Nonparametric Identification of the Production Functions

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***Abstract*—A class of semi-recursive kernel plug-in estimates of functions depending on multivariate density functionals and their derivatives is considered. The approach enables to estimate the production function, marginal productivity and marginal rate of technical substitution of inputs. The piecewise smoothed approximations of these estimates are proposed. The main parts of the asymptotic mean square errors (AMSE) of the estimates are found. The results are generalized to the production functions with the lagged values of the output.**

***Index Terms*—Almost surely convergence, kernel recursive estimator, mean square convergence, piecewise smooth approximation.**

I. Introduction

NUMEROUS statistical problems (such as identification, classification, filtering, prediction, etc.) are connected to estimation of certain characteristics of the following expressions:

$$J(x) = H\left(\{a_i(x)\},\ \{a_i^{(1j)}(x)\},\ i=\overline{1,s},\ j=\overline{1,m}\right) =$$

$$= H\left(a(x),\ a^{(1j)}(x)\right). \tag{1}$$

Here $x \in \mathsf{R}^m$, $H(\cdot) : \mathsf{R}^{(m+1)s} \to \mathsf{R}^1$ is a given function,

$$a^{(0j)}(x) = a(x) = (a_1(x), \dots, a_s(x)),$$

$$a^{(1j)}(x) = \left(a_1^{(1j)}(x), \dots, a_s^{(1j)}(x)\right),$$

$$a_i(x) = \int g_i(y) f(x,y)dy, \quad i=\overline{1,s},$$

$$a_i^{(1j)}(x) = \frac{\partial a_i(x)}{\partial x_j}, \quad i=\overline{1,s}, \quad j=\overline{1,m},$$

where $g_1, \dots, g_s$ are the known Borel functions, $\int \equiv \int_{\mathsf{R}^1}$, $f(\cdot,\cdot)$ is an unknown probability density function (p.d.f.) for the observed random vector $Z=(X,Y) \in \mathsf{R}^{m+1}$.

If $g_i(y) \equiv 1$, then $a_i(x) = \int f(x,y)dy = p(x)$, where $p(\cdot)$ is the marginal p.d.f. of the random variable X, and $f(y|\,x) = f(x,y)/p(x)$ is the conditional p.d.f.

Here are the well known examples of such functions:

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— the conditional initial moments

$$\mu_m(x) = \int y^m f(y|x)dy,\ H(a_1,a_2) = a_1/a_2,\ m \geq 1,$$

$g_1(y) = y^m,\ g_2(y) = 1;\ \mu_1(x) = r(x)$ is the regression line;

— the conditional central moments

$$V_m(x) = \int (y - r(x))^m f(y|x)dy,\ g_1(y) = y,$$

$$g_2(y) = y^2, \dots,\ g_m(y) = y^m,\ g_{m+1}(y) = 1;$$

$V_2(x) = D(x)$ is the conditional variance;

— the conditional coefficient of skewness

$$\beta_1(x) = \frac{\mathsf{E}((Y - r(x))|x)^3}{[\mathsf{D}(Y|x)]^{3/2}},\ b_i = a_i/a_1,\ g_i(y) = y^{i-1},$$

$$H(a_1,a_2,a_3,a_4) = (b_4 - 3b_3b_2 + 2b_2^3)/(b_3 - b_2^2)^{3/2};$$

— the sensitivity functions

$$T_j(x) = \frac{\partial r(x)}{\partial x_j},\ g_1(y) = 1,\ g_2(y) = y,$$

$$H\left(a_1, a_2, a_1^{(1j)}, a_2^{(1j)}\right) = \frac{a_1^{(1j)}}{a_2} - \frac{a_1 a_2^{(1j)}}{a_2^2} = b_2^{(1j)}.$$

II. Problem Statement

Take the following expression as an estimate of the functional $a(x) = a^{(0j)}(x)$ $(r=0)$ and its derivatives $a^{(1j)}(x)$ $(r=1)$ at a point x:

$$a_n^{(rj)}(x) = \frac{1}{n}\sum_{i=1}^{n} \frac{g(Y_i)}{h_i^{m+r}} \mathbf{K}^{(rj)}\left(\frac{x - X_i}{h_i}\right). \tag{2}$$

