# On The Price Sensitivities During Financial Crisis

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***Abstract*—The computation of the price sensitivities is of paramount importance in financial risk management. Several new approaches have been recently suggested in the literature for this purpose. However, there is lack of studies that investigate this issue during financial crises. During crises volatility is naturally higher than normal situations. This is going to affect the underlying option pricing. It is especially during the crisis that the investors require to have access to precise calculations in order to deal with the increased level of risk. This issue is especially relevant due to the globalization. Thus, to compute the price sensitivities in such a scenario is crucial. The issue that this paper addresses is the computation of sensitivities during the crisis period based on the Malliavin calculus.**

***Index Terms*—Malliavin Calculus, Crisis; Price Sensitivities, Options**
**JEL Classifications: F36, G15.**

## I. Introduction

A Financial derivative trader that sells an option to an investor in the over-the-counter encounters certain problems to manage its risk. This is due to the fact that in such cases the options are usually tailored to the needs of the investor and not the standardized ones that can easily be hedged by buying an option with the same properties that is sold. In such a customer tailored scenario, hedging the exposure is rather cumbersome. This problem can be dealt with by using the price sensitivities that are usually called "Greeks" in the financial literature. The price sensitivities can play a crucial role in financial risk management. The first price sensitivity is denoted by delta and it represents the rate of the value of the underlying derivative (in this case the price of the option) with regard to the price of the original asset, assuming the ceteris paribus condition. Delta is closely related to the Black and Scholes ([1])formula for option pricing. In order to hedge against this price risk it is desirable to create a delta-neutral or delta hedging position, which is a position with zero delta. This can be achieved by taking a position of minus delta in the original asset for each long option because the delta for the original asset is equal to one[1]. Therefore, calculating a correct value of the delta is of vital importance in terms of successful hedging. It should be mentioned that the delta of an option changes across time and for this reason the position in the original asset needs to be adjusted regularly. Theta represents the rate of the price of the option with respect to time. Gamma signifies the rate of change in delta with regard to the price of the original asset. Thus, if gamma is large in absolute terms then by consequence the delta is very sensitive to the price change of the asset, which implies that leaving a delta-neutral position unchanged during time is very risky. By implication, it means that there is need for creating a gamma-neutral position in such a situation. The sensitivity of the price of the option with respect to the volatility of the original asset is called vega. If the value of vega is high in absolute terms it means that the option price is easily affected by even a small change in the volatility. Hence, it is important to create a vega neutral position in this case. Finally, the sensitivity of the option value with regard to the interest rate as a measure of risk free return is denoted by rho. To neutralize these price sensitivities is the ultimate goal of any optimal hedging strategy. For these reason the computation of these price sensitivities in a precise manner is an integral part of successful financial risk management in order to monitor and neutralize risk. The efficient estimation of the price sensitivities is especially important during the periods when the market is under stress like during the recent financial crisis. Economic agents, including investors and policymakers, are interested in finding out whether there are spill-over effects from one market to another during such a period (see [2]). Because of globalization, with the consequent rise in integration between financial markets worldwide, this issue is becoming increasingly the focus of attention. It is during the crisis that the investors require to have access to precise calculations in order to deal with the increased level of risk. Thus, to compute the price sensitivities correctly in such a scenario is crucial. The issue that this paper addresses is to suggest an approach to compute sensitivities during the crisis period based on the Malliavin calculus.

Options pricing models coming from empirical studies on the dynamics of financial markets after the occurrence of a financial crash do not match with the stochastic models used in the literature. For instance, while the Black-Scholes model [1] assume that the underlying asset price follows a geometric Brownian motion, the work of [3] shows empirically that the post-crash dynamics follow a converging oscillatory motion. On the other hand, the paper of [4] shows that the financial markets follow power-law relaxation decay. Several ideas have been suggested to overcome this shortcoming of the Black-Scholes model. In fact, new option pricing models were suggested from empirical observations (see for instance [5], [6], [7], [8] and [9]). Recently, in [10], the authors suggest a new model which extends the Black-Scholes model. The extension takes into accounts the post-crash dynamics proposed by [3]. The authors derive the following stochastic differential equation that couples the post-crash market index to individual stock prices

$$\frac{dS_t}{S_t} = \left(a + \frac{bg(t)}{S_t}\right) dt + \left(\sigma + \frac{\gamma g(t)}{S_t}\right) dW_t,$$

where $t \in [0,T]$, $S_0 = x > 0$ and $g(t) = A + Be^{\alpha t} sin(\omega t)$. They obtain the following partial differential

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[1]That is, buy $-\Delta$ of the original asset for each long position of the option.

equation (P.D.E.) for the option price

$$\frac{\partial C}{\partial t}+rS\frac{\partial C}{\partial S}-rC+\frac{1}{2}\left(\sigma S+\gamma g(t)\right)^2\frac{\partial^2 C}{\partial S^2}=0,$$

with the terminal condition $C(S,T)=(S-K)^+$.
In this paper, we compute the prices sensitivities for options where the dynamic of the underlying asset price is driven by the model suggested in [10].
The rest of the paper is organized as follows. In section 2 we provide the computations of the Greeks using Malliavin calculus. The last section concludes the paper.

