Proceedings of the World Congress on Engineering 2011 Vol I
WCE 2011, July 6 - 8, 2011, London, U.K.

On Partial Fraction Decomposition of Rational Functions with Irreducible Quadratic Factors in the Denominators

Yiu-Kwong Man

Abstract—**In this paper, we present a new approach to partial fraction decomposition of rational functions with irreducible quadratic factors in the denominators. It improves the Heaviside's cover-up technique to handle this type of problem via polynomial divisions and substitutions only, with no need to solve for the complex roots of the irreducible quadratic polynomial involved, to use differentiation or to solve a system of linear equations. Some examples of its applications in engineering mathematics are included.**

***Index Terms*—partial fraction decomposition, the Heaviside's cover-up technique, inverse Laplace transforms, linear differential equations.**

I. Introduction

The problem of finding the partial fraction decomposition of rational functions is often encountered in the study of calculus, differential equations and certain topics of applied mathematics. One common approach is to use the method of undetermined coefficients to find the unknown coefficients of the partial fractions. Another approach is to use the Heaviside's cover-up technique, which uses substitutions to determine the unknown coefficients of the partial fractions with single poles, and successive differentiations to tackle those with multiple poles (see [1]). In [5], an improved Heaviside approach is proposed to handle the latter case by using substitutions and polynomial divisions only. However, the drawback is that this approach cannot be effectively applied to rational functions with only quadratic factors in the denominators. To fill-up this gap, we now introduce a further improved Heaviside approach in this paper, which can be used for finding the partial fraction decomposition of rational functions with irreducible quadratic factors in the denominators. Again, the unknown coefficients of the partial fractions can be determined by polynomial divisions and substitutions only, with no need to solve for the complex roots of the quadratic polynomial involved, to use differentiation or to solve a system of linear equations. Examples of its applications in some topics of engineering mathematics, such as inverse Laplace transforms and linear differential equations are included.

Manuscript submitted for review in March 2011. This work is supported partially by the HKIEd's Internal Research Grant.

The author is an Associate Professor in the Department of Mathematics and Information Technology, The Hong Kong Institute of Education, 10 Lo Ping Road, Tai Po, New Territories, Hong Kong (e-mail: ykman@ied.edu.hk).

II. Theoretical Background

The existence of partial fraction expansion of a given proper rational function is based on the following result and the discussion of its proof can refer to [4].

Theorem 2.1. Let F be a constant field and $a(x)$ and $b(x)$ be polynomials in $F[x]$ such that deg $a(x)$ < deg $b(x)$ and

$$b(x) = (x-\alpha_1)^{n_1}\cdots(x-\alpha_s)^{n_s}(x^2+b_1x+c_1)^{m_1}\cdots(x^2+b_rx+c_r)^{m_r}$$

where $n_1,\cdots,n_s,m_1,\cdots,m_r$ are positive integers and $\alpha_1,\cdots,\alpha_s,b_1,\cdots,b_r,c_1,\cdots,c_r,\in F$. Then $a(x)/b(x)$ has a unique partial fraction expansion of the form

$$\frac{a(x)}{b(x)} = \sum_{i=1}^{s}\left(\frac{a_{i,1}}{(x-\alpha_i)} + \frac{a_{i,2}}{(x-\alpha_i)^2} + \cdots + \frac{a_{i,n_i}}{(x-\alpha_i)^{n_i}}\right) + \sum_{i=1}^{r}\left(\frac{b_{i,1}x+c_{i,1}}{(x^2+b_ix+c_i)} + \frac{b_{i,2}x+c_{i,2}}{(x^2+b_ix+c_i)^2} + \cdots + \frac{b_{i,m_i}x+c_{i,m_i}}{(x^2+b_ix+c_i)^{m_i}}\right) \quad (1)$$

where $a_{i,j}$, $b_{i,j}$, $c_{i,j} \in F$.

Now, we proceed to describe a simple algorithm for computing the coefficients $a_{i,j}, b_{i,j}, c_{i,j}$ in the theorem below.

Theorem 2.2. The unknown coefficients in Theorem 2.1 can be found by the following method:

Case(I): Partial fractions with linear factors in the denominators.

$$a_{i,n_i} = \left.\frac{a(x)}{b(x)}.(x-\alpha_i)^{n_i}\right|_{x=\alpha_i};$$

$$a_{i,n_i-j} = \left.\left[\frac{a(x)}{b(x)} - \sum_{k=0}^{j-1}\frac{a_{i,n_i-k}}{(x-\alpha_i)^{n_i-k}}\right](x-\alpha_i)^{n_i-j}\right|_{x=\alpha_i},$$

where $1 \le i \le s$ and $1 \le j \le n_i - 1$.

