Exact Analytical Solutions of Three Nonlinear Heat Transfer Models

Mohammad Danish, Shashi Kumar and Surendra Kumar

Abstract— **Exact analytical solutions of three nonlinear heat transfer models of practical interests namely, steady state heat conduction in a rod, transient cooling of a lumped system and steady state heat transfer from a rectangular fin into the free space by the radiation mechanism, have been obtained. Recently, these three problems were investigated by several researchers by using homotopy perturbation, homotopy analysis and optimal homotopy analysis methods and the approximate series solutions were obtained. In this work, exact solutions of these three problems have been obtained in terms of a simple algebraic function, a Lambert *W* function and the Gauss's hypergeometric function, respectively. These exact solutions are superior to the available approximate solutions, agree very well with those obtained by the accurate numerical schemes and can also serve as the benchmarks for future testing of the approximate solutions.**

Index Terms— **Heat transfer, Conduction, Convection, Radiation, Exact solution**

I. Introduction

THIS communication primarily focuses on obtaining the exact analytical solution of three nonlinear heat transfer models having nonlinear temperature dependent terms. The first model describes the steady state heat conduction process in a metallic rod and is governed by a nonlinear BVP (boundary value problem) in ODE (ordinary differential equation). Recently, Rajabi *et al.* [1] have solved the resultant model equation by using a popular approximate scheme i.e. HPM (homotopy perturbation method), whereas, Sajid and Hayat [2] and Domairry and Nadim [3] have solved the same problem by using HPM and another very popular approximate scheme i.e. HAM (homotopy analysis method), and the results were obtained in the form of a truncated series. The second model, investigated by Ganji [4] using HPM, by Abbasbandy [5] using HAM and by Marinca and Herisanu [6] using OHAM (optimal HAM), portrays the unsteady heat convection from a lumped system.

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The corresponding controlling equation for this heat transfer problem is a nonlinear initial value problem (IVP) in ODE. Here again, the solutions were found in series form. The third model represents the steady state radiative heat transfer from a rectangular fin into the free space and the model equation results in a nonlinear BVP in ordinary differential equation. This model has also been studied recently by Ganji [4], Abbasbandy [5] and Marinca and Herisanu [6] by using HPM, HAM and OHAM, respectively and the solution in terms of the finite series are found.

It can be noted that the series solutions have varying degree of accuracy and radius of convergence, and are strongly dependent on the number of terms in the series as well as on the parameters' values. Due to this, there always exists a region beyond which the series solutions start diverging and this limits their regular use. However, in such cases efforts are made either to obtain the exact analytical solutions or to solve the problem with the help of some suitable numerical technique. Fortunately, the present work shows that all the above three mentioned models are exactly solvable in terms of algebraic function, Lambert *W* function [7] and hypergeometric function, respectively. These solutions have been obtained by using simple mathematical manipulations e.g. assuming an implicit form of the solution or reducing the equation into a simpler form by adding and subtracting certain terms, as elaborated in the following sections. The so obtained analytical solutions are quite valuable since:

(i) They provide better insight of the actual physical process.
(ii) They can directly be employed to find the accurate temperature profiles and temperature gradients for a complete range of parameters' values unlike their approximate counterparts (series solutions) that have convergence related issues for the entire range of parameters' values especially for the larger values of parameters.
(iii) These exact solutions can also be utilized in cross checking the accuracy of other approximate solutions.

Description of the mentioned processes, derivation of respective mathematical models and the approaches to obtain the exact solutions are discussed below, individually.

II. Model 1: Steady heat transfer in a metallic rod

This model basically describes the steady state heat conduction in a metallic rod and has the practical significance in estimating the thermal conductivity of metals e.g. heat flow meters [8]. In this process, the two ends of the rod are kept at different but fixed temperatures and heat transfer takes place from higher temperature to the lower by the mode of conduction. At present, it is assumed that the

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thermal conductivity varies linearly with temperature and there is no heat loss to the surrounding from the curved surface of the rod.