Here $Z_i = (X_i, Y_i),\ i=\overline{1,n},$ is the $(m+1)$-dimensional random sample from p.d.f. $f(\cdot,\cdot)$, (h_i) is a sequence of positive bandwidths tending to 0 as $i \to \infty$, $\mathbf{K}^{(0j)}(u) = \mathbf{K}(u) = \prod_{i=1}^{m} K(u_i)$ is a m-dimensional multiplicative function which does not need to possess the properties of p.d.f.,

$$\mathbf{K}^{(1j)}(u) = \frac{\partial \mathbf{K}(u)}{\partial u_j}, g(y) = (g_1(y), \dots, g_s(y)),$$

$$a_n^{(rj)}(x) = \left(a_{1n}^{(rj)}(x), \dots, a_{sn}^{(rj)}(x)\right).$$

Note that (2) can be computed recursively by

$$a_n^{(rj)}(x) = a_{n-1}^{(rj)}(x) -$$

$$-\frac{1}{n}\left[a_{n-1}^{(rj)}(x) - \frac{g(Y_n)}{h_n^{m+r}}\mathbf{K}^{(rj)}\left(\frac{x - X_n}{h_n}\right)\right]. \tag{3}$$

This property is particularly useful when the sample size is large since (3) can be easily updated with each additional observation.

The recursive kernel estimate of $p\,(x)$ (the case when $m = 1,\quad s = 1,\quad g(y) = 1,\quad H(a_1) = a_1)$ was introduced by Wolverton and Wagner in [1] and apparently independently by Yamato [2], and has been thoroughly examined in [3].

The semi-recursive kernel estimates of conditional functionals

$$b(x) = (b_1(x), \ldots, b_{s-1}(x)),$$

$$b_i(x) = a_i(x)/p\,(x) = \int g_i(y) f(y|\,x) dy$$

at a point x are designed as $(g_s(x) = 1)$

$$b_n(x) = \frac{\sum_{i=1}^{n} \frac{g(Y_i)}{h_i^m} \mathbf{K}\left(\frac{x - X_i}{h_i}\right)}{\sum_{i=1}^{n} \frac{1}{h_i^m} \mathbf{K}\left(\frac{x - X_i}{h_i}\right)} = \frac{a_n(x)}{p_n(x)} = \frac{a_n^{(0j)}(x)}{a_{sn}^{(0j)}(x)}.$$

Such estimates are called semi-recursive because they can be updated sequentially by adding extra terms to both the numerator and denominator when new observations became available. If $g_1(y) = y$ $(s = 2)$, we obtain semi-recursive kernel estimates of the regression line [4]– [6]. Weak and strong universal consistency of such estimates was investigated in [7]– [11].

For estimation of (1) we are going to use the following statistics

$$J_n(x) = H\left(\left\{a_n^{(rj)}(x)\right\},\quad j = \overline{1, m},\quad r = 0, 1\right). \qquad (4)$$

Plug-in estimates (4) are often used for the estimation of ratios. There are problems connected with unboundedness of the estimates at some points (see [12] for details). This problems can be solved by make using of the piecewise smooth approximation [13]

$$\widetilde{J}_n(x) = \frac{J_n(x)}{(1 + \delta_n |J_n(x)|^\tau)^\rho}, \qquad (5)$$

where $\tau > 0,\quad \rho > 0,\quad \rho\tau \geq 1,\quad (\delta_n) \downarrow 0$ as $n \to \infty$.

III. Mean square errors

Denote:

$$\sup_x = \sup_{x \in \mathsf{R}^m},\quad K^{(1)}(u) = \frac{dK(u)}{du},$$

$$T_j = \int u^j K(u) du,\ j = 1, 2, \ldots.$$

Definition 1. A function $H(\cdot) : \mathsf{R}^s \to \mathsf{R}^1$ belongs to the class $\mathcal{N}_\nu(t)$ $(H(\cdot) \in \mathcal{N}_\nu(t))$ if it is continuously differentiable up to the order ν at the point $t \in \mathsf{R}^s$. A function $H(\cdot) \in \mathcal{N}_\nu(\mathsf{R})$ if it is continuously differentiable up to the order ν for any $z \in \mathsf{R}^s$.