Case(II): Partial fractions with quadratic factors in the denominators.

$$b_{i,m_i}x + c_{i,m_i} = \left.\frac{a(x)}{b(x)}.(x^2+b_ix+c_i)^{m_i}\right|_{x^2=-b_ix-c_i};$$

$$b_{i,m_i-j}x + c_{i,m_i-j}$$

$$= \left.\left[\frac{a(x)}{b(x)} - \sum_{k=0}^{j-1}\frac{b_{i,m_i-k}x + c_{i,m_i-k}}{(x^2+b_ix+c_i)^{m_i-k}}\right](x^2+b_ix+c_i)^{m_i-j}\right|_{x^2=-b_ix-c_i},$$

where $1 \le i \le r$ and $1 \le j \le m_i - 1$.

ISBN: 978-988-18210-6-5
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

Proof. Case (I): By multiplying $(x-\alpha_i)^{n_i}$ to both sides of (1), we can eliminate the same factor in $b(x)$ and then include $(x-\alpha_i)$ or its higher power to each partial fraction, except the constant term a_{i,n_i}. Hence, a_{i,n_i} can be found by substituting $x=\alpha_i$ into the equation. In other words, we obtain

$$a_{i,n_i} = \frac{a(x)}{b(x)}.(x-\alpha_i)^{n_i}\bigg|_{x=\alpha_i}.$$

This technique is called *cover-up* in the original Heaviside's approach. Now, let us consider the function

$$\frac{a_{i,1}(x)}{b_{i,1}(x)} = \frac{a(x)}{b(x)} - \frac{a_{i,n_i}}{(x-\alpha_i)^{n_i}}.$$

Due to the uniqueness of the partial fraction expansion of a given rational function, the highest power of $(x-\alpha_i)$ in $b_{i,1}(x)$ is n_i-1. It means we can eliminate the common factor $(x-\alpha_i)$ when simplifying $a_{i,1}(x)/b_{i,1}(x)$. In other words, the multiplicity of the pole α_i is reduced by one via polynomial division. By using the cover-up technique again, we can find a_{i,n_i-1} as follows:

$$a_{i,n_i-1} = \frac{a_{i,1}(x)}{b_{i,1}(x)}.(x-\alpha_i)^{n_i-1}\bigg|_{x=\alpha_i}.$$

Similarly, we can consider the functions

$$\frac{a_{i,j}(x)}{b_{i,j}(x)} = \frac{a(x)}{b(x)} - \sum_{k=0}^{j-1}\frac{a_{i,n_i-k}}{(x-\alpha_i)^{n_i-k}}$$

and then repeat the above arguments to find a_{i,n_i-j} by the following formula

$$a_{i,n_i-j} = \frac{a_{i,j}(x)}{b_{i,j}(x)}.(x-\alpha_i)^{n_i-j}\bigg|_{x=\alpha_i},$$

where $1\le i\le s$ and $1\le j\le n_i-1$.

Case (II): By multiplying $(x^2+b_ix+c_i)^{m_i}$ to both sides of (1), we can eliminate the same factor in $b(x)$. Then, $(x^2+b_ix+c_i)$ or its higher power will be included to each partial fraction, except the term $b_{i,m_i}x+c_{i,m_i}$, which can be determined by the substitution below

$$b_{i,m_i}x+c_{i,m_i} = \frac{a(x)}{b(x)}.(x^2+b_ix+c_i)^{m_i}\bigg|_{x^2=-b_ix-c_i};$$

If the rational function on the right hand side is not in the form of a linear polynomial, we can use polynomial division to express its numerator and denominator in the forms

$$(x^2+b_ix+c_i)q_{i,1}(x)+(u_{i,1}x+v_{i,1}),$$

and $(x^2+b_ix+c_i)q_{i,2}(x)+(u_{i,2}x+v_{i,2})$, respectively.

Then, we can use the equation $x^2=-b_ix-c_i$ to eliminate $(x^2+b_ix+c_i)q_{i,1}(x)$ and $(x^2+b_ix+c_i)q_{i,2}(x)$ so that only the linear parts $u_{i,1}x+v_{i,1}$ and $u_{i,2}x+v_{i,2}$ are left. By multiplying p_ix+q_i to each one of them, where $p_i=1/u_{i,2}$ and $q_i=[b_i-(v_{i,2}/u_{i,2})]/u_{i,2}$, we can use $x^2=-b_ix-c_i$ again to reduce $(u_{i,2}x+v_{i,2})(p_ix+q_i)$ and $(u_{i,1}x+v_{i,1})(p_ix+q_i)$ into a constant and a linear polynomial, respectively. Thus, the term $b_{i,m_i}x+c_{i,m_i}$ is totally determined. Now, let us consider the function

$$\frac{a_{i,1}(x)}{b_{i,1}(x)} = \frac{a(x)}{b(x)} - \frac{b_{i,m_i}x+c_{i,m_i}}{(x^2+b_ix+c_i)^{m_i}}.$$

