Differentials of Eigenvalues and Eigenvectors in Undamped Discrete Systems under Alternative Normalizations

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***Abstract*—First-order differentials of a (simple) eigenvalue and the associated eigenvector in an undamped discrete system are investigated. We provide closed-form expressions under three alternative normalizations, namely the customary mass normalization, unit-length normalization and the normalization obtained setting an element equal to 1. The proposed formulas have no pretension to be computationally efficient in large systems, but may be useful for the interpretation of the results.**

***Index Terms*—undamped systems, generalized eigenvalue problem, matrix differential.**

I. Introduction

In the analysis of free undamped vibration and structural stability, the following generalized eigenvalue problem is often considered:

$$(\mathbf{K} - \lambda\mathbf{M})\,\mathbf{u} = \mathbf{0}.$$

$\mathbf{K}$ is called *stiffness matrix*, while $\mathbf{M}$ is called *mass matrix*. An interesting problem that has received attention in the literature (see e.g. [1], [2], [3], [4], [5], [6]) is the computation of the derivatives of the eigenvalues and of the eigenvectors of the problem. The aim of this paper is to propose new formulas for the differentials of the eigenvalues and of the eigenvectors of this generalized eigenvalue problem. A complication of this problem is that the eigenvector is usually normalized through the *mass normalization* $\sqrt{\mathbf{u}^{\star}\mathbf{M}\mathbf{u}} = 1$ (see [7], for the impact of alternative normalizations on the computation of the derivatives).

Most references obtain formulas for the derivatives of eigenvalues and eigenvectors using an approach that seems to be due to Nelson ([2]). When it was proposed, this method was a significant advance and spawned a large literature. All of the previous methods required calculation of a large number of eigenvectors, while Nelson method was the first to reduce the burden to the computation of only the selected eigenvalue and eigenvector. Moreover, the method preserves the structure (bandedness, symmetry, etc.) of the matrices involved in the computation, thus reducing notably the computational complexity. In practice, the method is based on taking the derivative of the generalized eigenequality $(\mathbf{K} - \lambda\mathbf{M})\,\mathbf{u} = \mathbf{0}$ and on supplementing this (singular) system of equalities with an additional equation allowing for removing this singularity. The method is very clever and has been extended to cover second-order derivatives ([4]), other kinds of normalization (see e.g. [7]), damped systems (see e.g. [5]). An earlier review paying particular attention to the pre-Nelson methods is [3].

There are two main differences of the present approach with respect to the existing ones. First of all, we use differentials of matrices and vectors instead of derivatives. Even if differentials can be easily obtained from derivatives using e.g. the technique in Section 5 in [8], the book [9] provides in our opinion a thorough defence of the approach using differentials. Moreover, obtaining derivatives from differentials using the classical definition of the derivative as ratio of differentials is very simple.

The second, and probably more profound, difference is that we obtain explicit expressions for differentials in terms of the original quantities of the problem. These formulas involve generalized inverses, in particular Moore-Penrose inverses. This constitutes a drawback from the computational point of view. For general matrices, the computation of the inverse of a $n \times n$ general matrix has complexity of $O\left(n^{2.376}\right)$ operations (the procedure attaining this rate, called Coppersmith-Winograd algorithm, is rarely used in practice: more common algorithms are the Strassen algorithm, with $O\left(n^{\log_2 7}\right)$ operations, and Gauss-Jordan elimination, with $O\left(n^3\right)$) while the computation of the Moore-Penrose inverse through the SVD requires $O\left(n^3\right)$ operations. On the other hand, if the matrices involved in the computations have a special structure, Nelson procedure can have a definite advantage on our formulas. However, even when computationally inefficient, a closed form expression can retain a certain appeal from the point of view of interpretation (see e.g. [5] for an example of a quadratic eigenvalue problem in which interpretation is more important than computational efficiency).

