On the Equilibrium Points and Some Properties of a SVEIRS Epidemic Model

M. De la Sen, S. Alonso-Quesada and A. Ibeas

Abstract- **This paper relies on the disease-free and endemic equilibrium points of SVEIRS epidemic model.**

Index terms- **Epidemic models, Equilibrium points, SEIRS epidemic models; SVEIRS epidemic models, positivity, vaccination control.**

I. INTRODUCTION

Important control problems nowadays related to Life Sciences are the control of ecological models like, for instance, those of population evolution (Beverton-Holt model, Hassell model, Ricker model etc.[1-5]) via the online adjustment of the species environment carrying capacity, that of the population growth or that of the regulated harvesting quota as well as the disease propagation via vaccination control. In a set of papers, several variants and generalizations of the Beverton-Holt model (standard time–invariant, time-varying parameterized, generalized model or modified generalized model) have been investigated at the levels of stability, cycle- oscillatory behavior, permanence and control through the manipulation of the carrying capacity (see, for instance, [1-5]). The design of related control actions has been proved to be important in those papers at the levels, for instance, of aquaculture exploitation or plague fighting. On the other hand, the literature about epidemic mathematical models is exhaustive in many books and papers. A non-exhaustive list of references is given in this manuscript, cf. [6-14] (see also the references listed therein). The sets of models include the most basic ones, [6-7]:

- SI- models where not removed- by – immunity population is assumed. In other words, only susceptible and infected populations are assumed.
- SIR -models, which include susceptible, infected and removed- by –immunity populations.
- SEIR- models where the infected populations is split into two ones (namely, the " infected" which incubate the disease but do not still have any disease symptoms and the " infectious" or " infective" which do exhibit the external disease symptoms).

The three above models have two possible major variants, namely, the so-called "pseudo-mass action models", where the total population is not taken into account as a relevant disease contagious factor or disease transmission power, and the so-called "true-mass action models", where the total population is more realistically considered as being an inverse factor of the disease transmission rates. There are other many variants of the above models, for instance, including vaccination of different kinds: constant [8], impulsive [12], discrete - time etc., by incorporating point or distributed delays [12-13], oscillatory behaviors [14] etc. . On the other hand, variants of such models become considerably simpler for the disease transmission among plants [6-7]. Some generalizations involve the use of a mixed regular continuous-time/ impulsive vaccination control strategies for generalized time-varying epidemic model which is subject to point and distributed time-varying delays, [12-13], [22-23], [25]. Another well-known types of epidemic models are the so-called SVEIRS epidemic models which incorporate the dynamics of a vaccinated population and the "infected" population without external symptoms of the SEIR-type models is replaced with an "exposed" population subject to a certain dynamics,[30-31].The infected population E(t) of SEIR models, which includes those infected which do not still have external disease symptoms (those having those symptoms are usually referred to as the " infectious" population I (t)), is replaced in the SVEIRS models by the so-called "exposed" population E (t) . Thus, in the context of SVEIRS models, the infected and infectious populations of the SEIR models are joined in a single "infected" population I (t) while there is an exposed population E (t) present in the model. In this paper, we focus on the existence and some properties of disease- free and endemic equilibrium points of a SVEIRS models subject to an eventual constant regular vaccination rather than to an impulsive vaccination type. Some issues about boundedness and positivity of the model are also investigated. The following SVEIRS epidemic model, of a modified true-mass action type, with regular constant vaccination is considered:

$$\dot{S}(t)=b(1-S(t))-\beta\frac{S(t)I(t)}{1+\eta S(t)}+\gamma I(t-\omega)e^{-b\omega}+\nu(1-V_c)N(t) \tag{1.1}$$

$$\dot{V}(t)=-\frac{\delta\beta V(t)I(t)}{1+\eta V(t)}-(\gamma_1+b)V(t)+\nu V_c N(t) \tag{1.2}$$

$$E(t)=\beta\int_{t-\omega}^{t}\left(\frac{S(u)I(u)}{1+\eta S(u)}+\frac{\delta V(u)I(u)}{1+\eta V(u)}\right)e^{-b(t-u)}du \tag{1.3}$$

$$\dot{I}(t)=\beta e^{-b\tau}\left(\frac{S(t-\tau)I(t-\tau)}{1+\eta S(t-\tau)}+\frac{\delta S(t-\tau)V(t-\tau)}{1+\eta V(t-\tau)}\right)-(\gamma+b+\alpha)I(t) \tag{1.4}$$

Manuscript received December, 1 2009. This work was supported in part by the Spanish Ministry of Science and Innovation through Grant DPI2009-07197 and also by Basque Government by its support through Grant GIC07143-IT-269-07.

M. De la Sen and S. Alonso-Quesada are with the Institute of Research and Development of Processes, Campus of Leioa , Bizkaia , Aptdo. 644-Bilbao. Spain (e-mail: manuel.delasen@ehu.es).

A. Ibeas is with Department of Telecom and Systems Eng. Universitat Autònoma de Barcelona, 08193- Bellaterra, Barcelona, Spain(e-*mail: Asier.Ibeas@uab.cat*)

ISBN: 978-988-19251-5-2
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$$\dot{R}(t) = -bR(t) + \gamma_1 V(t) + \gamma\left(I(t) - I(t-\omega)e^{-b\omega}\right) \quad (1.5)$$

where S, V, E , I and R are, respectively, the susceptible, vaccinated , exposed , infected (or infective or infectious) and recovered populations, $N(t)$ is the total population being the sum of the above ones, $V_c \in [0,1]$ is a constant vaccination action. There are potential latent and immune periods denoted by τ and ω, respectively, which are internal delays in the dynamic system (1.1)-(1.5), b is the natural birth rate and death rate of the population. $\nu < b$ takes into account a vaccination action on newborns which decreases the incremental susceptible population through time, γ_1 is the average rate for vaccines to obtain immunity and move into recovered population, β (disease transmission constant) and $\delta\beta$ are average numbers for contacts of an infective and a vaccinated individual per unit of time, [30]. The periodic impulsive, rather than regular, vaccination action of [1], can be got from (1.1)-(1.5) with $V_c = 0$. A variant of the above SVEIRS model (1.1)-(1.5) has been proposed in [31] without regular vaccination with the alternative incorporation of an impulsive periodic vaccination with a period $T > 0$.

II. DISEASE-FREE EQUILIBRIUM POINT

The potential existence of a disease-free equilibrium point is now discussed which asymptotically removes the disease if $\nu < b$.

