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Applying Variance Gamma Correlated to Estimate Optimal Portfolio of Variance Swap

Lingyan Cao, Zheng-Feng Guo*†‡§

Abstract—**In this paper, we focus on introducing the Variance Swap and estimating the portfolios. The portfolios of the Variance Swap are optimized based on maximizing the distorted expectation given the index of acceptability. The variance strike is calculated from the option surface calibration. The realized variance is constructed through Hardy-Littlewood transform considering the highly correlated autocorrelation and dependencies of cross assets. A non-Gaussian model: Varanance Gamma Correlated (VGC) is also applied to the residual data of the regression model.**

Keywords: Variance Swap, Variance Strike, Variance Gamma Correlated (VGC), Expected Distortion, Index of Acceptability

1 Introduction

This paper focuses on constructing an optimal portfolio of trading the Variance Swaps. Our approach is broadly similar to the classical portfolio theory for stock investment first proposed by Markowitz (1952). Madan (2010) first introduces the portfolio theory of Variance Swap, however, his method to deal with the realized variance is based on the Fully Gaussian Copula (FGC), proposed by Malevergne and Sornette (2003); while we apply another non-Gaussian model: Variance Gamma Correlated (VGC) model developed in Madan and Ajay (2009) and Eberlein and Madan (2009-2). Cherny and Madan (2009) introduce the optimization theory of performance evaluation, as well as distortion functions. Several ways of optimization are applied in Eberlein and Madan (2009-1) and Madan (2009), such as maximizing the index of acceptability, and maximizing the expected distortion given fixed acceptable index. In our paper, we make the index fixed and seek a portfolio that maximizes the expected distortion.

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Section 2 describes the definition of Variance Swap and gives a brief introduction on how to calculate the cash flow of the swap. Section 3 describes and derives the variance strike. Section 4 describes and derives the realized variance. Section 5 explains the distortion function and index of acceptability, and shows which methodology we will choose to run the optimization and get the optimal portfolio. Section 6 shows the numerical results in the process and the optimal portfolio given the index of acceptability.

2 Definition of Variance Swap

Variance Swap is an over-the-counter financial derivative that allows one to speculate on or hedge risks associated with the magnitude of movement, i.e. volatility, of some underlying product, like an exchange rate, interest rate. Through variance swap, investors could achieve long or short exposure to market volatility. In some sense, it could be considered as a kind of 'option', which is a contract signed by two parties that agree to exchange cash flows based on the measured variance of a specified underlying asset during a certain time period. On the trading day mentioned by the contract, the two parties will trade the variance swap according to the variance strike, the realized variance and the notional amount.

The features of a variance swap include the variance strike, which is also called fixed leg, the realized variance, i.e., the floating leg and the notional amount. The floating leg of the swap will pay an amount based on the realized variance of the price changes of the underlying product. The fixed leg of the swap will pay a fixed amount which is the strike quoted at the deal's inception. In general, the payoff of a variance swap would be:

$$P \times (\sigma_r^2 - \sigma_k^2),$$

where P is the notional pricipal, σ_r^2 is the annualized realized variance, and σ_k^2 is the variance strike. Madan(2007) shows that $\sigma_r^2 = \frac{252}{T}\sum_{t=1}^{T} x_t^2$, and $\sigma_k^2 = k^2$, where $x_t = log(\frac{S_t}{S_{t-1}})$, S_t is the stock price of the underlying asset at the end of day t, and k is the annualized volatility quotation. Thus, the variance swap pays at the end of day

T is:

$$P \times (\sum_{t=1}^{T} \frac{x_t^2}{T} 252 - k^2). \quad (1)$$

Moreover, r is the constant continuously compounded interest rate.

3 Variance Strike

Before we go to calculate the variance strike, let us introduce the Variance Gamma Specific Self-Decomposable Model (VGSSD), which is often used to calibrate the option surface. Thus, we calibrate this model to estimate the price of variance strike.