Definition 2. A Borel function $K(\cdot) \in \mathcal{A}^{(r)}$, $(\mathcal{A}^{(0)} = \mathcal{A})$ if $\int |\,K^{(r)}(u)|\,du < \infty$, and $\int K(u)\,du = 1$.

Definition 3. A Borel function $K(\cdot) \in \mathcal{A}_\nu^{(r)}, (\mathcal{A}_\nu^{(0)} = A_\nu)$ if $K(\cdot) \in \mathcal{A}^{(r)}$, $T_j = 0$, $j = 1, \ldots, \nu - 1$, $T_\nu \neq 0$, $\int |u^\nu K(u)| du < \infty$, and $K(u) = K(-u)$.

Definition 4. A sequence $(h_n) \in \mathcal{H}(m + r + q)$ if

$$(h_n + 1/(nh_n^{m+r+q})) \downarrow 0,\quad \frac{1}{n}\sum_{i=1}^{n} h_i^\lambda = S_\lambda h_n^\lambda + o(h_n^\lambda),$$

where λ is a real number, S_λ is a constant independent on n; $r, q = 0, 1$.

Definition 5. Let $t_n,\ X_1, \ldots, X_n$ are vectors, and $t_n = t_n(X_1, \ldots, X_n)$. A sequence of functions $\{H(t_n)\}$ belongs to the class $\mathcal{M}(\gamma)$ if for any possible values $X_1, \ldots, X_n$ the sequence $\{|H(t_n)|\}$ is dominated by a sequence of numbers $(C_0 d_n^\gamma),\ (d_n) \uparrow \infty$ as $n \to \infty$, $0 \leq \gamma < \infty$, C_0 is a constant.

Put for $r, q = 0, 1;\ t, p = \overline{1, s};\ j = \overline{1, m}:$

$$A = A(x) = \left\{a^{(rj)}(x)\right\};\ H_{tjr} = \partial H(A)/\partial a_t^{(rj)};$$

$$H\left(\left\{a_n^{(rj)}(x)\right\}\right) = H(A_n);\ a^{s+}(x) = \int |g^s(y)| f(x, y) dy;$$

$$a_{t,\,p}(x) = \int g_t\,(y) g_p\,(y)\, f(x, y)\, dy;$$

$$a_{t,\,p}^{1+}(x) = \int |g_t\,(y) g_p\,(y)|\, f(x, y)\, dy;$$

$$L^{(r,\,q)} = \int K^{(r)}(u) K^{(q)}(u)\, du;$$

$$\mathcal{B}_{t,\,p}^{(r,\,q)} = L^{(r,\,q)} \left(L^{(0,\,0)}\right)^{m-1} a_{t,\,p}\,(x);$$

$$\omega_{i\nu}^{(rj)}(x) = \frac{T_\nu}{\nu!} \sum_{l=1}^{m} \frac{\partial^\nu a_i^{(rj)}(x)}{\partial x_l^\nu};$$

the set

$$Q = \begin{cases} \{0\} & \text{if } \forall j\ \ r = 0; \\ \{1\} & \text{if } \forall j\ \ r = 1; \\ \{0, 1\} & \text{if } \exists j\ \ r = 0 \bigwedge r = 1. \end{cases}$$

Theorem 1 (the AMSE of the estimate $J_n(x)$). If for $t, p = \overline{1, s},\quad j = \overline{1, m},\quad r \in Q:$

1) the functions $a_{t,\,p}\,(\cdot) \in \mathcal{N}_0(\mathsf{R}),\ \sup_x a_{t,\,p}^{1+}\,(x) < \infty$, $\sup_x a_t^{1+}(x) < \infty$, $\sup_x a_t^{4+}(x) < \infty$;

2) the kernel function $K(\cdot) \in \mathcal{A}_\nu^{(\max(r))}$, $\sup_x \left|K^{(r)}(x)\right| < \infty$, if $Q = \{0, 1\}$ then $K^{(r)}(\cdot) \in \mathcal{N}_0(\mathsf{R})$, if $1 \in Q$ then $\lim_{|u| \to \infty} K(u) = 0$;

3) $a_t^{(rj)}(\cdot) \in \mathcal{N}_\nu(\mathsf{R})$, $\sup_x |\,a_t^{(rj)}(x)| < \infty$, $\sup_x \left|\frac{\partial^\nu a_t^{(rj)}(x)}{\partial x_l \partial x_t \ldots \partial x_q}\right| < \infty,\ \ l, t, \ldots, q = \overline{1, m}$;

4) the sequence $(h_n) \in \mathcal{H}(m + 2\max(r))$;

5) $H(\cdot) \in \mathcal{N}_2(A)$;

6) $\{H(A_n)\} \in \mathcal{M}(\gamma),\quad 0 \leq \gamma \leq 1/4$.