Proposition 1. Assume that $\nu < b$. Then the disease-free equilibrium point $E^* = I^* = 0$ fulfils :

$$R^* = \frac{\nu\gamma_1 V_c N^*}{b(\gamma_1 + b)} = \gamma_1 \frac{(b - \nu(1-V_c))N^* - b}{(\gamma_1 + b)b}$$

$$V^* = \frac{\nu V_c N^*}{\gamma_1 + b} = \frac{(b - \nu(1-V_c))N^* - b}{\gamma_1 + b} \quad , \quad S^* = 1 + \frac{\nu N^*(1-V_c)}{b}$$

which leads to the following further constraints:

$$N^* = \frac{b}{b-\nu}, \; S^* - 1 = \frac{\nu(1-V_c)}{b-\nu}$$

$$V^* + R^* = \frac{\nu V_c N^*}{b} = \frac{\nu V_c}{b-\nu}$$

Two particular disease-free equilibrium points are $S^* = N^* = \frac{b}{b-\nu}$, $E^* = I^* = V^* = R^* = 0$ if $V_c = 0$, and

$$S^* = 1, \; V^* = \frac{\nu N^*}{\gamma_1 + b} = \frac{\nu b}{(\gamma_1 + b)(b-\nu)}$$

$$R^* = \frac{\nu\gamma_1}{(\gamma_1 + b)(b-\nu)}, \; E^* = I^* = 0 \text{ if } V_c = 1$$

If $\nu \geq b$ then there is no disease-free equilibrium point. □

Proof: The equilibrium points are calculated from (1.1) as follows by zeroing (1.1), (1.2), (1.4) and (1.5) and making (1.3) identical to a disease-free equilibrium value E^* what leads to:

$$b - \left(b + \frac{\beta I^*}{1+\eta S^*}\right) S^* + \gamma I^* e^{-b\omega} + \nu N^*(1-V_c) = 0 \quad (2.1)$$

$$-\left(\frac{\delta\beta I^*}{1+\eta V^*} + \gamma_1 + b\right) V^* + \nu N^* V_c = 0 \quad (2.2)$$

$$E^* = \frac{\beta}{b}\left(1 - e^{-b\omega}\right)\left(\frac{S^*}{1+\eta S^*} + \frac{\delta V^*}{1+\eta V^*}\right) I^* \quad (2.3)$$

$$\beta e^{-b\tau}\left(\frac{S^*}{1+\eta S^*} + \frac{\delta V^*}{1+\eta V^*}\right) I^* - (\gamma + b + \alpha) I^* = 0 \quad (2.4)$$

$$\gamma_1 V^* - bR^* + \gamma\left(1 - e^{-b\omega}\right) I^* = 0 \quad (2.5)$$

The disease-free equilibrium point satisfies the constraints:

$$E^* = I^* = 0 \quad (2.6)$$

$$b(1 - S^*) + \nu N^*(1 - V_c) = 0 \Rightarrow S^* = 1 + \frac{\nu N^*(1-V_c)}{b} \quad (2.7)$$

$$\gamma_1 V^* - bR^* = 0 \Rightarrow V^* = \frac{bR^*}{\gamma_1} \quad (2.8$$

$$-(\gamma_1 + b) V^* + \nu N^* V_c = 0 \Rightarrow V^* = \frac{\nu N^* V_c}{\gamma_1 + b} = \frac{bR^*}{\gamma_1} \quad (2.9)$$

$$N^* = S^* + V^* + R^* = 1 + \frac{\nu N^*(1-V_c)}{b} + \left(1 + \frac{b}{\gamma_1}\right) R^* \quad (2.10)$$

$$= 1 + \frac{\nu N^*(1-V_c)}{b} + \frac{\nu N^* V_c}{b} = \frac{b + \nu N*}{b}$$

$$\Rightarrow N^* = \frac{b}{b-\nu} \text{ provided that } \nu < b \quad (2.11)$$

The proof follows directly from the above equations. □

Remark 1. Note that if $\gamma_1 = b$ then $R^* = V^* = \frac{\nu V_c N^*}{2b} = \frac{\nu V_c}{2(b-\nu)}$. Note also that if $\nu = 0$, as in the particular case of impulsive-free SVEIRS model obtained from that discussed in [1], then the disease-free equilibrium satisfies $E^* = V^* = I^* = R^* = 0$, $N^* = S^* = 1$. In such a case, the model can be ran out with population normalized to unity. □

Note that the exposed population at the equilibrium defined by (1.3) can be equivalently described by a differential equation obtained by applying the Leibniz differentiation rule under the integral symbol to yield:

$$\tilde{\dot{E}}(t) = -b\tilde{E}(t) + \beta\left(\frac{S^*}{1+\eta S^*} + \frac{\delta V^*}{1+\eta V^*}\right)\left(\tilde{I}(t) - \tilde{I}(t-\omega)e^{-b\omega}\right) \quad (2.12)$$

The following assumption of sufficiently small disease transmission constant is made:

Assumption 1. It is assumed that the diseasec transmission constant is small enough such that the inequality $\frac{S^*}{1+\eta S^*} + \frac{\delta V^*}{1+\eta V^*} \leq \frac{(\gamma + b + \alpha)e^{b\tau}}{\beta}$ holds. □

The local asymptotic stability of the disease-free equilibrium point is guaranteed by that of the linearized incremental system about it. The linearized model about the equilibrium becomes to be defined from (1.1)-(1.2), (2.12) and (1.4)-(1.5) by the state

ISBN: 978-988-19251-5-2
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

vector $\tilde{x}(t) := \left(\tilde{S}(t), \tilde{V}(t), \tilde{E}(t), \tilde{I}(t), \tilde{R}(t)\right)^T$ which satisfies the differential system:

$$\dot{\tilde{x}}(t) = A_0^* \tilde{x}(t) + A_\tau^* \tilde{x}(t-\tau) + A_\omega^* \tilde{x}(t-\omega);\ \tilde{x}(0) = \tilde{x}_0 \quad (2.13)$$

where, after using the identities in Proposition 1 and taking in mind Assumption 1 related to the disease-free equilibrium point:

$$A_0^* = A_{0d}^* + \tilde{A}_0^*$$

$$:= \begin{bmatrix} \nu(1-V_c)-b & \nu(1-V_c) & \nu(1-V_c) & \nu(1-V_c)-\frac{\beta S^*}{1+\eta S^*} & \nu(1-V_c) \\ \nu V_c & \nu V_c-(\gamma_1+b) & \nu V_c & \nu V_c-\frac{\delta\beta V^*}{1+\eta V^*} & \nu V_c \\ 0 & 0 & -b & \beta\left(\frac{S^*}{1+\eta S^*}+\frac{\delta V^*}{1+\eta V^*}\right) & 0 \\ 0 & 0 & 0 & -(\gamma+b+\alpha) & 0 \\ 0 & \gamma_1 & 0 & \gamma & -b \end{bmatrix}$$