## II. Computations of Greeks

The first part of this section gives an overview of the Malliavin derivative in Wiener space and of its adjoint: the Skorohod integral. We refer the reader to [11] and [12] for more details about the Malliavin calculus. The second Part of the section is dedicated to the computation of the price sensitivities.

We consider a market with two assets: the risky asset $S$ to which is related a European call option and a riskless one given by

$$dA_t=rA_tdt,\quad t\in[0,T],\quad A_0=1.$$

We work on a probability space $(\Omega,\mathcal{F},P)$, $(W_t)_{t\in[0,T]}$ denotes a Brownian motion and $(\mathcal{F}_t)_{t\in[0,T]}$ is the natural filtration generated by $(W_t)_{t\in[0,T]}$. Recall that a stochastic process is a function of two variables i.e time $t\in[0,T]$ and the event $\omega\in\Omega$. However in the literature it is common to write $S_t$ instead of $S_t(\omega)$. The same is true for $W_t$ or any other stochastic process in this paper. We assume that the probability $P$ is the risk-neutral probability and the stochastic differential equation for the underlying asset price under the risk-neutral probability $P$ is given as in [6] by

$$\frac{dS_t}{S_t}=rdt+\left(\sigma+\frac{\gamma g(t)}{S_t}\right)dW_t,\qquad (1)$$

where $t\in[0,T]$, and $S_0=x$. Let $(D_t)_{t\in[0,T]}$ be the Malliavin derivative on the direction of $W$. We denote by $\mathbf{V}$ the set of random variables $F:\Omega\to\mathbf{R}$, such that F has the representation

$$F(\omega)=f\left(\int_0^T f_1(t)dW_t,\ldots,\int_0^T f_n(t)dW_t\right),$$

where $f(x_1,\ldots,x_n)=\sum_\alpha a_\alpha x^\alpha$ is a polynomial in $n$ variables $x_1,\ldots,x_n$ and deterministic functions $f_i\in L^2([0,T])$ . Let $\|.\|_{1,2}$ be the norm

$$\|F\|_{1,2}:=\|F\|_{L^2(\Omega)}+\|D.F\|_{L^2([0,T]\times\Omega)},\quad F\in L^2(\Omega).$$

Thus the domain of the operator $D$, $\mathrm{Dom}(D)$, coincide with $\mathbf{V}$ w.r.t the norm $\|.\|_{1,2}$. The next proposition will be useful.

*Proposition 1:* Given $F=f\left(\int_0^T f_1(t)dW_t,\ldots,\int_0^T f_n(t)dW_t\right)\in\mathbf{V}$. We have

$$D_tF=\sum_{k=0}^{k=n}\frac{\partial f}{\partial x_k}\left(\int_0^T f_1(t)dW_t,\ldots,\int_0^T f_n(t)dW_t\right)f_k(t).$$

To calculate the Mallaivin derivative for integrals, we will use the following propositions

*Proposition 2:* Let $(u_t)_{t\in[0,T]}$ be a $\mathcal{F}_t-$adapted process, such that $u_t\in\mathrm{Dom}(D)$, we have then

$$D_s\int_0^T u_tdt=\int_s^T(D_su_t)dt,\quad s<T.$$

and

*Proposition 3:* Let $(u_t)_{t\in[0,T]}$ be a $\mathcal{F}_t-$adapted process, such that $u_t\in\mathrm{Dom}(D)$, we have

$$D_s\int_0^T u_tdW_t=\int_s^T(D_su_t)dW_t+u_s,\quad s<T.$$

From now on, for any stochastic process $u$ and for $F\in\mathrm{Dom}(D)$ such that $u_.D_.F\in L^2([0,T])$ we let

$$D_uF:=\langle DF,u\rangle_{L^2([0,T])}:=\int_0^T u_tD_tFdt.$$

Let $\delta$ be the Skorohod integral in Wiener space. We have $\delta$ is the adjoint of $D$ as showing in the next proposition, moreover it is an extension of the Itô integral

*Proposition 4:* a) Let $u\in\mathrm{Dom}(\delta)$ and $F\in\mathrm{Dom}(D)$, we have $E[D_uF]\le C(u)\|F\|_{1,2}$, and $E[F\delta(u)]=E[D_uF]$.
b) Consider a $L^2(\Omega\times[0,T])$-adapted stochastic process $u=(u_t)_{t\in[0,T]}$. We have $\delta(u)=\int_0^T u_tdW_t$.
c) Let $F\in\mathrm{Dom}(D)$ and $u\in\mathrm{Dom}(\delta)$ such that $uF\in\mathrm{Dom}(\delta)$ thus $\delta(uF)=F\delta(u)-D_uF$.

The computations of Greeks by Malliavin approach rest on a known integration by parts formula -cf. [13]- given in the following proposition.