Then AMSE of the estimate $J_n(x)$ as $n \to \infty$

$$\mathrm{u}^2(J_n(x)) = \sum_{t,\,p=1}^{s} \sum_{j,\,k=1}^{m} \sum_{r,\,q \in Q} H_{tjr} H_{pkq} \times$$

$$\times \left[S_{-(m+2\max(r,\,q))} \frac{\mathcal{B}_{t,\,p}^{(r,\,q)}}{nh_n^{m+r+q}} + S_\nu^2\, \omega_{t\nu}^{(rj)}(x) \omega_{p\ \nu}^{(qk)}(x) h_n^{2\nu}\right] +$$

$$+ O\left(\left[\frac{1}{nh_n^{m+2\max(r)}} + h_n^{2\nu}\right]^{\frac{3}{2}}\right).$$

It is important that we do not need condition 6) of Theorem 1 when piecewise smooth approximation (5) is used.

Theorem 2 (the AMSE of the piecewise smooth approximation $\widetilde{J}_{n,\nu}(x)$). Suppose that conditions 1)–5) of Theorem 1 hold and restriction 6) is replaced by
6*) $J(x) = H(A(x)) \neq 0$ or $\tau \geq 4$, τ is a positive integer. Then as $n \to \infty$ $\mathsf{u}^2(\widetilde{J}_n(x)) \sim \mathsf{u}^2(J_n(x))$.

The proofs are given in [14].

IV. Nonparametric semi-recursive identification of the production function and its characteristics

Apply the results to estimate the production function and its characteristics.

A. Estimation of the production function

Let $r(x)$, $x = (x_1, x_2, x_3) \in \mathsf{R}^3$ be the regression model of the three-factor production function, $a(x) = (a_1(x), a_2(x))$, $a_1(x) = \int y f(x,y)dy$, $a_2(x) = \int f(x,y)dy = p(x)$. Here $x_1 > 0$ is the capital input, $x_2 > 0$ is the labor input, $x_3 > 0$ is the nature input, $y > 0$ is a product, and $f(x,y) > 0$ only if $x_1 > 0$, $x_2 > 0$, $x_3 > 0$, $y > 0$. Then

$$J_n(x) = r_n(x) = \frac{\sum\limits_{i=1}^{n} \frac{Y_i}{h_i^3}\mathbf{K}\left(\frac{x - X_i}{h_i}\right)}{\sum\limits_{i=1}^{n} \frac{1}{h_i^3}\mathbf{K}\left(\frac{x - X_i}{h_i}\right)} =$$

$$= \frac{a_{1n}^{(0j)}(x)}{a_{2n}^{(0j)}(x)} = \frac{a_{1n}(x)}{p_n(x)}. \qquad (6)$$

Let $\mathbf{K}(u) = K(u_1)K(u_2)K(u_3)$, $K(\cdot) \in \mathcal{A}_\nu$, $\sup\limits_{u \in \mathsf{R}^1} |K(u)| < \infty$, and $(h_n) \in \mathcal{H}(3)$. To find the AMSE of the estimate $r_n(x)$, we use Theorem 1. In view of 1)–4) conditions of the theorem functions $a_i(z)$, $i = 1, 2$, and their derivatives are continuously differentiable up to the order ν for any $z \in \mathsf{R}^3$, and the function $\int y^4 f(x,y)dy$ is bounded on R^3. If $p(x) > 0$, then condition 5) is fulfilled. It seems impossible to find a majorizing sequence (d_n) (condition 6) of Theorem 1), since the denominator in (6) may be equal to zero. In some cases we can find a majorizing sequence according to Definition 5 with $\gamma = 0$ under $\nu = 2$ if, for example, $K(\cdot) \geq 0$, and $Y < \infty$ [15]. For $\nu > 2$ we can use the piecewise smooth approximation $\widetilde{r}_n(x)$:

$$\widetilde{r}_n(x) = \frac{r_n(x)}{(1 + \delta_{n,\nu} |r_n(x)|^{\tau})^{\rho}},$$

where $\tau > 0$, $\rho > 0$, $\rho\tau \geq 1$, $\delta_{n,\nu} = O\left(h_n^{2\nu} + 1/(nh_n^3)\right)$, $(\delta_{n,\nu}) \downarrow 0$ as $n \to \infty$.