$$= \begin{bmatrix} \nu(1-V_c)-b & \nu(1-V_c) & \nu(1-V_c) & \nu(1-V_c)-\frac{\beta\left(b+\nu(1-V_c)N^*\right)}{b+\eta\left(b+\nu(1-V_c)N^*\right)} & \nu(1-V_c) \\ \nu V_c & \nu V_c-(\gamma_1+b) & \nu V_c & \nu V_c-\frac{\delta\beta\nu V_c N^*}{\gamma_1+b+\delta\beta\nu V_c N^*} & \nu V_c \\ 0 & 0 & -b & (\gamma+b+\alpha)e^{b\tau}-\varepsilon & 0 \\ 0 & 0 & 0 & -(\gamma+b+\alpha) & 0 \\ 0 & \gamma_1 & 0 & \gamma & -b \end{bmatrix} \quad (2.14)$$

for some real $\varepsilon > 0$, where

$$A_{0d}^* := Diag\left(\ \nu(1-V_c)-b\ ,\ \nu V_c-(\gamma_1+b), -b, -(\gamma+b+\alpha), -b\ \right) \quad (2.15)$$

$$\tilde{A}_0^* := \begin{bmatrix} 0 & \nu(1-V_c) & \nu(1-V_c) & \nu(1-V_c)-\frac{\beta\left(b+\nu(1-V_c)N^*\right)}{b+\eta\left(b+\nu(1-V_c)N^*\right)} & \nu(1-V_c) \\ \nu V_c & 0 & \nu V_c & \nu V_c-\frac{\delta\beta\nu V_c N^*}{\gamma_1+b+\delta\beta\nu V_c N^*} & \nu V_c \\ 0 & 0 & 0 & (\gamma+b+\alpha)e^{b\tau}-\varepsilon & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma_1 & 0 & \gamma & 0 \end{bmatrix} \quad (2.16)$$

and the matrices A_τ^* and A_ω^* are entry-wise defined by:

$$\left(A_\tau^*\right)_{44} = \gamma+b+\alpha\ ,\quad \left(A_\omega^*\right)_{14} = \gamma e^{-b\omega}\ ,$$
$$\left(A_\omega^*\right)_{34} = -(\gamma+b+\alpha)e^{b(\tau-\omega)},\ \left(A_\omega^*\right)_{54} = -\gamma e^{-b\omega} \quad (2.17)$$

with all the remaining entries being zero. The following inequalities apply for equivalent norms of vectors and square matrices M of dimension or, respectively, order n:

$$n^{-1}\|M\|_2 \le n^{-1/2}\|M\|_\infty \le \|M\|_2 \le n^{1/2}\|M\|_1 \le n\|M\|_2 \quad (2.18)$$

Thus, one gets from the above inequalities (2.18) that

$$\left\|A_\tau^*\right\|_2 + \left\|A_\omega^*\right\|_2 \le \sqrt{5}\left(\left\|A_\tau^*\right\|_\infty + \left\|A_\omega^*\right\|_\infty\right) \le \sqrt{5}(\gamma+\beta+\alpha)\max\left(1, e^{b(\tau-\omega)}\right) \le \bar{\gamma} \quad (2.19)$$

where

$$\bar{\gamma} = \begin{cases} \sqrt{5}(\gamma+\beta+\alpha) & if \quad \tau \le \omega \\ \sqrt{5}(\gamma+\beta+\alpha)e^{b(\tau-\omega)} & if \quad \tau > \omega \end{cases} \quad (2.20)$$

Note that from (2.20) that $\sqrt{5}(\gamma+\beta+\alpha)e^{b(\tau-\omega)} \le b-b_0$ for a given b and any given positive real constant $b_0 < b$ if $(\gamma+\beta+\alpha)$ and $(\tau-\omega)$, if positive, are small enough such that, equivalently,

$$-\infty \le \frac{1}{2}\ln 5 + \ln(\gamma+\beta+\alpha) + b(\tau-\omega) \le \ln(b-b_0) \quad (2.21)$$

so that from (2.19)-(2.21)

$$-\infty \le \frac{1}{2}\ln 5 + \ln(\gamma+\beta+\alpha) + b(\tau-\omega)$$
$$\left\|A_\tau^*\right\|_2 + \left\|A_\omega^*\right\|_2 \le \bar{\gamma} \le b-b_0 \quad (2.22)$$

On the other hand, we can use from L´Hopital rule the following limit relations in the entries (1, 4) and (2, 4) of $\tilde{A}_0^*$:

$$\frac{\beta\left(b+\nu(1-V_c)N^*\right)}{b+\eta\left(b+\nu(1-V_c)N^*\right)} \to \frac{\beta}{1+\eta}$$

$$\frac{\delta\beta\nu V_c N^*}{\gamma_1+b+\eta\nu V_c N^*} \to 0 \text{ as } b \to \infty \quad (2.23)$$

if the remaining parameters remain finite and then $N^* = S^* = 1$ and $E^* = I^* = V^* = R^* = 0$ from Proposition 1. By continuity with respect to parameters, for any sufficiently large $M \in \boldsymbol{R}_+$, $\exists\ \varepsilon_{1,2} = \varepsilon_{1,2}(M) \in \boldsymbol{R}_+$ with $\varepsilon_{1,2} \to 0$ as $t \to \infty$ such that for $b \ge M$:

$$\frac{\beta\left(b+\nu(1-V_c)N^*\right)}{b+\eta\left(b+\nu(1-V_c)N^*\right)} \le \frac{\beta+\varepsilon_1}{1+\eta}$$

$$\frac{\delta\beta\nu V_c N^*}{\gamma_1+b+\eta\nu V_c N^*} \le \varepsilon_2 \quad (2.24)$$

and, one gets from (2.16),

$$\left|\tilde{A}_0^*\right| = \begin{bmatrix} 0 & \nu(1-V_c) & \nu(1-V_c) & \left|\nu(1-V_c)-\frac{\beta+\varepsilon_1}{1+\eta}\right| & \nu(1-V_c) \\ \nu V_c & 0 & \nu V_c & \left|\nu V_c-\varepsilon_2\right| & \nu V_c \\ 0 & 0 & 0 & (\gamma+b+\alpha)e^{b\tau} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma_1 & 0 & \gamma & 0 \end{bmatrix} \quad (2.25)$$

and for b being large enough such that it satisfies:

$$b \ge \max\left(\frac{1}{\tau}\max\left(\ln\frac{\gamma+\gamma_1}{\gamma+b+\alpha}, \ln\frac{4\max(1,\nu)}{\gamma+b+\alpha}\right), b_a\right) \quad (2.26)$$

with b_a being some existing real positive constant, depending on the vaccination constant V_c, such that $\nu(1-V_c) \ge \frac{\beta+\varepsilon_1}{1+\eta}$, it follows from inspection of (2.24)-(2.25) that $\left\|\tilde{A}_0^*\right\|_\infty = (\gamma+b+\alpha)e^{b\tau}$. Using again (2.18)-(2.19), it follows that the following close constraint to (2.21):

$$-\infty \le \frac{1}{2}\ln 5 + \ln(\gamma+\beta+\alpha) + b(\tau-\omega)$$
$$\le \frac{1}{2}\ln 5 + \ln(\gamma+\beta+\alpha) + b\tau + \ln\left(1+e^{-b\omega}\right) \le \ln(b-b_0) \quad (2.27)$$