3.1 Variance Gamma Specific Self-Decomposable Model

If $Y(t)$ follows VGSSD model, Carr, Geman, Madan and Yor (2007) show that the law of $Y(t)$ is equivalent to the law of $t^{\gamma}X(1)$ by the scaling property, where $X(t)$ follows the VG process and $X(1) = X(t)\big|_{t=1}$. Therefore, the characteristic function of $Y(t)$ is:

$$\begin{aligned} \phi_{Y(t)}(u) &= E[e^{iuY(t)}] \\ &= E[e^{iut^{\gamma}X(1)}] \\ &= (\frac{1}{1 - iu\theta\nu t^{\gamma} + 0.5 \times \sigma^2 \nu u^2 t^{2\gamma}})^{-\frac{1}{\gamma}}. \end{aligned}$$

Then the risk-neutral stock price $S(t)$ could be defined by:

$$S(t) = S(0)e^{(rt)} \frac{e^{Y(t)}}{E[e^{Y(t)}]}. \quad (2)$$

3.2 Calculation of Variance Strike

The price of a variance swap contract $\sigma_k^2 = k^2$ could be calculated as:

$$\sigma_k^2 t = 2i \frac{\partial \phi_M(u,t)}{\partial u}\bigg|_{u=0}, \quad (3)$$

where $\phi_M(u,t) = E^{\mathrm{Q}}[\exp(iu\ln(M(t)))]$. Denoting r the risk-free interest rate and q the dividend rate, we could define the stock price in the risk-neutral measure by $S(t) = S(0)e^{(r-q)t}M(t)$. Therefore, We could get the characteristic function of $M(t)$ from the characteristic function of $S(t)$, which can be calculated from equation 2. We would have

$$\begin{aligned} \sigma_k^2 t &= -2\mathrm{E}[\ln\mathrm{M}] \\ &= -2\mathrm{E}[\ln S_t - \ln S_0 - (r-q)t] \\ &= -2qt - 2\theta t^{\gamma} - \frac{2}{\nu}\ln(1 - \theta\nu t^{\gamma} - \frac{1}{2}\sigma^2 \nu t^{2\gamma}), \end{aligned}$$

where $r = 0, q = 0$ to make Equation (4) equivalent to the variance strike as proposed in Madan (2007). Therefore, the variance strike is

$$\sigma_k^2 = (-2\theta t^{\gamma} - \frac{2}{\nu}\ln(1 - \theta\nu t^{\gamma} - \frac{1}{2}\sigma^2 \nu t^{2\gamma}))t^{-1}.$$

The calibrated parameters are in table (1), the calibrated option surfaces are in figure (1) and the prices of the variance strike are in table (2).

4 Realized Variance

We download the data from the Wharton Research Data Services (WRDS) of the ten stock prices whose ticker are: xom, aapl, mmm, c, adbe, amzn, gs, coh, goog, bac on the S&P500 index as on November 18, 2007. Let $S_{i,t}$ denote the price of asset i at market close on day t for $i = 1, ..., 10$. As defined above, the daily realized variance for asset i on day t defined as $v_{i,t}$ is:

$$v_{i,t} = (\ln(\frac{S_{i,t}}{S_{i,t-1}}))^2. \quad (4)$$

4.1 Transform of log-Return

Recognizing the squared log returns are highly autocorrelated and will subject to some levels of clustering, we follow Madan (2010) and apply the 'similar to linear' transform: Hardy-Littlewood transform to the squared daily log returns. This transform can deal with the highly correlated autocorrelation, as well as map the $v_{i,t}$ from positive values to all real values, which is required by the linear regression model, since it is rather difficult to keep the linear regression model positive in the future simulation. We thought of the transform of taking the log of the $v_{i,t}$, however this would make a double exponent and result in pretty bad data for the linear regression.