In view of condition 6*) of Theorem 2 it is enough to take even $\tau \geq 4$, and as $n \to \infty$ $\mathsf{u}^2(\widetilde{r}_n(x)) =$

$$= \sum_{i,p=1}^{2} H_i H_p \left(S_{-3}\frac{CB_{i,p}}{nh_n^3} + S_\nu^2\, \omega_{i\nu}(x)\omega_{p\nu}(x) h_n^{2\nu}\right) +$$

$$+ O\left(\left[\frac{1}{nh_n^3} + h_n^{2\nu}\right]^{3/2}\right),$$

where

$$\omega_{1\nu}(x) = \frac{T_\nu}{\nu!}\left(\frac{\partial^\nu a_1(x)}{\partial x_1^\nu} + \frac{\partial^\nu a_1(x)}{\partial x_2^\nu} + \frac{\partial^\nu a_1(x)}{\partial x_3^\nu}\right),$$

$$\omega_{2\nu}(x) = \frac{T_\nu}{\nu!}\left(\frac{\partial^\nu p(x)}{\partial x_1^\nu} + \frac{\partial^\nu p(x)}{\partial x_2^\nu} + \frac{\partial^\nu p(x)}{\partial x_3^\nu}\right),$$

$$H_1 = \frac{1}{p(x)},\ H_2 = -\frac{r(x)}{p^2(x)};\ B_{1,1} = \int y^2 f(x,y)dy,$$

$$B_{1,2} = B_{2,1} = \int y f(x,y)dy,$$

$$B_{2,2} = p(x);\ C = \int K^2(u)du.$$

B. Estimation of the marginal productivity function

In the case of the marginal productivity function $T_1(x) = \dfrac{\partial r(x)}{\partial x_1}$ a dominant sequence finding difficulties force us to use the piecewise smooth approximation $\widetilde{T}_{1n}(x)$:

$$\widetilde{T}_{1n}(x) = \frac{T_{1n}(x)}{(1 + \delta_n |T_{1n}(x)|^{\tau})^{\rho}},$$

where

$$T_{1n}(x) = \left[\frac{\sum\limits_{i=1}^{n} \frac{Y_i}{h_i^4}\mathbf{K}^{(11)}\left(\frac{x - X_i}{h_i}\right)}{\sum\limits_{i=1}^{n} \frac{1}{h_i^3}\mathbf{K}\left(\frac{x - X_i}{h_i}\right)} - \right.$$

$$\left. - \frac{\sum\limits_{i=1}^{n} \frac{Y_i}{h_i^3}\mathbf{K}\left(\frac{x - X_i}{h_i}\right) \sum\limits_{i=1}^{n} \frac{Y_i}{h_i^4}\mathbf{K}^{(11)}\left(\frac{x - X_i}{h_i}\right)}{\left[\sum\limits_{i=1}^{n} \frac{1}{h_i^3}\mathbf{K}\left(\frac{x - X_i}{h_i}\right)\right]^2}\right], \qquad (7)$$

$\mathbf{K}^{(11)}(u) = K^{(1)}(u_1)K(u_2)K(u_3)$. The kernel has to satisfy the additional conditions: $\sup\limits_{u \in \mathsf{R}^1} |K^{(1)}(u)| < \infty$, $\lim\limits_{|u| \to \infty} K(u) = 0$, $K^{(\alpha)}(\cdot) \in \mathcal{N}_0(\mathsf{R})$, $\alpha = 1, 2$; functions $a_1(\cdot)$, $a_2(\cdot)$ and their derivatives up to the order $(\nu + 1)$ need to be continuous and bounded on R^3; the sequence $(h_n) \in \mathcal{H}(4)$.