ISBN: 978-988-19251-5-2
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

guarantees that

$$\left\| A_\tau^* \right\|_2 + \left\| A_\omega^* \right\|_2 + \left\| \tilde{A}_0^* \right\|_2 \le \sqrt{5}\left(\left\| A_\tau^* \right\|_\infty + \left\| A_\omega^* \right\|_\infty + \left\| \tilde{A}_0^* \right\|_\infty \right)_1$$
$$\le \sqrt{5}\left(\gamma + \beta + \alpha\right)\left(max\left(1, e^{b(\tau-\omega)}\right) + e^{b\tau}\right) \le \bar{\gamma} \quad (2.28)$$

where

$$\bar{\gamma}_1 (> \bar{\gamma}) = \begin{cases} \sqrt{5}\left(\gamma+\beta+\alpha\right)\left(1+e^{b\tau}\right) & if \quad \tau \le \omega \\ \sqrt{5}\left(\gamma+\beta+\alpha\right)\left(e^{b\tau}\left(1+e^{-b\omega}\right)\right) & if \quad \tau > \omega \end{cases} \quad (2.29)$$

Note that the linearized system about the disease-free equilibrium point) is asymptotically stable if and only if

$$det\left(sI - A_{0d}^* - \tilde{A}_0^* - \tilde{A}_\tau^* e^{-\tau s} - \tilde{A}_\omega^* e^{-\omega s}\right) \neq 0$$
$$; \quad \forall s \in \boldsymbol{C}_+ := \left\{ s \in \boldsymbol{C} : Re\, s \ge 0 \right\} \quad (2.30)$$

which is guaranteed under the two conditions below:

1) $det\left(sI - A_{0d}^*\right) \neq 0 \quad , \quad \forall s \in \boldsymbol{C}_+ := \left\{ s \in \boldsymbol{C} : Re\, s > 0 \right\}$, equivalently, A_{0d}^* is a stability matrix

2) The ℓ_2 -matrix measure $\mu_2\left(A_{0d}^*\right)$ of $\left(A_{0d}^*\right)$ is negative, and, furthermore, the following constraint holds

$$\bar{\gamma}_1 \le b - max\left(\left| \gamma_1 - \nu V_c \right|, \nu\left(1 - V_c\right) \right)$$

which guarantees the above stability condition 2 via (2.28)-(2.29) and (2.15):

$$\left\| \tilde{A}_0^* \right\|_2 + \left\| \tilde{A}_\tau^* \right\|_2 + \left\| \tilde{A}_\omega^* \right\|_2 \le \sqrt{5}\left(\gamma+\beta+\alpha\right)\left(max\left(1, e^{b(\tau-\omega)}\right) + e^{b\tau}\right) \le \bar{\gamma}_1$$

$$< \left| \mu_2\left(A_{0d}^*\right) \right| = \frac{1}{2}\left| \lambda_{max}\left(A_{0d}^* + A_{0d}^{*T} \right) \right| = \left| \lambda_{max}\left(A_{0d}^*\right) \right|$$
$$= b - max\left(\left| \gamma_1 - \nu V_c \right|, \nu\left(1 - V_c\right) \right) \quad (2.31)$$

The following result is proven from Proposition 1, the above asymptotic stability conditions for the linearized incremental system about the disease- free equilibrium point, which implies that of the nonlinear one (1.1)-(1.5) about the equilibrium point, and the related former discussion:

Proposition 2. It exists a sufficiently large $b > max\left(\left| \gamma_1 - \nu V_c \right|, \nu\left(1 - V_c\right) \right)$ such that the disease-free equilibrium point is locally asymptotically stable for any constant vaccination $V_c \in [0, 1]$ and a sufficiently small amount $\left(\gamma + \beta + \alpha\right)$, a sufficiently small delay τ and a sufficiently small $\left(\tau - \omega\right)$ (this being applicable if $\tau > \omega$). □

Note that the statement of Proposition 2 guarantees the local stability of the disease-free equilibrium point under its existence condition of Proposition 1 requiring $\nu < b$.

III. ENDEMIC EQUILIBRIUM POINT

The existence of one or two endemic equilibrium points which keep alive the disease propagation is now discussed:

Proposition 3. Assume that $\omega > 0$. Then, the following properties hold:

(i) $\beta \ge \dfrac{\eta e^{b\tau}\left(\gamma + b + \alpha\right)}{1+\delta}$ for $V_c > 0$ and $\beta \ge \eta e^{b\tau}\left(\gamma + b + \alpha\right)$ for $V_c = 0$. Thus, it exists at least one endemic equilibrium point at which the susceptible, vaccinated, infected, exposed and recovered populations are positive and the vaccinated population is zero if and only if $V_c = 0$ (i.e. in the absence of vaccination action). Furthermore, such an equilibrium point satisfies the constraints:

$$E^* = \frac{\beta}{b}\left(1 - e^{-b\omega}\right)\left(\frac{S^*}{1+\eta S^*} + \frac{\delta V^*}{1+\eta V^*} \right) I^* > 0$$

$$min\left(S^* + \delta V^*, \frac{1+\delta}{\eta} \right) \ge \frac{S^*}{1+\eta S^*} + \frac{\delta V^*}{1+\eta V^*} = \frac{e^{b\tau}\left(\gamma + b + \alpha\right)}{\beta} > 0$$

$$R^* = \frac{\gamma_1 V^* + \gamma\left(1 - e^{-b\omega}\right) I^*}{b} \ge \frac{\gamma\left(1 - e^{-b\omega}\right) I^*}{b} > 0$$

(ii) If the transmission constant is small enough satisfying $\beta < \bar{\beta} := \dfrac{\eta e^{b\tau}\left(\gamma + b + \alpha\right)}{1+\delta}$ for $V_c > 0$ and $\beta < \eta e^{b\tau}\left(\gamma + b + \alpha\right)$ for $V_c = 0$ then there is no reachable endemic equilibrium point. □