Let $f(x)$ be any symmetric density on the real line having finite expectation of absolute value of x. The Hardy-Littlewood transform is defined as:

$$g(x) = \frac{\int_x^{\infty} uf(u)du}{\int_x^{\infty} f(u)du}. \quad (5)$$

As $x \to -\infty$, $g(x)$ would be close to 0; while when x is large enough, $g(x)$ would behave like x, which means this transformation is close to 'linear'. It is easy to show the $g(x)$ is always positive, so that the inverse of $g(x)$ could transform the positive squared log return $v_{i,t}$ to all real values, which satisfies our requirement. In this paper, we set the density $f(x)$ be the standard normal density: $f(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$, then we could get the so-called Hardy-Littlewood Gauss transform:

$$g(x) = \frac{\int_x^{\infty} u\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}du}{\int_x^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}du} = \frac{\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}}{\int_x^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}du} \quad (6)$$

To calculate the realized variance, we suggest the following steps.

- Take historical data of stock prices of the 10 assets from Mar.18, 2007 to Nov.18, 2007 to calculate the series of real data of squared log-return $v_{i,t}$.

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- Take the Inverse Hardy-Littlewood Gauss transform to the historical data of $v_{i,t}$

$$x_{i,t} = g^{-1}(v_{i,t})$$

to get the series of newly generated data $x_{i,t}$.

- Considering the highly correlated autocorrelation of the series of data $x_{i,t}$, we take the linear regression to the newly generated time series data $x_{i,t}$:

$$x_{i,t} = a_i + \sum_{j=1}^{5} b_{i,j} x_{i,t-j} + u_{i,t}. \quad (7)$$

The results of the lags of the linear regression model are in table (3).

4.2 Variance Gamma Correlated Model

Recognizing that the time series of $u_{i,t}, i = 1, ..., M$ have excess kurtosis and possible skewness, we consider the newly generated data $u_{i,t}$ as correlated multi-dimensional variance gamma process. We apply Variance Gamma Correlated(VGC) proposed in Eberlein and Madan (2009-2) to deal with the residual data $u_{i,t}$, which can also be written as multi-dimensional correlated non-Gaussian process $U_i(t), i = 1, ..., N$. With the given series of data $u_{i,t}$,, we demean the original data first. Like the variance gamma process, $u_i(t)$ could be expressed as a gamma time changed Brownian motion which is subordinated by the gamma process $G_i(t)$. Note that $G_i(t)$ is a subordinating process which is a positive independent identical increasing process with unit expectation at a unit time. Therefore,

$$U_i(t) = \theta_i(G_i(t) - t) + \sigma_i W_i(G_i(t)), \quad (8)$$

where $\theta_i, \sigma_i > 0$. As $U_i(t)$ is correlated non-Gaussian process, we have to also put the correlation into the expression above. Eberlein and Madan(2009-2) showed that we can put the correlation in the time-changed Brownian motion. We can write the process at the unit time that $U_i = U_i(1)$ and:

$$U_i =^{(d)} \theta_i(g_i - 1) + \sigma_i \sqrt{g_i} Z_i,$$

where $g_i = G_i(1)$ and Z_i are standard normal variates with correlation ρ_{ij} between Z_i and Z_j for $i \neq j$, and g_i are independent gamma variates with unit mean and variance ν_i. We will have $E(Z_i) = 0$ and $\mathrm{Var}(Z_i) = 1$. Hence,

$$\mathrm{Cov}(Z_i, Z_j) = \mathrm{Corr}(Z_i, Z_j) = \rho_{ij},$$

and

$$\mathrm{Cov}(U_i, U_j) = \sigma_i \sigma_j \mathrm{E}(\sqrt{g_i}) \mathrm{E}(\sqrt{g_j}) \rho_{ij}. \quad (9)$$

We could use the series of $u_i(t)$ derived from the historical data and estimate the parameters $\sigma_i, \nu_i, \theta_i$ and calculate the covariance from Equation (9). Then we could simulate the multi-dimensional non-Gaussian process $U_i(t)$ which are correlated with each other. The estimating procedure for applying the VGC model to the residual variates is summarized as following:

- Apply MLE to the time series data $U = u_{i,t}, i = 1, ..., N$, in each dimension separately; each would follow variance gamma distribution with the corresponding parameters $\sigma_i, \nu_i, \theta_i$.