C. Estimation of the marginal rate of technical substitution

Let $T_j(x) = \partial r(x)/\partial x_j$ and

$$MRTS_{12,n}(x) = T_{1n}(x)/T_{2n}(x)$$

be the estimate of the marginal rate of technical substitution of an input x_2 with an input x_1, where the denominator $T_{2n}(x)$ is given by (7), where $\mathbf{K}^{(11)}(u)$ is replaced by $\mathbf{K}^{(12)}(u) = K(u_1)K^{(1)}(u_2)K(u_3)$.

The piecewise smooth approximation of the estimate $MRTS_{12,n}(x)$ can be written easily. In view of condition 5) of Theorem 1 the condition $r(x) \neq \dfrac{\partial a_1(x)}{\partial x_2} \Big/ \dfrac{\partial p(x)}{\partial x_2}$ has to hold in addition to the previous restrictions.

V. Nonparametric semi-recursive identification of the dynamic production function

Note that above results are given for independent observations (random samples). The results can be generalized to time series. In [16] an autoregressive heteroscedastic model satisfying geometric ergodicity conditions is considered. The approach allows us to estimate a dynamic production functions with lagged values of the output.

Suppose that a sequence $(Y_t)_{t=\dots,-1,0,1,2,\dots}$ is generated by a nonlinear homoscedastic ARX process of order (m,s)

$$Y_t = \Psi(Y_{t-i_1},\dots,Y_{t-i_m},X_t)+\xi_t = \Psi(U_t)+\xi_t, \qquad (8)$$

where $X_t=(X_{1t},\dots,X_{st})$ are exogenous variables, $U_t = (Y_{t-i_1},\dots,Y_{t-i_m},X_t)$, $1\le i_1<i_2<\dots<i_m$ is the known subsequence of natural numbers, (ξ_t) is a sequence of independent identically distributed (with density positive on R^1) random variables with zero mean, finite variance, zero third, and finite fourth moments, $\Psi(\cdot)$ is an unknown nonperiodic function bounded on compacts. Assume that the process is strictly stationary.

Criteria for geometric ergodicity of a nonlinear heteroscedastic autoregression and ARX models which in turn imply α-mixing have been given by many authors (see for example [17]– [21]).

Let $Y_1,\dots,Y_n$ be observations generated by the process (8). The conditional expectation $\Psi(x,z)=\Psi(u)=\mathsf{E}(Y_t|U_t=u)=\mathsf{E}(Y_t|u)$, $(x,z)=u\in\mathsf{R}^{m+s}$ we estimate by the statistic, which is a semi-recursive counterpart of the Nadaraya–Watson estimate [22], [23] (similarly to (6)):

$$\Psi_{n,\,m+s}(u)=\frac{\sum_{t=2}^{n}\dfrac{Y_t}{h_t^{m+s}}\mathbf{K}\left(\dfrac{u-U_t}{h_t}\right)}{\sum_{t=2}^{n}\dfrac{1}{h_t^{m+s}}\mathbf{K}\left(\dfrac{u-U_t}{h_t}\right)}. \qquad (9)$$

This quantity may be interpreted as the predicted value based on the past information.

Since the observations are dependent, investigation of the estimates properties becomes much harder. For example, the main part of the Nadaraya-Watson estimate's AMSE for strongly mixing (s. m.) sequences was found only in 1999 [24]; the authors also proved that this estimate converges with probability one.

We proved in [16] that if the observed sequence satisfies the s. m. condition with a s. m. coefficient $\alpha(\tau)$ such that

$$\int_0^\infty \tau^2[\alpha(\tau)]^{\frac{\delta}{2+\delta}}\,d\tau<\infty \qquad (10)$$

for some $0<\delta<2$, then Theorems 1–3 hold. Note that a s. m. coefficient with the geometric rate satisfies condition (10).