As an illustration of the use of the method, we provide a comparison of the formulas for the first-order differentials of a (simple) eigenvalue and the associated eigenvector under three alternative normalizations, namely the customary mass normalization, the unit-length normalization and the normalization obtained setting an element equal to 1. Further topics, such as the derivation of second-order differentials and the treatment of damped linear discrete systems through quadratic eigenvalue problems, will be dealt with in a forthcoming paper.

II. General Results

The following result, whose proof is contained in [10], will be the starting point on which we will base our analysis.

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Theorem. Consider the eigenvalue problems $\mathbf{A}_0 \cdot \mathbf{u}_0 = \lambda_0 \cdot \mathbf{u}_0$ and $\mathbf{A} \cdot \mathbf{u} = \lambda \cdot \mathbf{u}$ where $\mathbf{A} \simeq \mathbf{A}_0 + \mathrm{d}\mathbf{A}$ and $\mathrm{d}\mathbf{A}$ is a matrix differential in the sense of [9]. λ_0 is a simple eigenvalue of $\mathbf{A}_0$ with right eigenvector $\mathbf{u}_0$ and left eigenvector $\mathbf{v}_0$. Suppose that the right eigenvectors $\mathbf{u}_0$ and $\mathbf{u}$ take their values in the complex plane C^n. Denote as $\mathbf{u}_0$ and $\mathbf{u}$ the eigenvectors normalized as $n(\mathbf{u}_0) = 1$ and $n(\mathbf{u}) = 1$, where $n : C^n \rightarrow R$ is a function infinitely differentiable in the interior of its domain with Taylor expansion:

$$n(\mathbf{u}) \simeq n(\mathbf{u}_0) + \mathbf{n}_0^\star \cdot \mathrm{d}\mathbf{u} + \mathrm{d}\eta(\mathbf{u}_0)$$

with $\mathbf{n}_0^\star \mathbf{u}_0 \neq 0$ and $\mathrm{d}\eta(\mathbf{u}_0)$ positively homogeneous of degree 1. The following expansions hold:

$$\begin{aligned}\lambda(\mathbf{A}) &\simeq \lambda_0 + \mathrm{d}\lambda, \\ \mathbf{u}(\mathbf{A}) &\simeq \mathbf{u}_0 + \mathrm{d}\mathbf{u},\end{aligned}$$

where:

$$\begin{aligned}\mathrm{d}\lambda &= \frac{\mathbf{v}_0^\star \cdot \mathrm{d}\mathbf{A} \cdot \mathbf{u}_0}{\mathbf{v}_0^\star \mathbf{u}_0} \\ \mathrm{d}\mathbf{u} &= (\mathbf{I}_n - \mathbf{u}_0 \mathbf{n}_0^\star) \cdot (\lambda_0 \mathbf{I}_n - \mathbf{A}_0)^+ \left(\mathbf{I}_n - \frac{\mathbf{u}_0 \mathbf{v}_0^\star}{\mathbf{v}_0^\star \mathbf{u}_0}\right) \\ &\quad \cdot \mathrm{d}\mathbf{A} \cdot \mathbf{u}_0 - \mathbf{u}_0 \cdot \mathrm{d}\eta(\mathbf{u}_0).\end{aligned}$$

III. Differentials under Alternative Normalizations

Consider the previous generalized eigenvalue problem. The unperturbed problem is:

$$(\mathbf{K}_0 - \lambda_0 \mathbf{M}_0)\,\mathbf{u}_0 = \mathbf{0}.$$

Supposing, as we will do in the following, that $\mathbf{M}_0$ is a full rank matrix, this can be cast as a classical eigenvalue problem $(\mathbf{A}_0 - \lambda_0 \mathbf{I}_n)\,\mathbf{u}_0 = \mathbf{0}$, with $\mathbf{A}_0 = \mathbf{M}_0^{-1}\mathbf{K}_0$, premultiplying the previous equation by $\mathbf{M}_0^{-1}$. Consider now two matrices $\mathbf{M}$ and $\mathbf{K}$ that are obtained as perturbations of the matrices $\mathbf{M}_0$ and $\mathbf{K}_0$, say $\mathbf{M} = \mathbf{M}_0 + \mathrm{d}\mathbf{M}$ and $\mathbf{K} = \mathbf{K}_0 + \mathrm{d}\mathbf{K}$ where $\mathrm{d}\mathbf{M}$ and $\mathrm{d}\mathbf{K}$ are matrix differentials. This means that the classical eigenvalue problem becomes $(\mathbf{A} - \lambda \mathbf{I}_n)\,\mathbf{u} = \mathbf{0}$, where $\mathbf{A} = \mathbf{A}_0 + \mathrm{d}\mathbf{A}$ and (see [9] for the differentials):

$$\begin{aligned}\mathrm{d}\mathbf{A} &= \mathrm{d}\mathbf{M}^{-1}\mathbf{K}_0 + \mathbf{M}_0^{-1}\mathrm{d}\mathbf{K} \\ &= -\mathbf{M}_0^{-1} \cdot \mathrm{d}\mathbf{M} \cdot \mathbf{M}_0^{-1}\mathbf{K}_0 + \mathbf{M}_0^{-1}\mathrm{d}\mathbf{K}.\end{aligned}$$

On the other hand, the mass normalization admits the expansion:

$$\begin{aligned}n(\mathbf{u}) &= n(\mathbf{u}_0) + \frac{\mathbf{u}_0^\star (\mathbf{M}_0 + \mathbf{M}_0^\star)\,\mathrm{d}\mathbf{u}}{2\sqrt{\mathbf{u}_0^\star \mathbf{M}_0 \mathbf{u}_0}} + \frac{\mathbf{u}_0^\star \mathrm{d}\mathbf{M}\mathbf{u}_0}{2\sqrt{\mathbf{u}_0^\star \mathbf{M}_0 \mathbf{u}_0}} \\ &\quad + o(\|\mathrm{d}\mathbf{u}\|) + o(\|\mathrm{d}\mathbf{M}\|).\end{aligned}$$

Therefore, $\mathbf{n}_0 = \frac{(\mathbf{M}_0 + \mathbf{M}_0^\star)\mathbf{u}_0}{2\sqrt{\mathbf{u}_0^\star \mathbf{M}_0 \mathbf{u}_0}}$ and $\mathrm{d}\eta = \frac{\mathbf{u}_0^\star \mathrm{d}\mathbf{M}\mathbf{u}_0}{2\sqrt{\mathbf{u}_0^\star \mathbf{M}_0 \mathbf{u}_0}}$.

As concerns the eigenvalue, we get:

$$\mathrm{d}\lambda = \frac{\mathbf{v}_0^\star \cdot \mathbf{M}_0^{-1} \cdot (\mathrm{d}\mathbf{K} - \lambda_0 \cdot \mathrm{d}\mathbf{M}) \cdot \mathbf{u}_0}{\mathbf{v}_0^\star \mathbf{u}_0}$$

where we have used the fact that $\mathbf{M}_0^{-1}\mathbf{K}_0\mathbf{u}_0 = \lambda_0 \mathbf{u}_0$. Remark that this formula coincides with the formula of the derivative given e.g. in [7]. Moreover, the differential of the eigenvalue is independent of the normalization. As concerns the eigenvector:

$$\begin{aligned}\mathrm{d}\mathbf{u} &= \left(\mathbf{I}_n - \mathbf{u}_0 \mathbf{u}_0^\star \frac{(\mathbf{M}_0 + \mathbf{M}_0^\star)}{2}\right) \cdot \left(\lambda_0 \mathbf{I}_n - \mathbf{M}_0^{-1}\mathbf{K}_0\right)^+ \\ &\quad \cdot \left(\mathbf{I}_n - \frac{\mathbf{u}_0 \mathbf{v}_0^\star}{\mathbf{v}_0^\star \mathbf{u}_0}\right) \cdot \mathbf{M}_0^{-1} \cdot (\mathrm{d}\mathbf{K} - \lambda_0 \cdot \mathrm{d}\mathbf{M}) \cdot \mathbf{u}_0 \\ &\quad - \frac{\mathbf{u}_0^\star \mathrm{d}\mathbf{M}\mathbf{u}_0}{2} \cdot \mathbf{u}_0.\end{aligned}$$

If $\mathbf{v}_0^\star \mathbf{u}_0 = 1$, as is often supposed, then the formulas simplify accordingly.