Due to the uniqueness of the partial fraction expansion of a given rational function, the highest power of $(x^2+b_ix+c_i)$ in $b_{i,1}(x)$ is m_i-1. So, we can eliminate the common factor $(x^2+b_ix+c_i)$ when simplifying $a_{i,1}(x)/b_{i,1}(x)$ and the power of $(x^2+b_ix+c_i)$ is reduced by one. By using similar arguments described above and the cover-up technique repeatedly, we can determine $b_{i,m_i-j}x+c_{i,m_i-j}$ by the formula below:

$$b_{i,m_i-j}x+c_{i,m_i-j} = \left[\frac{a(x)}{b(x)} - \sum_{k=0}^{j-1}\frac{b_{i,m_i-k}x+c_{i,m_i-k}}{(x^2+b_ix+c_i)^{m_i-k}}\right](x^2+b_ix+c_i)^{m_i-j}\bigg|_{x^2=-b_ix-c_i},$$

where $1\le i\le r$ and $1\le j\le m_i-1$. It completes the proof.

III. Examples

We now provide some examples of this new approach, including its applications in certain topics of engineering mathematics, such as inverse Laplace transforms and linear differential equations (see [2], [3], [6]).

Example 3.1 Find the partial fraction expansion of the rational function $(5x^3+9x-4)/(x(x-1)(x^2+4))$.

Solution. Let $F(x)=\dfrac{5x^3+9x-4}{x(x-1)(x^2+4)}=\dfrac{a}{x}+\dfrac{b}{x-1}+\dfrac{cx+d}{x^2+4}$,

where *a*, *b*, *c*, *d* are unknown constants to be determined. Using the cover-up technique, we have

$$a = xF(x)\big|_{x=0} = \frac{-4}{(-1)(4)} = 1$$

$$b = (x-1)F(x)\big|_{x=1} = \frac{5+9-4}{5} = 2$$

$$cx+d = (x^2+4)F(x)\big|_{x^2=-4} = \frac{5x^3+9x-4}{x(x-1)}\bigg|_{x^2=-4} = \frac{-20x+9x-4}{-4-x}\bigg|_{x^2=-4}$$

$$= \frac{(11x+4)(x-4)}{(x+4)(x-4)}\bigg|_{x^2=-4} = \frac{-44-40x-16}{-4-16}\bigg|_{x^2=-4} = 2x+3.$$

Hence, $F(x)=\dfrac{1}{x}+\dfrac{2}{x-1}+\dfrac{2x+3}{x^2+4}$.

Example 3.2 Find the partial fraction expansion of the rational function $(3x^3+2x^2+5x+2)/((x^2+1)(x^2+x+1))$.

Solution.

Let $F(x)=\dfrac{3x^3+2x^2+5x+2}{(x^2+1)(x^2+x+1)}=\dfrac{ax+b}{x^2+1}+\dfrac{cx+d}{x^2+x+1}$, where *a*, *b*, *c*, *d* are unknown constants to be determined. Using the cover-up technique, we have

$$ax+b = (x^2+1)F(x)\big|_{x^2=-1} = \frac{3x^3+2x^2+5x+2}{x^2+x+1}\bigg|_{x^2=-1} = \frac{-3x-2+5x+2}{-1+x+1}\bigg|_{x^2=-1} = 2.$$

$$cx+d = (x^2+x+1)F(x)\big|_{x^2=-x-1} = \frac{3x^3+2x^2+5x+2}{x^2+1}\bigg|_{x^2=-x-1} = \frac{3-2(x+1)+5x+2}{-x}\bigg|_{x^2=-x-1}$$

$$= \frac{(3x+3)(x+1)}{-x(x+1)}\bigg|_{x^2=-x-1} = (3x^2+6x+3)\big|_{x^2=-x-1} = -3x-3+6x+3 = 3x.$$

ISBN: 978-988-18210-6-5
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

Hence, $F(x)=\dfrac{2}{x^2+1}+\dfrac{3x}{x^2+x+1}.$

Example 3.3 Find the partial fraction expansion of the rational function $(x^3+3x^2+11x+1)/(x^2+3x+9)^2$.

Solution.