Consider a rod of length, L and uniform cross sectional area, A_c with its end maintained at two different temperatures i.e. $T(x=0)=T_a$ and $T(x=L)=T_b$. For the above stated assumptions, the steady state energy balance over the rod yields the following dimensional equation and the allied BCs (boundary conditions):

$$\frac{d}{dx}\left(A_c k(T)\frac{dT}{dx}\right)=0 \tag{1a}$$

BCI: $T=T_a$ at $x=0$ (1b)

BCII: $T=T_b$ at $x=L$ (1c)

Where, $k(T)=k_a\left(1+\beta\frac{T-T_a}{T_b-T_a}\right)$ is the temperature dependent thermal conductivity of the rod. With the introduction of the following dimensionless variables, the governing equation and the associated BCs i.e. (1a)-(1c), transform into the following equations i.e. (2a)-(2c):

$$\xi=\frac{x}{L},\ \theta=\frac{T-T_a}{T_b-T_a}$$

$$(1+\beta\theta)\theta''+\beta(\theta')^2=0 \tag{2a}$$

BCI: $\theta(0)=0$ (2b)

BCII: $\theta(1)=1$ (2c)

Where, θ' & θ'' represents the 1st and 2nd order derivatives of θ with respect to ξ, respectively. Following two different approaches can be adopted to obtain the exact solution of the above equation, as demonstrated below:

A. Approach 1

A careful visualization of (2a) shows that it can conveniently be expressed in the following form:

$$((1+\beta\theta)\theta')'=0 \tag{3}$$

Integrating the above equation two times with respect to ξ, one obtains the following quadratic equation in θ:

$$\left(\theta+\beta\frac{\theta^2}{2}\right)=C_1\xi+C_2 \tag{4}$$

Where, $C_1=\left(1+\beta/2\right)$ and $C_2(=0)$ are the constants of integration and have been found from the associated BCs i.e. (2b) & (2c). Substituting these values in (4) and solving for θ, one finds the following two explicit solutions; two solutions appear because of the nonlinear nature of the equation.

$$\theta=\frac{-1+\sqrt{1+2\beta\xi+\beta^2\xi}}{\beta} \tag{5a}$$

$$\theta=\frac{-1-\sqrt{1+2\beta\xi+\beta^2\xi}}{\beta} \tag{5b}$$

Since, 2nd solution is unrealistic and does not satisfy the BCs therefore, it is discarded. If one expands (5a) around $\beta=0$ using Taylor series the following approximate series is obtained.

$$\theta\approx\xi+\frac{1}{2}\beta\left(\xi-\xi^2\right)+\frac{1}{2}\beta^2\left(\xi^3-\xi^2\right)+\ldots$$

On comparing it with the approximate HPM solution [(47)] of Rajabi *et al.* [1] and approximate HAM solution of Domairry and Nadim [3] for the convergence control parameter $h=-1$ (used therein), an exact conformity is observed.

Similar comparison could not be performed with the results of Sajid and Hayat [2] as no such solution expression was provided. However, in this case the results were judged against those of Sajid and Hayat [2] by tabulating the values of temperature gradients at $\xi=0$ and $\xi=1$ (see Table 1). An excellent agreement is observed between these values.

The results obtained by the present method i.e. (5a) have also been successfully cross-examined against those obtained by (47) of Rajabi *et al.* [1] and those obtained by the accurate numerical methods, as shown in Fig. 1. Fig. 1 clearly illustrates that the approximate temperature profile obtained by Rajabi e. al. [1] deviates appreciably even for moderate values of β and becomes redundant for larger values of β. Although not shown, however, the same characteristics can also be attributed to the HAM solution of Domairry and Nadim [3] for the convergence control parameter $h=-1$. On the other hand, no deviation is observed in the present solution, even for higher values of β. It is also clear from Fig. 1 that as β varies from 0 to ∞, the temperature of the rod tends to reach the higher temperature ($\theta=1$) and thus establishes the fact that with the increase in thermal conductivity the temperature of the rod also rises.

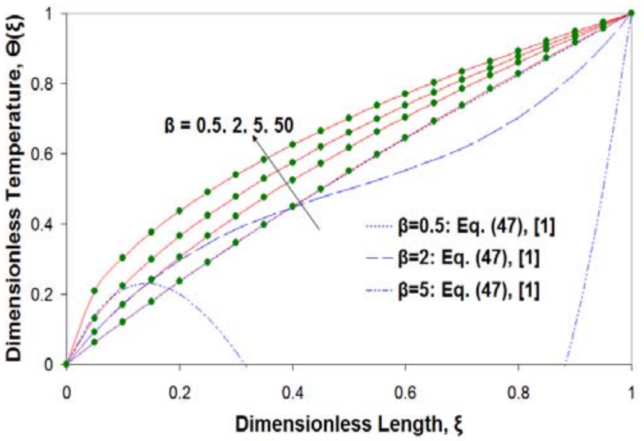

Fig. 1 Dimensionless temperature profiles along the length of the rod (model 1), solid lines: exact solution; filled circle: numerical solution