- Apply the calculated covariance of $u_{i,t}$ to the Equation (9) to get the correlation $\rho_{i,j}$ of the standard normal variable Z_i.

- Simulate the N-dimensional correlated standard normal variable $\hat{Z}$ with the correlation $\rho_{i,j}$ between different assets.

- Use the parameters we estimated and the newly simulated Z_i, and plug back into the equation 8, we will get the newly simulated series data $\hat{U}$.

The estimated parameters of the independent VG variates Y are in table (4), and the covariance matrix of the standard normal variable Z are in table (5).

Therefore, we will simulate 10000 times, and the annualized unit realized variance of the asset i on day t on sample path s is

$$\sigma_{i,t,s}^2 = \frac{252}{21} \sum_{j=t+1}^{t+21} v_{i,j,s}, \quad (10)$$

and $T = 21$ in this paper. Note that we are using trading day, which means that each year has 252 days and each month has 21 days.

The simulated cash flow to asset i on the variance swap on path s is then obtained as

$$c_{i,s} = \sigma_{i,t,s}^2 - k_{i,t,s}^2. \quad (11)$$

5 Optimization

Eberlein and Madan (2009-1) show that concave distortion function $\phi^\gamma(y)$ is defined on the unit interval with values in the unit interval that is point wise increasing in the level of the distortion γ. A random variable X with distribution function $F(x)$ is accepted at level γ if

$$\int_{-\infty}^{\infty} x d\phi^\gamma(F(x)) \geq 0, \quad (12)$$

which means that the expected value of the cash flow under the distortion Φ^γ is nonnegative. Cherny and Madan (2009) propose the MINMAXVAR distortion function ϕ^γ at level γ as:

$$\phi^\gamma(u) = 1 - (1 - u^{\frac{1}{1+\gamma}})^{1+\gamma}, 0 \leq u \leq 1.$$

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We optimize the portfolios to maximize the distorted expectation in Equation (12), given some acceptability index γ. The distorted expectation would be

$$\int_{\infty}^{\infty} c d\phi^{\gamma}(F(c)), \qquad (13)$$

where F is the cumulative distribution function of the portfolio cash flow c. The computation of distorted expectation is facilitated in terms of an ordered sample from the relevant distribution with $c_{(1)} < c_{(2)} < ... < c_{(N)}$ as:

$$\sum_{i=1}^{N} c_{(i)}(\phi(\frac{i}{N}) - \phi(\frac{i-1}{N})).$$

When we do the optimization, we restrict the portfolios on the unit sphere by the condition that:

$$\sum_{i=1}^{51} a_i^2 = 1.$$

Moreover, the aggregated portfolio is zero dollar:

$$\sum_{i=1}^{51} k_{i,t}^2 a_i = 0.$$

In addition, we have to have a zero Vega constraint as:

$$\sum_{i=1}^{51} k_{i,t} a_i = 0.$$

Therefore, we can apply the restrictions above to do the optimization showed above. We set the acceptable index to be some constant, i,e, $\gamma = 0.6$ and maximize the expected distortion value.

6 Numerical Results

Setting index of acceptability $\gamma = 0.6$, and maximizing the expected distortion, we will get the maximized expected distortion 0.2136 and the optimal portfolio results are in table (6). All the tables of results are on the last pages.

