Buckling Analysis of Nonuniform Nanowires under Axial Compression

Haw-Long Lee, Ruo-Ping Chang, and Win-Jin Chang

Abstract—**The buckling behavior of a nonuniform nanowire under axial compression is studied using the nonlocal elasticity theory. We obtain an analytical expression for calculating the critical buckling load of the nanowire. The expression can be simplified for the special case of the nanowire with uniform cross section. Based on the simplified equation, the analyzed result agrees with the previous work. In addition, the influences of nanolocal and surface effects on the critical buckling load of the nonuniform nanowire are analyzed by using the Rayleigh-Ritz method. According to the analyzed result, it can be found that the critical buckling load decreases with an increase of nonlocal parameter. Furthermore, the surface effects are more significant for a slender nanowire with a higher diameter ratio.**

KEYWORDS: Surface effect; Nonlocal effect; nanowire; Buckling analysis.

I. INTRODUCTION

For nanostructure materials such as nanowires, surface effects become important and can influence their physical and chemical properties because of the high surface-to-volume ratio. In the recent years, much theoretical and experimental work has been done in recent years to investigate the surface effects on nanowires.[1-5] For example, Jing et al. [1] measured the elastic properties of silver nanowires with outer diameters ranging from 20 to 140 nm using contact atomic force microscopy and found that the size dependence of the apparent Young modulus of the nanowires was attributed to the surface effect. He and Lilley [3] studied the surface effects on the elastic behavior of static bending nanowires using the Euler beam theory.

However, significant increase in theoretical research is due to the fact that the experimental study of surface effects on the nanoscale is still difficult. Recently, Zheng et al. [6] utilized the core–shell model to study the surface effect on the elastic property of nanowires and found that the influence of surface elasticity on the elastic moduli can be well characterized by two dimensionless material and geometric parameters. Song et al. [7] developed a high-order continuum model to investigate high-frequency wave propagation in nanowires with surface effects

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In addition, some researchers studied the buckling properties of nanowires. For example, Wang and Feng [8] derived an analytical expression for the axial buckling of nanowires with consideration of surface effects and analyzed the buckling behavior of nanowires subjected to surface elasticity and residual surface tension. Xiao et al. [9] established continuum mechanics theory to analyze the in-surface buckling of one-dimensional nanomaterials on compliant substrates. They found that the energy for in-surface buckling is lower than that for normal buckling.

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II. ANALYSIS

A cantilever nanowire with nonuniform cross-section is subjected to a uniaxial compression P as depicted in Fig. 1. The maximum diameter at the fixed end is D_0 and the minimum diameter at the free end is D_1. The nanowire diameter is linearly varied with its length L and has the Young's modulus E and volume density ρ. In this work, the surface and small scale effects on the buckling of the nanowire are studied by using the nonlocal elastic theory. The analysis for the surface effects is considered as a uniform surface layer with infinitesimal thickness.[18]

The buckling equation based on the Euler beam theory for a nanowire is expressed by

$$V = (P - H)\frac{dY}{dX} \tag{1}$$

$$V = \frac{dM}{dX} \tag{2}$$

where Y is transverse displacement depends on the spatial coordinate along the longitudinal axis X; M is the resultant bending moment; V is the is the resultant shear force, H is the constant parameter which is determined by the residual surface tension, and P is the axial compression force.

The nonlocal constitutive relations for one-dimensional case can be written as [13]

$$M - (e_0 a)^2 \frac{d^2 M}{dX^2} = -E^* I^* \frac{d^2 Y}{dX^2} \tag{3}$$

where $E^* I^*$ is the effective flexural rigidity which includes the surface bending elasticity and bending rigidity [8], and $e_0 a$ is the nonlocal constant which is used to modify the classical elasticity theory and is limited to apply to a device on the nanometer scale.

Using Eqs.(1)-(3), the nonlocal bending moment M can be expressed as:

$$M = -E^* I^* \frac{d^2 Y}{dX^2} + (e_0 a)^2 \frac{d}{dX}[(P - H)\frac{dY}{dX}] \tag{4}$$

Therefore, the governing equation of buckling for the nonuniform nanowire with consideration of both surface and nonlocal effects can be expressed as

$$\frac{d^2}{dX^2}(E^*I^*\frac{d^2Y}{dX^2})+[1-(ea_0)^2\frac{d^2}{dX^2}][\frac{d}{dX}(P-H)\frac{dY}{dX}]=0 \tag{5}$$

The diameter $D(X)$ of the nanowire varies linearly along the longitudinal axis X. Here, the parameters $E^*I^*(X)$ and $H(X)$ are defined as [8]

$$E^*I^*(X)=\frac{E}{64}\pi D(X)^4+\frac{E^S}{8}\pi D(X)^3 \tag{6}$$

$$H(X)=2\tau D(X) \tag{7}$$

where E^S and τ are the surface elasticity modulus and residual surface tension of the nanowire per length, respectively.