B. Approach 2

In this approach we assume that the solution of (2a) exhibits an implicit form i.e. $f(\theta)=\xi$, in other words, the derivative θ' is a function of θ only i.e. $\theta'=p(\theta)$. Therefore, $\theta''=\frac{1}{2}\frac{d\left(p^2\right)}{d\theta}$, where, p (still unknown) is a function of θ, only. It is worthwhile to mention that this approach is quite helpful whenever the independent variable ξ is absent in the concerned equation. Replacing θ' & θ'' in (2a) by the above respective definitions, one obtains:

$$\left(1+\beta\theta\right)\frac{d\left(p^2\right)}{d\theta}+2\beta p^2=0 \tag{6}$$

Now, substituting $p^2 = y$ and after little manipulations the above equation reduces to the following 1st order linear ODE:

$$\left(1+\beta\theta\right)y'+2\beta y=0 \tag{7}$$

Solving the above 1st order linear ODE by integrating factor method one finds:

$$y=\frac{C_1}{\left(1+\beta\theta\right)^2} \tag{8}$$

Or

$$p\left(\theta\right)=\frac{d\theta}{d\xi}=\frac{\sqrt{C_1}}{\left(1+\beta\theta\right)} \tag{9}$$

Where, C_1 is a constant of integration. Integrating the above (9) once more, one finds the expression for θ [note that the equation below is similar, in form, to the (4)]:

$$\left(\theta+\beta\frac{\theta^2}{2}\right)=\sqrt{C_1}\xi+C_2 \tag{10}$$

C_2 is another constant of integration and $C_1=\left(1+\beta/2\right)^2$ and $C_2\left(=0\right)$ are evaluated from the associated BCs, like in the first approach. Substituting the values of these constants in (10) and solving for θ, one arrives at the following two solutions which are exactly same as those given in (5a) & (5b).

$$\theta=\frac{-1+\sqrt{1+2\beta\xi+\beta^2\xi}}{\beta} \tag{11a}$$

$$\theta=\frac{-1-\sqrt{1+2\beta\xi+\beta^2\xi}}{\beta} \tag{11b}$$

2nd solution does not satisfy the BCs so discarded. Rest of the discussion remains same as presented in approach 1.

TABLE I
COMPARISON OF SLOPES OF THE DIMENSIONLESS TEMPERATURE PROFILE AT BOTH THE ENDS OF THE ROD (MODEL 1)

S. No.	β	θ′(1)			θ′(0)	
		Numerical solution	Sajid & Hayat [2]	Exact solution (5a)	Numerical solution	Exact solution (5a)
1	0.5	0.833333	0.833333	0.833333	1.250000	1.250000
2	2	0.666667	0.666667	0.666667	2.000000	2.000000
3	5	0.583333	0.583333	0.583333	3.500000	3.500000
4	50	26/51	26/51	26/51	26.000000	26.000000

III. MODEL 2: TRANSIENT COOLING OF A LUMPED SYSTEM

This model represents the transient cooling of a lumped parameter system with specific heat as a linear function of temperature. In practice, this situation arises in the cooling of heated stirred vessels and cooling of electronic components with high thermal conductivity etc [8]. Ganji [4], Abbasbandy [5] and Marinca and Herisanu [6] have worked out this model by using HPM, HAM and OHAM, respectively and the solutions were obtained in the form of series. The problem can be stated as: at the outset of the experiment, a system with density ρ, volume V and heat transfer area A, is exposed to an environment at different temperature (T_a) and heat is transferred from the system to the surrounding by convection. The governing model equation is derived by applying the unsteady energy balance over the system and is described by the following nonlinear IVP (initial value problem) in 1st order ODE:

$$\rho Vc(T)\frac{dT}{dt}+hA(T-T_a)=0 \tag{12a}$$

IC: $T(0)=T_b$ (12b)

Where, $c(T)=c_a\left(1+\beta\frac{T-T_a}{T_b-T_a}\right)$ is the heat capacity of the system showing linear dependency on temperature and h is the constant heat transfer coefficient. With the assistance of the following dimensionless quantities, (12a) & (12b) attain the dimensionless form given by (13a) & (13b), respectively.