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Table 1: VGSSD Parameters on 20071019

ticker	σ	ν	θ	γ	RMSE	APE
xom	0.2511	0.3708	-0.2480	0.4961	0.0511	0.0320
aapl	0.4207	0.0648	-0.3021	0.3912	0.2460	0.0413
mmm	0.2013	0.1177	-0.4167	0.4685	0.0518	0.0395
c	0.2978	0.3147	-0.3405	0.4427	0.0348	0.0432
adbe	0.3182	0.2501	-0.2370	0.5628	0.0277	0.0289
amzn	0.4711	0.1265	-0.6917	0.4135	0.1993	0.0492
gs	0.3236	0.4495	-0.3659	0.4406	0.1526	0.0241
coh	0.3566	0.1027	-0.7032	0.4132	0.0658	0.0372
goog	0.3006	0.3513	-0.1737	0.4954	0.3885	0.0341
bac	0.2540	0.5081	-0.2585	0.5405	0.0425	0.0419

Table 2: One Month Variance Strike on 20071019

xom	aapl	mmm	c	adbe	amzn	gs	coh	goog	bac
0.06919	0.28499	0.05088	0.13431	0.07342	0.35493	0.13865	0.23385	0.11399	0.05944

Table 3: Robust Regression Results

ticker	Constant	Lag1	Lag2	Lag3	lag4	Lag5
xom	-3.044282304	0.122916681	-0.022811004	0.133290834	0.034290086	0.012055923
aapl	-2.722644539	0.078059138	-0.022363233	0.090387532	0.160207687	0.015064441
mmm	-4.624608398	-0.061968352	0.105659475	-0.001571275	-0.056962574	-0.032638583
c	-2.277948738	-0.068889361	0.050149556	0.140823951	0.189014766	0.144177176
adbe	-4.48662104	-0.123990041	-0.019817241	0.048802074	0.028637862	-0.001471024
amzn	-2.979800894	0.135376415	-0.040781192	0.101940809	0.112595202	-0.051193212
gs	-2.613411184	-0.01689588	0.06347052	0.238165455	0.159326734	-0.088686314
coh	-2.839041356	0.043968516	0.09038196	0.003050781	0.207789308	-0.061070552
goog	-3.748142097	0.025429933	-0.026313457	0.081221404	-0.098445657	0.139077882
bac	-2.808136424	0.11446335	0.160697946	0.032296285	0.00548911	0.040231169

Table 4: VG estimates for variates Y for VGC

ticker	σ	ν	θ
xom	0.346247456	0.118161236	-1.066018343
aapl	0.402793432	0.276506222	-0.830714211
mmm	0.399104493	0.117985537	-0.636605618
c	0.507005417	0.337929748	-0.182348617
adbe	0.181295155	0.157614674	-1.097035718
amzn	0.584271328	0.424370606	-0.155372743
gs	0.286851575	0.10806575	-1.308065579
coh	0.393375704	0.417369534	-0.576416845
goog	0.496208941	0.054063569	-0.422749397
bac	0.211200295	0.088339226	-1.484604124

Table 5: The Covariance Matrix for Standard Normal Variates Z

2.3313	0.1848	0.2617	0.2570	0.5993	0.2871	0.6386	0.1729	0.1490	0.9059
0.1848	2.2776	0.2799	0.1420	0.5811	0.5530	0.2907	0.0289	0.1640	0.4178
0.2617	0.2799	1.3459	0.1469	0.4177	0.1317	0.2606	0.1992	0.1948	0.6956
0.2570	0.1420	0.1469	0.8496	0.3184	0.2681	0.6036	0.2162	0.2105	1.0608
0.5993	0.5811	0.4177	0.3184	6.9588	0.0314	0.3632	0.2397	0.4339	1.3021
0.2871	0.5530	0.1317	0.2681	0.0314	1.1467	0.2536	0.1850	0.1943	0.4705
0.6386	0.2907	0.2606	0.6036	0.3632	0.2536	3.3790	0.3895	0.3428	1.3985
0.1729	0.0289	0.1992	0.2162	0.2397	0.1850	0.3895	1.9780	0.2554	0.3340
0.1490	0.1640	0.1948	0.2105	0.4339	0.1943	0.3428	0.2554	0.9470	0.5116
0.9059	0.4178	0.6956	1.0608	1.3021	0.4705	1.3985	0.3340	0.5116	5.3622