The previous formulas can be simplified if the matrices $\mathbf{M}$, $\mathbf{K}$, $\mathbf{M}_0$ and $\mathbf{K}_0$ are real symmetric. In this case, if $\mathbf{u}$ and $\mathbf{u}_0$ are the right eigenvectors, the relations $\mathbf{v} \propto \mathbf{M}\mathbf{u}$ and $\mathbf{v}_0 \propto \mathbf{M}_0 \mathbf{u}_0$ hold for the left ones, independently of the normalization chosen for them. Then, we get:

$$\begin{aligned}\mathrm{d}\lambda &= \mathbf{u}_0^\mathsf{T} \cdot (\mathrm{d}\mathbf{K} - \lambda_0 \cdot \mathrm{d}\mathbf{M}) \cdot \mathbf{u}_0 \\ \mathrm{d}\mathbf{u} &= \left(\mathbf{I}_n - \mathbf{u}_0 \mathbf{u}_0^\mathsf{T} \mathbf{M}_0\right) \cdot \left(\lambda_0 \mathbf{I}_n - \mathbf{M}_0^{-1}\mathbf{K}_0\right)^+ \\ &\quad \cdot \left(\mathbf{M}_0^{-1} - \mathbf{u}_0 \mathbf{u}_0^\mathsf{T}\right) \cdot (\mathrm{d}\mathbf{K} - \lambda_0 \cdot \mathrm{d}\mathbf{M}) \cdot \mathbf{u}_0 \\ &\quad - \frac{\mathbf{u}_0^\mathsf{T} \mathrm{d}\mathbf{M}\mathbf{u}_0}{2} \cdot \mathbf{u}_0.\end{aligned}$$

Now, we consider the unit-length normalization, obtained setting $n(\mathbf{u}) = \sqrt{\mathbf{u}^\star \mathbf{u}}$. Then $\mathbf{n}_0 = \frac{\mathbf{u}_0}{\sqrt{\mathbf{u}_0^\star \mathbf{u}_0}}$ and $\mathrm{d}\eta = 0$. Therefore, using the formula $\mathbf{M}_0^{-1}\mathbf{K}_0\mathbf{u}_0 = \lambda_0 \mathbf{u}_0$ we get:

$$\begin{aligned}\mathrm{d}\mathbf{u} &= \left(\lambda_0 \mathbf{I}_n - \mathbf{M}_0^{-1}\mathbf{K}_0\right)^+ \left(\mathbf{I}_n - \frac{\mathbf{u}_0 \mathbf{v}_0^\star}{\mathbf{v}_0^\star \mathbf{u}_0}\right) \\ &\quad \cdot \mathbf{M}_0^{-1} \cdot (\mathrm{d}\mathbf{K} - \lambda_0 \cdot \mathrm{d}\mathbf{M}) \cdot \mathbf{u}_0.\end{aligned}$$

If the matrices $\mathbf{M}$, $\mathbf{K}$, $\mathbf{M}_0$ and $\mathbf{K}_0$ are real symmetric:

$$\begin{aligned}\mathrm{d}\mathbf{u} &= \left(\lambda_0 \mathbf{I}_n - \mathbf{M}_0^{-1}\mathbf{K}_0\right)^+ \cdot \left(\mathbf{M}_0^{-1} - \frac{\mathbf{u}_0 \mathbf{u}_0^\mathsf{T}}{\mathbf{u}_0^\mathsf{T} \mathbf{M}_0 \mathbf{u}_0}\right) \\ &\quad \cdot (\mathrm{d}\mathbf{K} - \lambda_0 \cdot \mathrm{d}\mathbf{M}) \cdot \mathbf{u}_0.\end{aligned}$$