$$\tau=\frac{hAt}{\rho Vc_a},\ \theta=\frac{T-T_a}{T_b-T_a}$$

$$\left(1+\beta\theta\right)\theta'+\theta=0 \tag{13a}$$

IC: $\theta(0)=1$ (13b)

A simple rearrangement of the above (13a) yields:

$$\frac{\theta'}{\theta}+\beta\theta'=-1 \tag{14}$$

Integrating (14) with respect to τ results in:

$$Log[\theta]+\beta\theta=-\tau+C_1 \tag{15}$$

Where, C_1 is the constant of integration and using IC, it is found to be $C_1=\beta$. Substituting back the so found value of C_1 in (15), provides the following exact analytical solution.

$$Log[\theta]+\beta\theta=\beta-\tau \tag{16}$$

It can be noted that due to the above implicit form of θ, it has to be found for each and every τ by solving (16) with the help of some suitable iterative numerical scheme. This feature limits the repeated use of the above formula. Keeping this in view, we now develop, from (16), the explicit solution form. A constant term $Log[\beta]$ is added and subtracted in (16) and after performing a little modification, (17) is obtained.

$$Log\left[\beta\theta e^{\beta\theta}\right]=\beta-\tau+Log\left[\beta\right] \tag{17}$$

Equation (17) can be further expressed as:

$$\left(\beta\theta\right)e^{(\beta\theta)}=\beta e^{\beta-\tau} \tag{18}$$

The L.H.S. of (18) can be replaced by the Lambert *W* function (implemented as *ProductLog* function in some mathematical softwares e.g. Mathematica). A Lambert *W* function is basically the inverse function of $x=ye^y$ i.e. y= *Lambert*(x) and is symbolized by $y=W(x)$. In general, the domain and range of the function is the set of complex values however, for $x\in[0,\infty)$ Lambert *W* function yields single real values. For $x\in(-\infty,-1/e)$, Lambert *W* function does not evaluate to any real value whereas, for $x\in[-1/e,0)$ it computes two real values. Now, with this function available, the transient dimensionless temperature profile is given by:

$$\theta = \frac{1}{\beta} ProductLog\left[\beta e^{\beta-\tau}\right] \qquad (19)$$

Expanding θ around $\beta = 0$ by using Taylor series, yields the following expansion which harmonizes well with the (18) of Ganji [4] and (9) of Abbasbandy [5] for convergence parameter $h = -1$.

$$\theta \approx e^{-\tau} + \beta\left(e^{-\tau} - e^{-2\tau}\right) + \frac{\beta^2}{2}\left(e^{-\tau} - 4e^{-2\tau} + 3e^{-3\tau}\right) + \ldots$$

Fig. 2 compares that the transient temperature profiles obtained by the present (19), HPM solution obtained by Ganji [4] and those obtained by numerical scheme. It is clear that the present solution match very well with the numerical solution whereas, the solutions obtained by Ganji [4] show considerable discrepancies except for $\beta = 0$ where the (13a) becomes linear. Fig. 2 also supports the fact that with the increase in β, the specific heat increases which in turn causes the decrease in temperature gradient.

Extending the comparison, the initial rates of temperature change, given by the following (20), have also been found using (19) and plotted in Fig. 3 along with those obtained by Abbasbandy [5].

$$\theta'(0) = \frac{-1}{1+\beta} \qquad (20)$$

Accuracy is evident by the overlapping profiles. Similar comparisons with the OHAM solution of Marinca and Herisanu [6] have been avoided due to their more involved solution expression. However, it can be shown that our present solution, being exact in form, is superior to the approximate solution of Marinca and Herisanu [6].

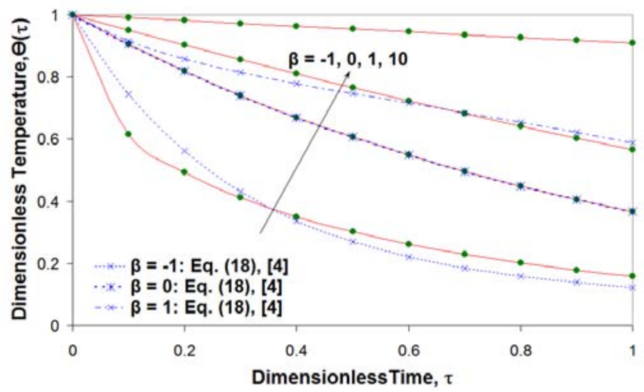

Fig. 2. Transient profile of the dimensionless temperature (model 2), solid lines: exact solution; filled circle: numerical solution