The corresponding boundary conditions are

$$Y(0)=0 \tag{8}$$

$$\frac{dY(0)}{dX}=0 \tag{9}$$

$$-E^*I^*(L)\frac{d^2Y(L)}{dX^2}+(e_0a)^2\frac{d}{dX}[(P-H(L))\frac{d^2Y(L)}{dX^2}]=0 \tag{10}$$

$$P-H(L)\frac{dY(L)}{dX}=0 \tag{11}$$

The nanowire is fixed at the end of $X = 0$, then boundary conditions given by Eqs. (8) and (9) correspond to conditions of zero displacement and zero slope at $X = 0$. Moreover, the Eqs. (10) and (11) correspond to zero moment and shear force at $X = L$, respectively.

The dimensionless variables are defined as follows:

$$x=\frac{X}{L},\quad y=\frac{Y}{L},\quad \alpha(x)=\frac{D(x)}{D_0},\ \varepsilon=\frac{e_0a}{L}$$

$$\xi(x)=\frac{E^*I^*(x)}{EI_0},\ h(x)=\frac{H(x)L^2}{EI_0}\quad p=\frac{PL^2}{EI_0} \tag{12}$$

where D_0 is the maximum diameter of the nonuniform nanowire at the fixed end, therefore, the flexural rigidity EI_0 of the nanowire at the fixed end should be $\pi ED_0^4/64$.

Meanwhile, $\alpha(x)$, ε , $\xi(x)$, $h(x)$ and p denote the diameter ratio, dimensionless nonlocal parameter, dimensionless flexural rigidity, dimensionless residual surface tension, and dimensionless buckling load, respectively.

Using the dimensionless variables given by (12), the governing equations and associated boundary conditions can be simplified to the following dimensionless forms:

$$\frac{d^2}{dx^2}[\xi(x)\frac{d^2y}{dx^2}]+(1-\varepsilon^2\frac{d^2}{dx^2})[\frac{d}{dx}(p-h(x)\frac{dy}{dx}]=0 \tag{13}$$

$$y(0)=0 \tag{14}$$

$$\frac{dy(0)}{dx}=0 \tag{15}$$

$$\xi\frac{d^2y(1)}{dx^2}-\varepsilon^2\frac{d}{dx}[(p-h(1))\frac{dy(1)}{dx}]=0 \tag{16}$$

$$(p-h(1))\frac{dy(1)}{dx}=0 \tag{17}$$

Since the parameters $\alpha(x)$, $\xi(x)$ and $h(x)$ are dependent on the position x along the nanowire, the method of Rayleigh-Ritz is used to determine the critical buckling load. In order to solve Eqs.(13)-(17) by the Rayleigh–Ritz method, we set

$$y=\sum_{i=1}^{n}u_i\gamma_i(x) \tag{18}$$

where u_i are constants, and $\gamma_i(x)$ is the admissible function which required to satisfy the geometric boundary conditions, but need not satisfy the natural boundary conditions, In this present work, $\gamma_i(x)=x^{i+1},\quad i=1,2,3,...,10$ was chosen. Then, substituting Eqs. (18) into Eqs.(13)-(17) and applying the Rayleigh quotient, the following eigenvalue problem can be obtained:

$$Ku=pMu \tag{19}$$

where u is the eigenvector of expansion coefficients, and

$$K_{ij}=\int_0^1[\frac{d^2\gamma_j}{dx^2}(\xi(x)\frac{d^2\gamma_i(x)}{dx^2}+\varepsilon^2\frac{d}{dx}(h(x)\frac{d\gamma_i(x)}{dx})+h(x)\frac{d\gamma_j(x)}{dx}\frac{d\gamma_i(x)}{dx}]dx$$
$$-h(1)\gamma_i(1)\frac{d\gamma_i(1)}{dx} \tag{20}$$

$$M_{ij}=\int[\varepsilon^2\frac{d^2\gamma_j(x)}{dx^2}\frac{d^2\gamma_i(x)}{dx^2}+\frac{d\gamma_j(x)}{dx}\frac{d\gamma_i(x)}{dx}]dx-\gamma_j(1)\frac{d\gamma_i(1)}{dx} \tag{21}$$

Furthermore, $\xi(x)$ and $h(x)$ given by Eq. (12) can be expressed in terms of $\alpha(x)$ as

$$\xi(x)=\alpha(x)^4+\beta\alpha(x)^3 \tag{22}$$

$$h(x)=\gamma(\frac{L}{D_0})^2\alpha(x) \tag{23}$$

where $\beta=\frac{8E^S}{ED_0}$, $\gamma=\frac{128\tau}{\pi ED_0}$. (24)