Proof: The endemic equilibrium point is calculated as follows:

$$\beta e^{-b\tau}\left(\frac{S^*}{1+\eta S^*} + \frac{\delta V^*}{1+\eta V^*} \right) - \left(\gamma + b + \alpha\right) = 0 \quad (3.1)$$

$$E^* = \frac{\beta}{b}\left(1 - e^{-b\omega}\right)\left(\frac{S^*}{1+\eta S^*} + \frac{\delta V^*}{1+\eta V^*} \right) I^* > 0 \quad (3.2)$$

with

$$E^* > 0, \; I^* > 0 \quad (3.3)$$

$$\frac{S^*}{1+\eta S^*} + \frac{\delta V^*}{1+\eta V^*} = \frac{e^{b\tau}\left(\gamma + b + \alpha\right)}{\beta} > 0 \text{ for } \omega \neq 0 \quad (3.4)$$

(since, otherwise, the above disease – free equilibrium point would be being considered). $S^* > 0$ since, otherwise, the following contradiction would follow:

$$0 < b + \gamma I^* e^{-b\omega} + \nu N^*\left(1 - V_c\right) = 0 \quad (3.5)$$

$V^* = 0$ if and only if $V_c = 0$, since otherwise for $V_c > 0$ and $V^* = 0$, it would follow that $\nu N^* V_c = 0$ which is only possible in the disease-free equilibrium point if the total population is extinguished what is a contradiction at the endemic point.

$$R^* = \frac{\gamma_1 V^* + \gamma\left(1 - e^{-b\omega}\right) I^*}{b} \ge \frac{\gamma\left(1 - e^{-b\omega}\right) I^*}{b} > 0 \quad (3.6)$$

for $\omega \neq 0$

Remark 2. Note that if $\omega = 0$ then it follows from (1.3) and (2.3) that $E(t) = E^* = 0$; $\forall t \in \boldsymbol{R}_{0+}$ so that the SVEIRS model (1.1)-(1.5) becomes a simpler SVIRS one without specification of the infected population dynamics. □

ISBN: 978-988-19251-5-2
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

Remark 3. Note that, under the constraints in Proposition 3 (ii) for β, if there is no reachable endemic equilibrium point because $\beta < \overline{\beta}$ then the solution trajectory of (1.1)-(1.5) can only either converge to the disease-free equilibrium point provided that it is at least locally asymptotically stable or to be bounded converging or not to an oscillatory solution or to diverge to an unbounded total population depending on the values of the parameterization of the model (1.1)-(1.5). Note that the endemic free transmission constant upper-bound $\overline{\beta}$ increases as η, τ and $(\gamma + b + \alpha)$ increase and also as δ decreases. □

If $V_c > 0$ then it follows from Proposition 3 that there exist positive constants α_S, α_V, α_E, α_I and α_R satisfying $\alpha_S^{-1} + \alpha_V^{-1} + \alpha_E^{-1} + \alpha_I^{-1} + \alpha_R^{-1} = 1$ such that the endemic equilibrium points, if any, satisfy the constraints:

$$N^* = \alpha_S S^* = \alpha_V V^* = \alpha_E E^* = \alpha_I I^* = \alpha_R R^* \quad (3.7)$$

so that, one gets from (3.2)- (3.6) that

$$R^* = \frac{\gamma_1/\alpha_V + \gamma\left(1 - e^{-b\omega}\right)/\alpha_I}{b}\alpha_R R^*$$
$$= \frac{\gamma_1 \alpha_I + \gamma\left(1 - e^{-b\omega}\right)\alpha_V}{b\,\alpha_I \alpha_V}\alpha_R R^* \quad (3.8)$$

$$\frac{\beta}{b}\left(1 - e^{-b\omega}\right)\frac{1+\delta}{1+\eta} \le E^*/I^* = \alpha_I/\alpha_E = \frac{\beta}{b}\left(1 - e^{-b\omega}\right)\left(\frac{S^*}{1+\eta S^*} + \frac{\delta V^*}{1+\eta V^*}\right) \le \frac{\beta}{b}\left(1 - e^{-b\omega}\right)\frac{1+\delta}{\eta} \quad (3.9)$$

if $\min\left(S^*, V^*\right) \ge 1$, otherwise if $\nu = 0$, then only the lower-bounding constraint holds in (3.9);

$$b - \left(b + \frac{\beta\alpha_S S^*}{\alpha_I\left(1+\eta S^*\right)}\right)S^* + \gamma\frac{\alpha_S}{\alpha_I}S^* e^{-b\omega} + \nu\alpha_S S^*\left(1 - V_c\right) = 0 \quad (3.10)$$

$$\frac{\alpha_V V^*}{\alpha_S + \eta\alpha_V V^*} + \frac{\delta V^*}{1+\eta V^*} = \frac{e^{b\tau}(\gamma + b + \alpha)}{\beta} \quad (3.11)$$

Eq. (3.8) is equivalent, since $R^* > 0$ at the endemic equilibrium point, to

$$\frac{\gamma_1 \alpha_I \alpha_R + \gamma\left(1 - e^{-b\omega}\right)\alpha_V \alpha_R}{b\,\alpha_I \alpha_V} = 1 \quad (3.12)$$

Eq. (3.10) is equivalent to

$$\left[\alpha_S\,\eta\left(\nu\alpha_I\left(1 - V_c\right) + \gamma e^{-b\omega}\right) + \beta\alpha_S - b\alpha_I\eta\right]S^{*2} + \left[\alpha_S\left(\gamma e^{-b\omega} + \nu\alpha_I\left(1 - V_c\right)\right) + b\alpha_I(\eta - 1)\right]S^* + b\alpha_I = 0 \quad (3.13)$$