Let $F(x)=\dfrac{x^3+3x^2+11x+1}{(x^2+3x+9)^2}=\dfrac{ax+b}{x^2+3x+9}+\dfrac{cx+d}{(x^2+3x+9)^2}$,

where *a*, *b*, *c*, *d* are unknown constants to be determined. Using the cover-up technique, we have

$$cx+d=(x^2+3x+9)^2F(x)\Big|_{x^2=-3x-9}=(x^3+3x^2+11x+1)\Big|_{x^2=-3x-9}=2x+1.$$

$$ax+b=(x^2+3x+9)\Big(F(x)-\frac{2x+1}{(x^2+3x+9)^2}\Big)\Bigg|_{x^2=-3x-9}=\frac{x^3+3x^2+9x}{(x^2+3x+9)}\Bigg|_{x^2=-3x-9}=x.$$

Hence, $F(x)=\dfrac{x}{x^2+3x+9}+\dfrac{2x+1}{(x^2+3x+9)^2}.$

Example 3.4 Find the inverse Laplace transform of the rational function $1/((s^2+9)(s^2+4))$.

Solution.

Let $F(s)=\dfrac{1}{(s^2+9)(s^2+4)}=\dfrac{as+b}{s^2+9}+\dfrac{cs+d}{s^2+4}$, where *a*, *b*, *c*, *d* are unknown constants to be determined. Using the cover-up technique, we have

$$as+b=(s^2+9)F(s)\Big|_{s^2=-9}=\frac{1}{s^2+4}\Bigg|_{s^2=-9}=\frac{-1}{5},$$

$$cs+d=(s^2+4)F(s)\Big|_{s^2=-4}=\frac{1}{(s^2+9)}\Bigg|_{s^2=-4}=\frac{1}{5}.$$

Hence, $F(s)=\dfrac{-1}{5(s^2+9)}+\dfrac{1}{5(s^2+4)}$ and the inverse Laplace transform is given by $L^{-1}(F(s))=\dfrac{-1}{15}\sin 3t+\dfrac{1}{10}\sin 2t.$

Example 3.5 Solve the differential equation $y''+y=e^t$, such that $y(0)=y'(0)=0$ and $y''(0)=1$.

Solution. Let $F(s)=L[y(t)]$. Applying Laplace transform, we have $(s^2+1)F(s)=1/(s-1)$.

Let $F(s)=\dfrac{1}{(s-1)(s^2+1)}=\dfrac{as+b}{s-1}+\dfrac{cs+d}{s^2+1}$, where *a*, *b*, *c*, *d* are unknown constants to be determined. Using the cover-up technique, we have

$$as+b=(s-1)F(s)\Big|_{s=1}=\frac{1}{s^2+1}\Bigg|_{s=1}=\frac{1}{2},$$

$$cs+d=(s^2+1)F(s)\Big|_{s^2=-1}=\frac{1}{(s-1)}\Bigg|_{s^2=-1}=\frac{s+1}{(s-1)(s+1)}\Bigg|_{s^2=-1}=-\left(\frac{s+1}{2}\right).$$

Hence, $F(s)=\dfrac{1}{2(s-1)}-\dfrac{s+1}{2(s^2+1)}$. Applying the inverse Laplace transform, we obtain $y(t)=\dfrac{1}{2}(e^t-\cos t-\sin t)$, which satisfies $y(0)=y'(0)=0$ and $y''(0)=1$.

IV. Concluding Remarks

In this paper, we have introduced a simple approach for computing the partial fraction decompositions of rational functions with irreducible quadratic factors in the denominators. By using this method, there is no need to solve for the complex roots of the quadratic polynomials involved, to use differentiation or to solve a system of linear equations. Due to its simplicity and useful applications in applied and engineering mathematics, this method can introduced to higher school or undergraduate students, as an alternative to those classic techniques mentioned in most mathematics textbooks.

References

[1] L. B. Norman, *Discrete Mathematics*, 2nd ed. New York: Oxford University Press, 1990, pp. 403–407.
[2] E. Kreyszig, *Advanced Engineering Mathematics*, New York: John Wiley & Sons, 1999, pp. 284–293.
[3] D. W. Jordan, P. Smith, *Mathematical Techniques: An Introduction for the Engineering,* Oxford: Oxford University Press, 2002, pp. 448–464.
[4] J.S. Cohen, *Computer Algebra and Symbolic Computation: Mathematical Methods*, Massachusetts: A K Peters, 2003, pp. 166–173.
[5] Y.K.Man, “A simple algorithm for computing partial fraction expansions with multiple poles,” *International Journal of Mathematical Education in Science & Technology*, 38, 2007, pp. 247–251.
[6] Y. K.Man, “Partial Fraction Decomposition by Synthetic Division and Its Applications,” in *Current Themes in Engineering Science*, New York: American Institute of Physics, 2009, pp. 71–82,.