Solving Eq.(19) and using the dimensionless variables given by Eq. (12), the critical buckling load can be expressed as

$$P_{cr}=\frac{pEI_0}{L^2} \tag{25}$$

In case of setting $\alpha(x)=1$ and $\varepsilon=0$ in the above analysis, it implies that a uniform nanowire with surface effects but without nonlocal effect is assumed. For the case, the critical buckling load of a uniform nanowire can be obtained based on the analysis, and the result can also be yielded from the previous study [8]. In addition, the critical buckling load of a uniform nanowire without both surface and nonlocal effects can be written as [19]

$$P_{cr}^0=\frac{\pi^2EI_0}{4L^2} \tag{26}$$

III. Results and Discussion

In order to examine the expression derived as above, we compare this work with the previous study, the following surface properties are used in the analysis: τ =0.89 N m and E^s =1.22 N m. [8] It implies the nanowire is a uniform cross-section when the value of diameter ratio D_1/D_0 is assumed to be unity. Fig. 2 shows the dimensionless critical buckling load P_{cr}/P_{cr}^0 for the uniform nanowire with surface effects but without nonlocal effect for the different aspect ratios L/D_0. When the aspect ratio values of 20 and 30 were considered, it can be seen that the result excellently agrees with the previous exact result [8].

In addition, the influences of surface effects, nonlocal parameter and diameter ratio on the buckling of the nonuniform nanowire are analyzed. Fig. 3 depicts the

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dimensionless critical buckling load of a nonuniform nanowire for different nonlocal parameters and diameter ratios. For different diameter ratios, it can be seen that the influence of surface effects on the critical buckling load is significant. When the diameter ratio value is large, the influence becomes more and more prominent. For a slender nanowire with a larger value of L/D_0, the surface effects are more significant. In addition, no matter whether the surface effects is considered or not, increasing the nonlocal parameter decreases the value of P_{cr}/P_{cr}^0. This is because the internal interaction force increases as the nonlocal parameter increases.

IV. Conclusions

In this article, we studied the axial buckling of a nonuniform nanowire. The Rayleigh-Ritz method was used to analyze the influences of nonlocal and surface effects on the buckling behavior of the nanowire. According to the analysis, the following results are obtained:

1. The dimensionless critical buckling load increased with an increase of aspect ratio value.
2. When the value of diameter ratio increased, the surface effects on the critical buckling load of the nonuniform nanowire becomes more and more prominent.
3. When the nonlocal effect was taken into account, the critical buckling load increased with decreasin**g** nonlocal parameter.

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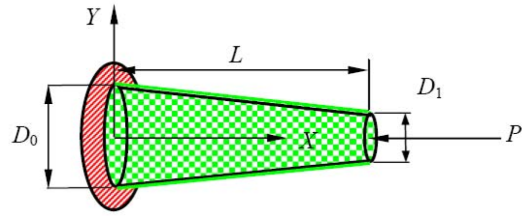

Fig. 1. A cantilever nanowire with a varying diameter and a uniform surface layer.

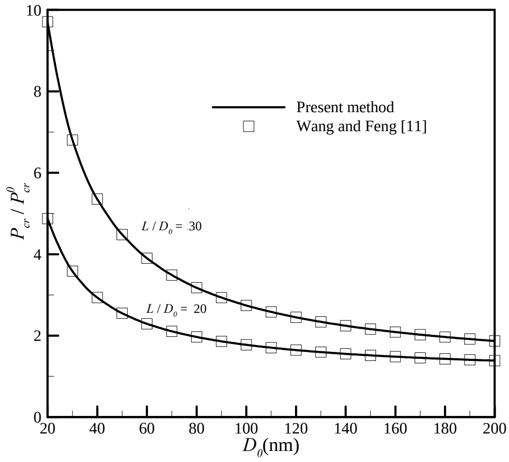

Fig. 2. The comparison of dimensionless critical buckling load of a uniform nanowire with surface effects but without nonlocal effect.

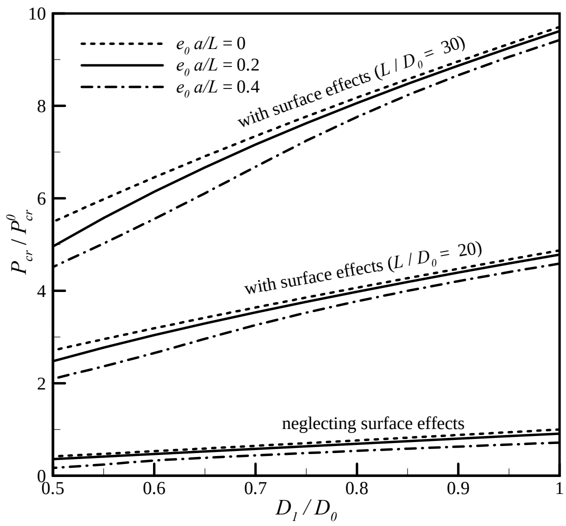

Fig. 3. The dimensionless critical buckling load of a nonuniform nanowire for different nonlocal parameters and diameter ratios