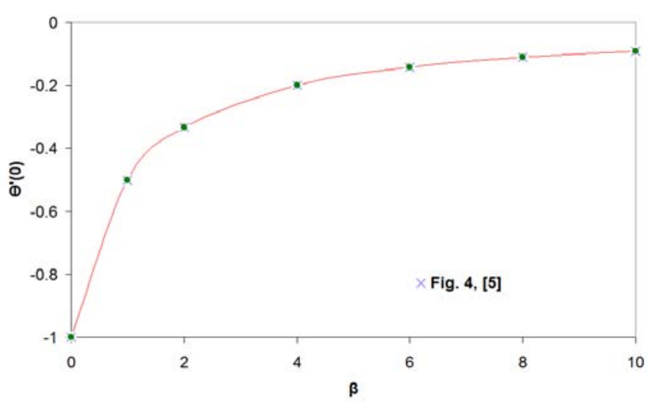

Fig. 3. Initial rate of change of dimensionless temperature vs. β (model 2 solid lines: exact solution; filled circle: numerical solution

IV. Model 3: Radiative heat transfer from a rectangular fin

This model represents the steady state heat transfer from a rectangular fin to the free space by the radiation mechanism. Such situations appear in the cooling of the heated parts of the space vehicles. This problem, too, has been tackled by Ganji [4], Abbasbandy [5] and Marinca and Herisanu [6] with the help of HPM, HAM and OHAM, respectively and the solutions were obtained in the form of series. We consider a rectangular fin having cross sectional area A_c, perimeter P, length L and the constant thermal conductivity and emissivity as k and $\in$, respectively. The fin base is maintained at a higher temperature T_b and the fin is transmitting the heat energy into the space by the mode of radiation. It is assumed that the steady state is prevailing and the negligible heat transfer takes place from fin end. Keeping these assumptions in view, the governing model equation is derived by applying the steady energy balance over the fin element and is described by the following nonlinear BVP in 2nd order ODE:

$$\frac{d}{dx}\left(A_c k \frac{dT}{dx}\right) = P\sigma \in (T^4 - T_s^4) \qquad (21a)$$

BCI: $T = T_b$ at $x = L$ (at fin base) (21b)

BCII: $\frac{dT}{dx} = 0$ at $x = 0$ (at fin end) (21c)

It is worthwhile to note that the space temperature can very well be replaced by the absolute zero temperature i.e. $T_s = 0$ [4]-[6]. Taking this fact into account and defining the following dimensionless variables, the above equations are conveniently expressed into the dimensionless form given by (22a) - (22c).

$$\theta = \frac{T}{T_b},\quad \xi = \frac{x}{L},\quad \varepsilon = \frac{\sigma \in P T_b^3 L^2}{k A_c}$$

And the (21a) - (21c) become

$$\frac{d^2\theta}{d\xi^2} = \varepsilon\theta^4 \qquad (22a)$$

BCI: $\theta = 1$ at $\xi = 1$ (at fin base) (22b)

BCII: $\frac{d\theta}{d\xi} = 0$ at $\xi = 0$ (at fin end) (22c)

To solve the above BVP, the same approach has been followed as adopted previously for the solution of problem 1, and here also, it is assumed that the derivative $\frac{d\theta}{d\xi}$ is a function of θ only i.e. $\frac{d\theta}{d\xi} = p(\theta)$ where, p is yet to be found. This assumption leads to $\theta'' = \frac{1}{2}\frac{d\left(p^2\right)}{d\theta}$. Replacing θ'' in (22a) by this relation, one obtains:

$$\frac{d\left(p^2\right)}{d\theta} = 2\varepsilon\theta^4 \qquad (23)$$

Now, replacing p^2 with y, the (23) attains the following 1st order linear ODE:

$$\frac{dy}{d\theta} = 2\varepsilon\theta^4 \qquad (24)$$

Integrating the above equation, one finds

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$$y = \frac{2}{5}\varepsilon\theta^5 + C_1 \tag{25}$$

C_1 is constant of integration and can be evaluated with the help of BCII i.e. (22c) and is found to be $C_1 = -\frac{2}{5}\varepsilon\theta_0^5$; where, θ_0 is the unknown dimensionless temperature at the fin base. Substituting this value of C_1 in (25), one gets

$$y = \left(\frac{d\theta}{d\xi}\right)^2 = \frac{2}{5}\varepsilon\left(\theta^5 - \theta_0^5\right) \tag{26}$$