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Figure 1: Graph of Fitted Option Surface on 20071019

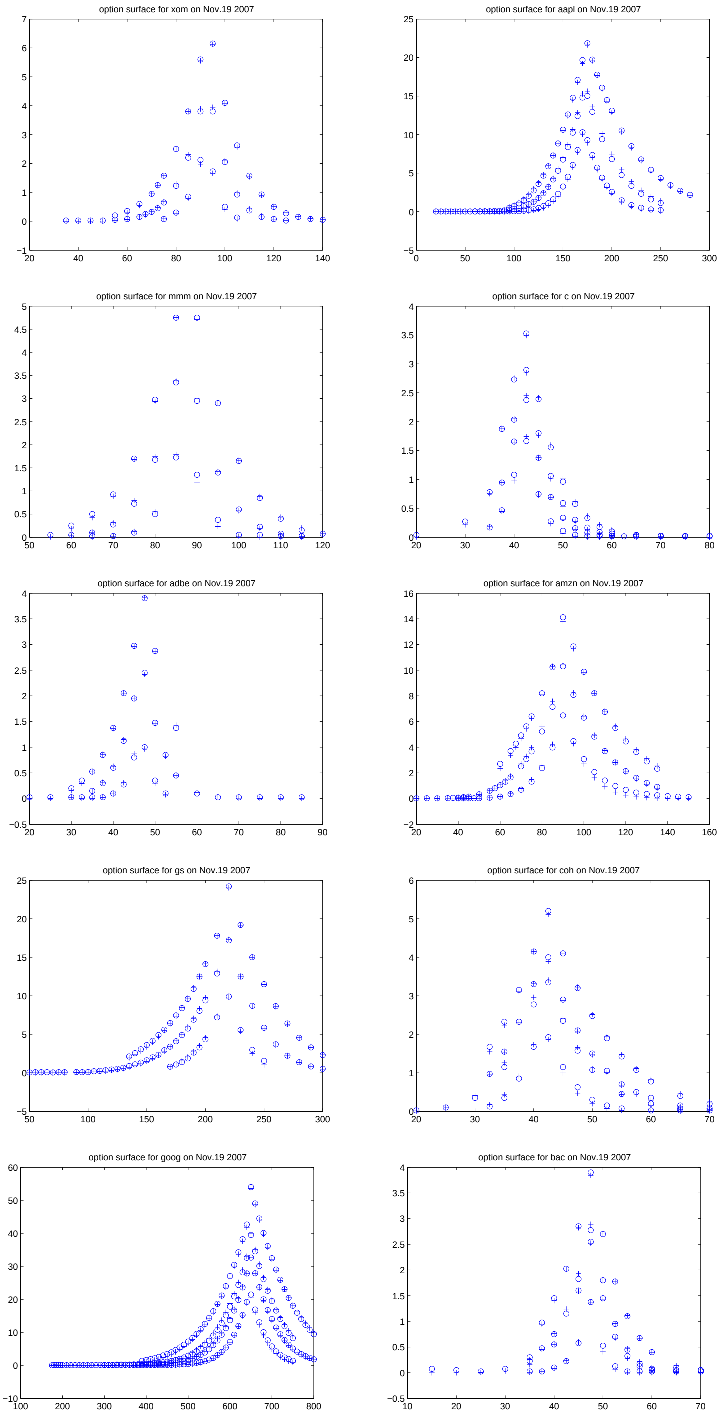

Table 6: Portfolio to Maximize the Expected Distortion

xom	aapl	mmm	c	adbe	amzn	gs	coh	goog	bac
-0.1584	-0.1187	-0.2532	0.2937	0.0700	0.0024	0.7776	-0.1848	-0.3749	-0.1618

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