We apply (9) under

$$U_t=(Y_{t-1},X_{1(t-1)},X_{2(t-1)},X_{3(t-1)})$$

to investigate the dependence of Russian Federation's Industrial Production Index Y on the dollar exchange rate X_1, direct investment X_2, and export X_3 for the period from September 1994 till March 2004. The data are available from: http://www.gks.ru and http://sophist.hse.ru/. The estimate

$$\Psi_{n,\,4}(Y_{n-1},X_{1(n-1)},X_{2(n-1)},X_{3(n-1)}) = \frac{\sum_{t=2}^{n-1}\dfrac{K_t}{H_t}Y_tK\left(\dfrac{Y_{n-1}-Y_{t-1}}{h_{1t}}\right)}{\sum_{t=2}^{n-1}\dfrac{K_t}{H_t}K\left(\dfrac{Y_{n-1}-Y_{t-1}}{h_{1t}}\right)},$$

where

$$H_t=\prod_{j=1}^{4}h_{jt},\quad K_t=\prod_{j=1}^{3}K\left(\frac{X_{j(n-1)}-X_{j(t-1)}}{h_{(j+1)t}}\right).$$

To find the AMSE of the estimate $\Psi_{n,\,4}(u)$ we use Theorem 2 [16].

Let $f(\cdot,\cdot)$ be the stationary distribution of the vector (U_t,Y_t). Suppose that $K(\cdot)\in\mathcal{A}_\nu$, $\mathbf{K}(u)=\prod_{i=1}^{4}K(u_i)$, $\sup_{u\in\mathsf{R}^1}|K(u)|<\infty$, the sequence $(h_n)\in\mathcal{H}(4)$, and $\lambda=-4$. Let functions $a_i(u)$, $i=0,1$, and their derivatives up to and including the order ν be continuous and bounded on R^4; functions $\int y^2f(u,y)\,dy$ and $\int y^4f(u,y)\,dy$ be bounded on R^4; and, moreover, $\int y^2f(u,y)\,dy$ and $\int |y|^{2+\delta}f(u,y)\,dy$ be continuous at the point u. Then conditions (1)–(5) of Theorem 2 [16] hold; we also suppose that condition (6) (Theorem 2 [16]) holds. If $p(u)>0$, then condition (7) (Theorem 2 [16]) holds too.

If the random variables Y_t are uniformly bounded, and we select a nonnegative kernel, then it is easy to show that $\Psi_{n,\,4}(u)$ are bounded for $\nu=2$. By condition (8) (Theorem 2 [16]), this is equivalent to the existence of a majorizing sequence with $\gamma=0$.

For $\nu>2$ the piecewise smooth approximation solves the problem (see the previous section).

In Table 1 the relative errors of the forecast (REF) obtained with $\Psi_{n,\,4}(\cdot)$ for each year from 1995 till 2004 are given. Total REF is 6.68%.

TABLE 1

Errors of Forecasts

1995	1996	1997	1998	1999
0.189	0.057	0.04	0.133	0.05

2000	2001	2002	2003	2004
0.055	0.047	0.042	0.057	0.043

The result of 1998 can be explained by 1998 Russian financial crisis ("Ruble crisis") in August 1998. The kernel used is the Gaussian kernel and the bandwidths $h_{jt}=0.17\hat\sigma_jt^{-1/8}$, where $\hat\sigma_j$, $j=1,2,3,4$ are the corresponding sample mean square deviations, the constant 0.17 is chosen subjectively.

The marginal productivity function and marginal rate of technical substitution are estimated in the same way on the base of (7).

VI. Conclusion

This work presents a unifying approach to estimating the dynamic production function and its characteristics (the marginal productivity function, marginal rate of technical substitution). The approach is based on plug-in estimating of functions depending on functionals of the joint stationary distribution of the vector of explanatory variables

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$U_t = (Y_t, Y_{t-i_1}, \ldots, Y_{t-i_m}, X_{1t}, \ldots, X_{st})$, where $X_t = (X_{1t}, \ldots, X_{st})$ are exogenous variables, Y_t is an output (product), $i_2 < ... < i_m$ is the known subsequence of natural numbers. Note that i_m may be large, while m is small. We assume that the process Y_t is a nonlinear homoscedastic strictly stationary ARX process, which satisfies to the s. m. condition with the geometric rate. The plug-in estimates are semirecursive, i.e., we recursively compute only the kernel estimates of functionals (3). By using the piecewise smooth approximations of the estimates, we have managed to avoid the problems concerning to the majorizing sequence's existence needed for obtaining of the main parts of the estimate's AMSE.

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