A minor rearrangement of the above equation yields

$$\frac{d\theta}{\sqrt{\frac{2}{5}\varepsilon\left(\theta^5 - \theta_0^5\right)}} = d\xi \tag{27}$$

Integrating the above equation between the limits prescribed by the BCs I & II, following definite integral is found.

$$\int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{\frac{2}{5}\varepsilon\left(\theta^5 - \theta_0^5\right)}} = \int_0^{\xi} d\xi \tag{28}$$

The integration of the above equation gives the following result.

$$\sqrt{\frac{5}{2\varepsilon\left(\theta^5 - \theta_0^5\right)}}\,\theta\sqrt{1 - \frac{\theta^5}{\theta_0^5}}\,HG_2F_1\left[\frac{1}{5}, \frac{1}{2}, \frac{6}{5}, \frac{\theta^5}{\theta_0^5}\right]$$

$$-i\sqrt{\frac{5\pi}{2\varepsilon}}\,\frac{1}{\theta_0^{3/2}}\,\frac{\Gamma\left[\frac{6}{5}\right]}{\Gamma\left[\frac{7}{10}\right]} = \xi \tag{29}$$

The unknown θ_0 is computed by solving the following nonlinear equation which has been obtained by forcing (29) to satisfy the unutilized BCI i.e. $\theta = 1$ at $\xi = 1$.

$$\sqrt{\frac{5}{2\varepsilon\left(1 - \theta_0^5\right)}}\sqrt{1 - \frac{1}{\theta_0^5}}\,HG_2F_1\left[\frac{1}{5}, \frac{1}{2}, \frac{6}{5}, \frac{1}{\theta_0^5}\right]$$

$$-i\sqrt{\frac{5\pi}{2\varepsilon}}\,\frac{1}{\theta_0^{3/2}}\,\frac{\Gamma\left[\frac{6}{5}\right]}{\Gamma\left[\frac{7}{10}\right]} = 1 \tag{30}$$

Where, $\Gamma[z]$ and $HG_2F_1[a,b,c,z]$ are the well known Gamma and the Gauss' Hypergeometric functions, respectively and are defined as follows:

$$\Gamma[z] = \int_0^{\infty} t^{z-1} e^{-t}\, dt$$

$$HG_2F_1[a,b,c,z] = \frac{\Gamma[c]}{\Gamma[b]\,\Gamma[c-b]}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt$$

Ganji [4], Abbasbandy [5] and Marinca and Herisanu [6] have solved this problem by using HPM, HAM and OHAM, respectively and solutions are obtained in terms of the series. For comparison purposes, the two terms HPM and HAM solutions of Ganji [4] and Abbasbandy [5] are reproduced below, however, because of complexity in the expression of Marinca and Herisanu [6], it has not been considered here.

$$\theta_{Ganji} \cong 1 + \varepsilon\left(\frac{x^2 - 1}{2}\right) + \varepsilon^2\left(\frac{x^4 - 6x^2 + 5}{6}\right) \tag{31}$$

$$\theta_{Abbasbandy} \cong 1 - \varepsilon h\left(\frac{x^2 - 1}{2}\right) - \varepsilon h(1+h)\left(\frac{x^2 - 1}{2}\right)$$

$$+\varepsilon^2 h^2\left(\frac{x^4 - 6x^2 + 5}{6}\right) \tag{32}$$

Figs. 4 & 5, plot the dimensionless temperature profiles obtained by the above approximate series solutions, the accurate numerical scheme as well as those obtained by the presently obtained exact solution i.e. (29) & (30). It can be noted that the same value of the parameter ε have been taken as those considered by Ganji [4] and Abbasbandy [5] i.e. $\varepsilon = 0.09$ and $\varepsilon = 0.7$, respectively. It is clearly visible in Fig. 4 that the profile obtained by Ganji [4] slightly deviates with the numerical solution whereas, the profile obtained by the exact analytical solution shows an excellent matching with its numerical counterpart.

