*^{\mu} f\right)(x):=\frac{1}{\Gamma(m-\mu)}\int_0^x (x-t)^{m-\mu-1} f^{(m)}(t)dt$$

$$=\frac{1}{\Gamma(m-\mu)}\left(\int_0^x (x-t)^{m-\mu-1}\bar{f}^{(m)}(t)\,dt\right.$$
$$+\sum_{i\in I(x)}(-1)^{m-j(x_i)-1}(x-x_i)^{m-\mu-1}$$
$$\left.\frac{(m-j(x_i)-1)!}{(x-x_i)^{m-j(x_i)-1}}\left(f^{(j(x_i))}(x_i^+)-f^{(j(x_i))}(x_i)\right)\hat{\delta}(x-x_i)\right)$$

$$=\frac{1}{\Gamma(m-\mu)}\sum_{i\in I(x)\cup\{0\}}\int_{x_i^+}^{x_{i+1}}(x-t)^{m-\mu-1}\bar{f}^{(m)}(t)dt$$
$$+\frac{1}{\Gamma(m-\mu)}\int_{x_{n(x)}^+}^{x}(x-t)^{m-\mu-1}\bar{f}^{(m)}(t)dt$$
$$+\frac{1}{\Gamma(m-\mu)}\sum_{i\in I(x)}(-1)^{m-j(x_i)-1}(x-x_i)^{m-\mu-1}$$
$$\times\frac{(m-j(x_i)-1)!}{(x-x_i)^{m-j(x_i)-1}}\left(f^{(j(x_i))}(x_i^+)-f^{(j(x_i))}(x_i)\right) \quad (23)$$

$$\left(D_*^{\mu} f\right)(x^+):=\frac{1}{\Gamma(m-\mu)}\int_0^{x^+}(x-t)^{m-\mu-1} f^{(m)}(t)dt$$
$$=\frac{1}{\Gamma(m-\mu)}\int_0^{x}(x-t)^{m-\mu-1}\bar{f}^{(m)}(t)dt$$
$$+\frac{1}{\Gamma(m-\mu)}\sum_{i\in I(x^+)}(-1)^{m-j(x_i)-1}(x-x_i)^{m-\mu-1}$$
$$\times\frac{(m-j(x_i)-1)!}{(x-x_i)^{m-j(x_i)-1}}\left(f^{(j(x_i))}(x_i^+)-f^{(j(x_i))}(x_i)\right)$$

$$=\frac{1}{\Gamma(m-\mu)}\sum_{i\in I(x^+)\cup\{0\}}\int_{x_i^+}^{x_{i+1}}(x-t)^{m-\mu-1}\bar{f}^{(m)}(t)dt$$
$$+\frac{1}{\Gamma(m-\mu)}\sum_{i\in I(x^+)}(-1)^{m-j(x_i)-1}(x-x_i)^{m-\mu-1}$$
$$\times\frac{(m-j(x_i)-1)!}{(x-x_i)^{m-j(x_i)-1}}\left(f^{(j(x_i))}(x_i^+)-f^{(j(x_i))}(x_i)\right) \quad (24)$$

□

Note that $\left|\left(D_*^{\mu} f\right)(x^+)\right|=\infty$ if $x=x_i\in IMP$, as expected.

V. CONCLUDING REMARKS

This manuscript has investigated some simple formulas for Riemann-Liouville impulsive fractional integral calculus and for Riemann-Liouville and Caputo impulsive fractional derivatives. It can be asserted that the method of fractional calculus is promising to be applied for some applied problems of dynamic systems, in particular, those involving delays. See, for instance, [6-9] which can be potentially reformulated accordingly to the formalism described in this manuscript.

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