Eq. (3.13) is an algebraic equation of real coefficients of the form $aS^{*2} + dS^* + c = 0$ with $c > 0$. Such an equation has two positive real roots if $a > 0, d < 0$ and $d^2 \ge 4ac$ and one positive real root if $a < 0$ and $d > 0$. Thus, since there is a nonzero susceptible population at an endemic equilibrium point then either (3.14)-(3.16) below hold:

$$\alpha_S\,\eta\left(\nu\alpha_I\left(1 - V_c\right) + \gamma e^{-b\omega}\right) + \beta\alpha_S > b\alpha_I\eta \quad (3.14)$$

$$\alpha_S\left(\gamma e^{-b\omega} + \nu\alpha_I\left(1 - V_c\right)\right) < d\alpha_I\left(1 - \eta\right) \quad (3.15)$$

provided that $\eta < 1$

$$\left[\alpha_S\left(\gamma e^{-b\omega} + \nu\alpha_I\left(1 - V_c\right)\right) + d\alpha_I(\eta - 1)\right]^2 \ge 4b\alpha_I\left[\alpha_S\,\eta\left(\nu\alpha_I\left(1 - V_c\right) + \gamma e^{-b\omega}\right) + \beta\alpha_S - b\alpha_I\eta\right] \quad (3.16)$$

or, alternatively,

$$\beta < \frac{\alpha_I}{\alpha_S}b\eta - \left(\nu\alpha_I\left(1 - V_c\right) + \gamma e^{-b\omega}\right)\eta = \frac{\eta}{I^*}\left[bS^* - \left(\nu N^*\left(1 - V_c\right) + \gamma e^{-b\omega}\right)\right] \quad (3.17)$$

and

$$b < \frac{\alpha_S\left(\gamma e^{-b\omega} + \nu\alpha_I\left(1 - V_c\right)\right)}{\alpha_I\left(1 - \eta\right)} = \frac{\gamma e^{-b\omega} I^* + \nu N^*\left(1 - V_c\right)}{S^*\left(1 - \eta\right)} \quad (3.18)$$

with $\eta < 1$ hold. On the other hand, Eq. (3.11) is equivalent to

$$\alpha_V\beta_0\left(1 + \eta V^*\right)V^* + \delta\beta_0 V^*\left(\alpha_S + \eta\alpha_V V^*\right) = \left(1 + \eta V^*\right)\left(\alpha_S + \eta\alpha_V V^*\right) \quad (3.19)$$

where $\beta_0 := \dfrac{\beta}{e^{b\tau}(\gamma + b + \alpha)}$ so that (3.19) is of the form $aV^{*2} + dV^* + c = 0$ specifically as follows

$$\left(\eta - (1+\delta)\beta_0\right)\alpha_V\eta V^{*2} + \left(\alpha_V(\eta - \beta_0) + (\eta - \delta\beta_0)\alpha_S\right)V^* + \alpha_S = 0 \quad (3.20)$$

Now, the same reasoning as that used for the susceptible endemic equilibrium component is applied to (3.20) to yield that since there is a nonzero vaccinated population at two endemic equilibrium points then

$$(\gamma + b + \alpha)e^{b\tau}\frac{\eta}{1+\delta} \le \beta \le (\gamma + b + \alpha)e^{b\tau}\frac{1+\eta}{1+\delta} \quad (3.21)$$

which is obtained from (3.1), and either

$$\alpha_V(\beta_0 - \eta) + (\delta\beta_0 - \eta)\alpha_S = \left(\alpha_V + \alpha_S\delta\right)\beta_0 - \eta\left(\alpha_V + \alpha_S\right) > 0 \quad (3.22)$$

$$\beta_0 < \frac{\eta\alpha_V + \eta\alpha_S}{\alpha_V + \alpha_S\delta} \Leftrightarrow \beta < \frac{\eta\alpha_V + \eta\alpha_S}{\alpha_V + \alpha_S\delta}e^{b\tau}(\gamma + b + \alpha) \quad (3.23)$$

$$\left(\alpha_V(\eta - \beta_0) + (\eta - \delta\beta_0)\alpha_S\right)^2 > 4\left(\eta - (1+\delta)\beta_0\right)\alpha_V\alpha_S\eta \quad (3.24)$$

or, $a < 0$, $d > 0$ in (3.20). The above discussion concerning the endemic equilibrium point is summarized as follows:

Proposition 4. Assume that $V_c \in (0, 1]$ and that

$$(\gamma + b + \alpha)e^{b\tau}\frac{\eta}{1+\delta} \le \beta \le (\gamma + b + \alpha)e^{b\tau}\frac{1+\eta}{1+\delta}$$

(the upper-bounding condition does not hold if $\min\left(S^*, R^*\right) < 1$) so that

$N^* = \alpha_S S^* = \alpha_V V^* = \alpha_E E^* = \alpha_I I^* = \alpha_R R^*$ for some positive constants $\alpha_S, \alpha_V, \alpha_E, \alpha_I$ and α_R. Then, it exists at least one endemic equilibrium point, and at most two endemic equilibrium points, with all the corresponding partial populations being positive and the following parametrical constraints hold:

$$\alpha_S^{-1} + \alpha_V^{-1} + \alpha_E^{-1} + \alpha_I^{-1} + \alpha_R^{-1} = 1$$

ISBN: 978-988-19251-5-2
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$$\frac{\beta}{b}\left(1-e^{-b\omega}\right)\frac{1+\delta}{1+\eta} \le \alpha_I / \alpha_E \le \frac{\beta}{b}\left(1-e^{-b\omega}\right)\frac{1+\delta}{\eta}$$

The constants α_S, α_I and α_V satisfy either (3.14)-(3.16), or (3.18), and (3.22)-(3.24). □

IV. INFECTION PROPAGATION AND POSITIVITY

This section discuses briefly the monotone increase of the infected population and the boundedness of the total population as well as the positivity of the model

Proposition 5. If the infection propagates through $(t-\tau, t)$ with the infected population being monotone increasing then $\frac{S(\sigma)}{1+\eta S(\sigma)}+\frac{\delta V(\sigma)}{1+\eta V(\sigma)} \ge \frac{\gamma+b+\alpha}{\beta} e^{b\sigma}$; $\forall \sigma \in (t^*-2\tau, t^*-\tau)$

Proof: Note from (1.4) that for $t \in (t^*-2\tau, t^*)$

$$\dot{I}(t)>0 \Leftrightarrow \frac{I(t)}{I(t-\tau)} < \frac{\beta e^{-b\tau}}{\gamma+b+\alpha}\left(\frac{S(t-\tau)}{1+\eta S(t-\tau)}+\frac{\delta V(t-\tau)}{1+\eta V(t-\tau)}\right)$$

and if, furthermore, $I(t)>I(t-\tau)$ for $t \in (t^*-\tau, t^*)$, thus

$$1 < \frac{I(t)}{I(t-\tau)} < \frac{\beta e^{-b\tau}}{\gamma+b+\alpha}\left(\frac{S(t-\tau)}{1+\eta S(t-\tau)}+\frac{\delta V(t-\tau)}{1+\eta V(t-\tau)}\right) \quad \square$$

Now, rewrite (1.3) in differential equivalent form by using Leibniz´s rule as follows:

$$\dot{E}(t) = -bE(t) + \beta\left[\left(\frac{S(t)}{1+\eta S(t)}+\frac{\delta S(t)}{1+\eta V(t)}\right)I(t) - \left(\frac{S(t-\omega)}{1+\eta S(t-\omega)}+\frac{\delta S(t-\omega)}{1+\eta V(t-\omega)}\right)I(t-\omega)e^{-b\omega}\right] \quad (4.1)$$

Proposition 6. Assume that $\nu < b$. Then, the following properties hold:

(i) The total population is uniformly bounded for all time, irrespective of the susceptible and vaccinated populations, for any bounded initial conditions with

$$N(\infty) \le \frac{1}{b-\nu}\left(b+\beta\frac{1+\delta}{1+\eta}e^{-b\tau}\right) < \infty$$

(ii) Assume that $\beta \ge \frac{1}{1+\delta}\left(\frac{b\eta(1+\eta)}{\eta e^{-b\omega}-(1+\eta)e^{-b\tau}}\right)$ subject to $\frac{\eta}{1+\eta} > e^{b(\omega-\tau)}$ and $\omega<\tau$. Then, $N: \boldsymbol{R}_{0+} \to$ is monotone decreasing and exponentially vanishing.