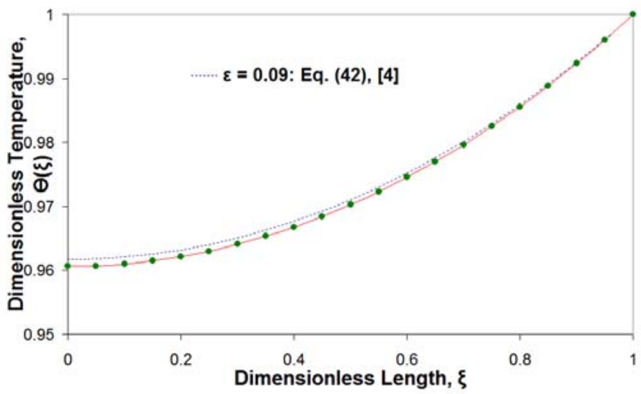

Fig. 4. Dimensionless temperature profiles along the length of the fin (model 3), solid lines: exact solution; filled circle: numerical solution

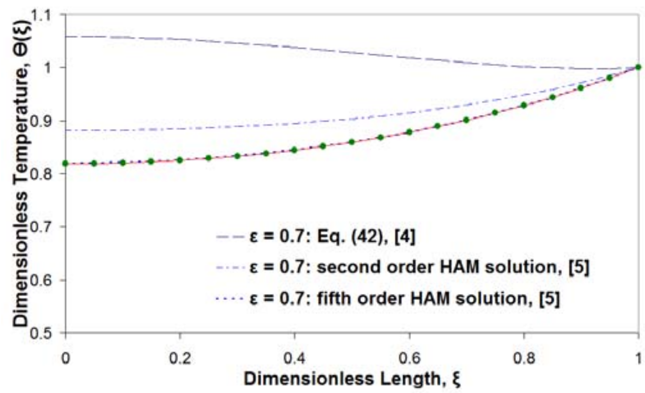

Fig. 5. Dimensionless temperature profiles along the length of the fin (model 3), solid lines: exact solution; filled circle: numerical solution

Similarly, in Fig. 5, the two terms HPM solution of Ganji [4] yields divergent results whereas, the two term HAM solution of Abbasbandy [5] show minor deviations with the numerically obtained accurate profile. However, the five term HAM solution obtained by Abbasbandy [5] matches well with the numerical solution. In contrast to this, the exact analytical solution i.e. (29) & (30) are in complete agreement with the numerical solution. It can be verified that the deviations in the series solutions of Ganji [4] and Abbasbandy [5], will increase with the increase in the value of ε, however, this is not true for the currently derived

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exact solution. The true profiles signify the sharp decrease in temperature with the increase in the parameter ε. This observation is in compliance with the physics of the problem.

V. Conclusion

Exact analytical solutions of the three nonlinear heat transfer models of real significance and arising in heat transfer have been obtained in the form of elementary algebraic and transcendental functions. These problems represent steady state heat conduction in a solid rod, the unsteady cooling of a lumped parameter system and the steady state radiative heat transfer from a rectangular fin to the space, respectively. The corresponding exact solutions have been obtained in terms of a simple algebraic function, Lambert W function and Gauss's hypergeometric function, respectively. These obtained exact solutions agree very well with the corresponding true numerical solutions and are found to be superior to the previously available approximate HPM, HAM and OHAM solutions. These exact solutions provide better insight of the physical process and are valid for all parameter ranges unlike their approximate alternatives; moreover, these can be pretty useful in judging the accuracy of other approximate solutions.

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Nomenclature

A	[m^2]	heat transfer area
A_c	[m^2]	cross-sectional area
a,b,c	[-]	constants
c_a	[J/kg.K]	specific heat at temperature T_a
$c(T)$	[J/kg.K]	specific heat at temperature T
C_1, C_2	[-]	constants of integration
h	[J/s.m^2.K]	heat transfer coefficient
k_a	[J/s.m.K]	thermal conductivity at temperature T_a
$k(T)$	[J/s.m.K]	thermal conductivity at temperature T
L	[m]	length of rod
p	[-]	function of θ
t	[s]	time
T	[K]	temperature
T_s	[K]	radiation sink temperature
u	[-]	dummy variable
V	[m^3]	volume
x	[m]	distance variable
y	[-]	function of θ
z	[-]	dummy variable

Greek letters

β	[-]	dimensionless parameter for $k(T)$ and $c(T)$
$\in$	[-]	emissivity
ε	[-]	conduction radiation parameter
θ	[-]	dimensionless temperature
ρ	[kg/m^3]	density
σ	[W/m^2.K^4]	Stephan-Boltzmann constant (=5.669×10^{-8})
τ	[-]	dimensionless time
ξ	[-]	dimensionless distance

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