A last kind of normalization is obtained setting an element of the vector, say the $j-$th, to a fixed value, say 1. Let $\mathbf{e}_j$ be the vector with the $j-$th element equal to 1 and all other elements equal to 0. Then $\mathbf{n}_0 = \mathbf{e}_j$ and $\mathrm{d}\eta = 0$:

$$\begin{aligned}\mathrm{d}\mathbf{u} &= \left(\mathbf{I}_n - \mathbf{u}_0 \mathbf{e}_j^\mathsf{T}\right) \cdot \left(\lambda_0 \mathbf{I}_n - \mathbf{M}_0^{-1}\mathbf{K}_0\right)^+ \cdot \left(\mathbf{I}_n - \frac{\mathbf{u}_0 \mathbf{v}_0^\star}{\mathbf{v}_0^\star \mathbf{u}_0}\right) \\ &\quad \cdot \mathbf{M}_0^{-1} \cdot (\mathrm{d}\mathbf{K} - \lambda_0 \cdot \mathrm{d}\mathbf{M}) \cdot \mathbf{u}_0.\end{aligned}$$

Under real symmetry:

$$\begin{aligned}\mathrm{d}\mathbf{u} &= \left(\mathbf{I}_n - \mathbf{u}_0 \mathbf{e}_j^\mathsf{T}\right) \cdot \left(\lambda_0 \mathbf{I}_n - \mathbf{M}_0^{-1}\mathbf{K}_0\right)^+ \\ &\quad \cdot \left(\mathbf{M}_0^{-1} - \frac{\mathbf{u}_0 \mathbf{u}_0^\mathsf{T}}{\mathbf{u}_0^\mathsf{T} \mathbf{M}_0 \mathbf{u}_0}\right) \cdot (\mathrm{d}\mathbf{K} - \lambda_0 \cdot \mathrm{d}\mathbf{M}) \cdot \mathbf{u}_0.\end{aligned}$$

References

[1] R. L. Fox and M. P. Kapoor, "Rates of change of eigenvalues and eigenvectors." *AIAA Journal*, vol. 6, no. 12, pp. 2426–2429, 1968.
[2] R. B. Nelson, "Simplified calculation of eigenvector derivatives," *AIAA Journal*, vol. 14, no. 9, pp. 1201–1205, 1976.
[3] H. M. Adelman and R. T. Haftka, "Sensitivity analysis of discrete structural systems," *AIAA Journal*, vol. 24, no. 5, pp. 823–832, 1986.
[4] M. I. Friswell, "Calculation of 2nd-order and higher-order eigenvector derivatives," *Journal of guidance, control, and dynamics*, vol. 18, no. 4, pp. 919–921, 1995.
[5] S. Adhikari, "Rates of change of eigenvalues and eigenvectors in damped dynamic system," *AIAA Journal*, vol. 39, no. 11, 1999.

[6] ——, "Derivative of eigensolutions of non-viscously damped linear systems," *AIAA Journal*, vol. 40, no. 10, pp. 2061–2069, October 2002.

[7] V. Siddhi, "A generalized approach for calculation of the eigenvector sensitivity for various eigenvector normalizations," Master's thesis, University of Missouri - Columbia, 2005.

[8] C. D. Meyer and G. W. Stewart, "Derivatives and perturbations of eigenvectors," *SIAM J. Numer. Anal.*, vol. 25, no. 3, pp. 679–691, 1988.

[9] J. R. Magnus and H. Neudecker, *Matrix differential calculus with applications in statistics and econometrics*, ser. Wiley Series in Probability and Statistics. Chichester: John Wiley & Sons Ltd., 1999, revised reprint of the 1988 original.

[10] M. Bernasconi, C. Choirat, and R. Seri, "Differentials of eigenvalues and eigenvectors under nonstandard normalizations," Tech. Rep., 2011.

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