Proof: Consider the SVEIRS model in differential form described by (1.1), (1.2), (1.4)-(1.5) and (4.1). Summing up the five equations, one gets directly:

$$\begin{aligned}\dot{N}(t) &= (\nu-b)N(t)+b-\alpha I(t) \\ &+\beta\left(\frac{S(t-\tau)I(t\tau)}{1+\eta S(t-\tau)}+\frac{\delta V(t-\tau)I(t-\tau)}{1+\eta V(t-\tau)}\right)e^{-b\tau} \\ &-\beta\left(\frac{S(t-\omega)I(t-\omega)}{1+\eta S(t-\omega)}+\frac{\delta V(t-\omega)I(t-\omega)}{1+\eta V(t-\omega)}\right)e^{-b\omega} \\ &= (\nu-b)N(t)+b-\alpha I(t)+\frac{1+\delta}{\eta} \end{aligned} \quad (4.2)$$

since $\frac{S(t)}{1+\eta S(t)}+\frac{\delta V(t)}{1+\eta V(t)} \le \frac{1+\delta}{\eta}$

This yields directly Property (i). On the other hand, note from (4.2) that:

a) If $min(S(t), V(t)) \ge 1$ then

$$0 < \frac{1+\delta}{1+\eta} \le \frac{S(t)}{1+\eta S(t)}+\frac{\delta V(t)}{1+\eta V(t)} \le \frac{1+\delta}{\eta} \quad (4.3)$$

so that, since $\beta \ge \frac{1}{1+\delta}\left(\frac{b\eta(1+\eta)}{\eta e^{-b\omega}-(1+\eta)e^{-b\tau}}\right)$,

$$\begin{aligned}\dot{N}(t) &\le (\nu-b)N(t)+b-\alpha I(t)+\beta\left[\frac{1+\delta}{\eta}e^{-b\tau}-\frac{1+\delta}{1+\eta}e^{-b\omega}\right] \\ &< (\nu-b)N(t)-\alpha I(t) \end{aligned} \quad (4.4)$$

b) If $max(S(t), V(t)) \le 1$ then the same conclusion as for Case a arises so that Property (ii) is proven. □

A brief discussion about positivity is summarized in the next result:

Proposition 7. Assume that $V_c \in [0, 1]$. Then, the SVEIRS epidemic model (1.1)-(1.5) is positive in the sense that no partial population is negative at any time if its initial conditions are non-negative and the vaccinated population exceeds a certain minimum measurable threshold in the event that the recovered population is zero as follows: $V(t) \ge max\left(\frac{\gamma}{\gamma_1}\left(I(t-\omega)e^{-b\omega}-I(t)\right), 0\right)$ if $R(t)=0$. The susceptible, vaccinated, exposed and infected populations are nonnegative for all time irrespective of the above constraint. If, in addition, Proposition 6(i) holds then all the partial populations of the SVEIRS model are uniformly bounded for all time.

Proof: First note that all the partial populations are defined by continuous- time differentiable functions from (1.1)-(1.5). Then, if any partial population is negative, it is zero at some previous time instant. Assume that S(σ) ≥ 0 for $\sigma < t$ and S(t)=0 at some time instant t, Then from (1.1):

$$\dot{S}(t) = b+\gamma I(t-\omega)e^{-b\omega}+\nu(1-V_c)N(t) \ge 0;\ \forall V_c \in [0, 1]$$

Thus, $S(t^+) \ge 0$. As a result, S(t) cannot reach negative values at any time instant. Assume that V(σ) ≥ 0 for $\sigma < t$ and V(t)=0 at some time instant t. Then, $\dot{V}(t) = \nu V_c N(t) \ge 0$ from (1.2) so that $V(t^+) \ge 0$. As a result, V (t) cannot reach negative values at any time. $E(t) \ge 0$ for any time instant t from (1.3). Assume that I(σ) ≥ 0 for $\sigma < t$ and $I(t)=0$ at some time instant t. Then, $\dot{I}(t) \ge 0$ from (1.4). As a result, I(t) cannot reach negative values at any time. Finally, assume that R(σ) ≥ 0 for $\sigma < t$ and R(t)=0 at some time instant t. Thus, $\dot{R}(t) = \gamma_1 V(t)+\gamma\left(I(t)-I(t-\omega)e^{-b\omega}\right) \ge 0$ from (1.5) if $V(t) \ge max\left(\frac{\gamma}{\gamma_1}\left(I(t-\omega)e^{-b\omega}-I(t)\right), 0\right)$. Thus, if $V(t) \ge max\left(\frac{\gamma}{\gamma_1}\left(I(t-\omega)e^{-b\omega}-I(t)\right), 0\right)$ when R(t)=0 then all the partial populations are uniformly bounded, since they are nonnegative and the total population N(t) is uniformly bounded from Proposition 6 (i). □

ISBN: 978-988-19251-5-2
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

V. CONCLUSION

This paper discusses the disease-free and endemic equilibrium points of a SVEIRS propagation disease model which potentially involves a regular constant vaccination. The positivity of such a model is also discussed as well as the boundedness of the total and partial populations. The model takes into consideration the natural population growing and the mortality associate to the disease as well as the lost of immunity of newborns. It is assumed that there are two finite delays, being associated with latent and immune periods, affecting to the susceptible, recovered, exposed and infected population dynamics.

Acknowledgements

The authors thank to the Spanish Ministry of Education by Grant DPI2009-07197. They are also grateful to the Basque Government by its support through Grants IT378-10, SAIOTEK S-PE08UN15, and SAIOTEK SPE07UN04 and SPE09UN12.

References

[1] M. De la Sen and S. Alonso-Quesada, "A control theory point of view on Beverton-Holt equation in population dynamics and some of its generalizations", *Applied Mathematics and Computation*, Vol. 199, No. 2, pp. 464-481, 2008.

[2] M. De la Sen and S. Alonso-Quesada, "Control issues for the Beverton-Holt equation in population in ecology by locally monitoring the environment carrying capacity: Non-adaptive and adaptive cases", *Applied Mathematics and Computation*, Vol. 215, No. 7, pp. 2616-2633, 2009.

[3] M. De la Sen and S. Alonso-Quesada, "Model-matching-based control of the Beverton-Holt equation in Ecology", *Discrete Dynamics in Nature and Society*, Article number 793512, 2008.

[4] M. De la Sen, "About the properties of a modified generalized Beverton-Holt equation in ecology models", *Discrete Dynamics in Nature and Society*, Article number 592950, 2008.

[5] M. De la Sen, "The generalized Beverton- Holt equation and the control of populations", *Applied Mathematical Modelling*, Vol. 32, No. 11, pp. 2312-2328, 2008.

[6] D. Mollison Ed., *Epidemic Models: Their Structure and Relation to Data*, Publications of the Newton Institute, Cambridge University Press, 1995, Denis Mollison Editor (transferred to digital printing 2003).

[7] M. J. Keeling and P. Rohani, *Modeling Infectious Diseases in Humans and Animals*, Princeton University Press, Princeton and Oxford, 2008.

[8] A. Yildirim and Y. Cherruault, "Anaytical approximate solution of a SIR epidemic model with constant vaccination strategy by homotopy perturbation method", *Kybernetes*, Vol. 38, No. 9, pp. 1566-1575, 2009.

[9] V.S. Erturk and S. Momani, "Solutions to the problem of prey and predator and the epidemic model via differential transform method", *Kybernetes*, Vol. 37, No. 8 , pp. 1180-1188, 2008.

[10] N. Ortega, L.C. Barros and E. Massad, "Fuzzy gradual rules in epidemiology", *Kybernetes*, Vol. 32, Nos. 3-4, pp. 460-477, 2003.

[11] H. Khan, R.N. Mohapatra, K. Varajvelu and S, J. Liao, "The explicit series solution of SIR and SIS epidemic models", *Applied Mathematics and Computation*, Vol. 215, No. 2, pp. 653-669, 2009.

[12] X. Y. Song, Y. Jiang and H.M. Wei, "Analysis of a saturation incidence SVEIRS epidemic model with pulse and two time delays", *Applied Mathematics and Computation*, Vol. 214, No. 2 , pp. 381-390, 2009.

[13] T.L. Zhang, J.L. Liu and Z.D. Teng, "Dynamic behaviour for a nonautonomous SIRS epidemic model with distributed delays", *Applied Mathematics and Computation*, Vol. 214, No. 2, pp. 624-631, 2009.

[14] B. Mukhopadhyay and R. Battacharyya, "Existence of epidemic waves in a disease transmission model with two-habitat population", *International Journal of Systems Science*, Vol. 38, No. 9, pp. 699-707, 2007.

[15] T. Zhang, J. Liu and Z.Teng, "Dynamic behavior for a nonautonomous SIRS epidemic model with distributed delay ", *Applied Mathematics and Computation*, Vol. 214, No.1, pp. 624-631, 2009.

[16] S. Gao, Z. Teng and D. Xie, "The effects of pulse vaccination on SEIR model with two time delays ", *Applied Mathematics and Computation*, Vol. 201, Nos.1-2 pp. 282-292, 2008.

[17] Q.J.A. Khan, E.V. Krishnan and Al- Khodh, "An epidemic model with a time-delay transmission", *Applications of Mathematics*, Vol. 48, No. 3 , pp. 193-203, 2003.

[18] D.J. Daley and J. Gani, *Epidemic Modelling. An Introduction*, Cambridge Studies in Mathematical Biology: 15, Cambridge University Press, New York, 2005.

[19] C. Piccardi and S. Lazzaris, "Vaccination policies for chaos reduction in childhood epidemics", *IEEE Transactions on Biomedical Engineering*, Vol. 45, No. 5, pp. 591-595, 1998.

[20] V. M. Marchenko and Z. Zaczkiewicz, "Representation of solutions of control hybrid differential-difference impulse systems ", *Differential Equations* , Vol. 45, No. 12, pp 1811-1822, 2009.

[21] V. M. Marchenko and J.J. Loiseau, " On the stability of hybrid difference-differential systems", *Differential Equations* , Vol. 45, No. 12, pp 1811-1822, 2009.

[22] A. Barreiro and A. Banos, "Delay-dependent stability of reset systems", *Automatica*, Vol. 46, No. 1, pp 216-221, 2010.

[23] M. De la Sen, "On positivity of singular regular linear time-delay time-invariant systems subject to multiple internal and external incommensurate point delays", *Applied Mathematics and Computation*, Vol. 190, No. 12, pp. 382-401, 2007.

[24] M. De la Sen, "A method for general design of positive real functions", *IEEE Transactions on Circuits and Systems I – Fundamental Theory and Applications*, Vol. 45, No. 7, pp. 764-769, 1998.

[25] M. De la Sen, "Quadratic stability and stabilization of switched dynamic systems with uncommensurate internal point delays", *Applied Mathematics and Computation*, Vol. 185, No. 1, pp. 508-526, 2007.

[26] M. De la Sen and S. Alonso-Quesada, "On vaccination control tools for a general SEIR- epidemic model", *Proceedings of the 18th IEEE Mediterranean Conference on Control and Automation*, Vol. 1, pp. 1322-1328, 2010.

[27] HG Yu, SM Zhong, RP Agarwal and LL Xiong, "Species permanence and dynamical behaviour analysis of an impulsively controlled ecological system with distributed time delay", *Computers & Mathematics with Applications*, Vol. 59, No. 2, pp. 3824- 3835, 2010.

[28] A Boichuk, M Langerová and J Skorková , "Solutions of linear impulsive differential systems bounded on the entire real axis", *Advances in Difference Equations*, Article number 494379, 2010.

[29] JJ Nieto and CC Tisdell, "On exact controllability of first-order impulsive differential equations", *Advances in Difference Equations*, Article number 136504, 2010.

[30] Y Jiang, W. Huiming, X. Song, L. Mei, G. Su and S. Qiu, " Global attractivity and permanence of a delayed SVEIR epidemic model with pulse vaccination and saturation incidence" *Applied Mathematics and Computation*, Vol. 213, pp. 312-321, 2009.

[31] X. Song, Y. Jiang and H. Wei, " Analysis of a saturation incidence SVEIRS epidemic model with pulse and two time delays" *Applied Mathematics and Computation*, Vol. 214, pp. 381-390, 2009.