*Proposition 5:* Let $I$ be an open interval of $\mathbf{R}$. Let $(F^\zeta)_{\zeta\in I}$ and $(H^\zeta)_{\zeta\in I}$, be two families of random functionals, continuously differentiable in $\mathrm{Dom}(D)$ in the parameter $\zeta\in I$. Let $(u_t)_{t\in[0,T]}$ be a process satisfying

$$D_uF^\xi\neq 0,\quad a.s.\text{ on }\{\partial_\zeta F^\zeta\neq 0\},\quad \zeta\in I,$$

and such that $uH^\zeta\partial_\zeta F^\zeta/D_uF^\zeta$ is continuous in $\zeta$ in $\mathrm{Dom}(\delta)$. We have

$$\begin{aligned}\frac{\partial}{\partial\zeta}E\left[H^\zeta f\left(F^\zeta\right)\right] &= E\left[f\left(F^\zeta\right)\delta\left(uH^\zeta\frac{\partial_\zeta F^\zeta}{D_uF^\zeta}\right)\right]\\ &\quad+E\left[f\left(F^\zeta\right)\partial_\zeta H^\zeta\right].\end{aligned}$$

for any function $f$ such that $f\left(F^\zeta\right)\in L^2(\Omega)$, $\zeta\in I$.

Our aim is to compute the Greeks for options with payoff $f(S_T)$, where $(S_t)_{t\in[0,T]}$ denotes the underlying asset price given by

$$S_T=x+r\int_0^T S_sds+\int_0^T\left(\sigma S_s+\gamma g(s)\right)dW_s.\qquad (2)$$

Let $\zeta$ be a parameter taking the values: the initial asset price $x=S_0$, the volatility $\sigma$, or the interest rate $r$. Let $C=e^{-rT}E[f(S_t^\zeta)]$ be the price of the option. We will compute the following Greeks:

$$\text{Delta}=\frac{\partial C}{\partial x},\quad \text{Gamma}=\frac{\partial^2 C}{\partial x^2},$$
$$\text{Rho}=\frac{\partial C}{\partial r},\quad \text{Vega}=\frac{\partial C}{\partial\sigma}\quad\text{and}\quad \text{Theta}=\frac{\partial C}{\partial T}.$$

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Delta, Rho, Vega

We have using Proposition 5 and c) in Proposition 4

$$\frac{\partial}{\partial \zeta} E\left[H^\zeta f\left(S_T^\zeta\right)\right] = E\left[f(S_T)\left(L^\zeta \delta(u) - D_u L^\zeta + \partial_\zeta H^\zeta\right)\right] \tag{3}$$

where

$$L^\zeta := \frac{H^\zeta \partial_\zeta S_T^\zeta}{D_u S_T^\zeta}. \tag{4}$$

and

$$D_u L^\zeta = D_u \frac{H^\zeta \partial_\zeta S_T^\zeta}{D_u S_T^\zeta} = \frac{D_u\left(H^\zeta \partial_\zeta S_T^\zeta\right) - D_u D_u S_T^\zeta}{(D_u S_T^\zeta)^2}. \tag{5}$$

The Delta and Vega are the first order derivatives of $E[H^\zeta f(S_T^\zeta)]$ with respect to $\zeta = x$ and $\zeta = \sigma$ respectively, with $H^\zeta = e^{-rT}$ and so $\partial_\zeta H^\zeta = 0$, we have

$$\frac{\partial}{\partial \zeta} E\left[e^{-rT} f(S_T^\zeta)\right] = E\left[f(S_T^\zeta)\left(L^\zeta \delta(u) - D_u L^\zeta\right)\right], \tag{6}$$

where $L^\zeta$ is given by (4). For instance the Delta is computed by

$$\mathrm{Delta} = e^{-rT} E\left[f(S_T)\left(\frac{\partial_x S_T}{D_u S_T}\delta(u) - D_u\left(\frac{\partial_x S_T}{D_u S_T}\right)\right)\right].$$

The Rho and Theta are calculated by using equation (3) with $H^\zeta = e^{-rT}$, then $\partial_r e^{-rT} = -re^{-rT}$ and $\partial_T e^{-rT} = -Te^{-rT}$. The Rho for example is given by

$$\mathrm{Rho} = e^{-rT} E\left[f(S_T)\left(\frac{\partial_r S_T}{D_u S_T}\delta(u) - D_u\left(\frac{\partial_r S_T}{D_u S_T}\right) - r\right)\right].$$

Gamma

The Gamma is the second order derivative of $C = E[e^{-rT} f(S_T)]$ with respect to $x$ and it is obtained by differentiating Delta with respect to $x$. Using twice equation (3)

$$\begin{aligned}\frac{\partial^2}{\partial x^2} E[e^{-rT} f(S_T)] &= \frac{\partial}{\partial x} E\left[f(S_T^x)\left(L^x \delta(u) - D_u L^x\right)\right] \\ &= \frac{\partial}{\partial x} E\left[f(S_T)\left(G^x \delta(u) - D_u G^x + \partial_\zeta G^x\right)\right],\end{aligned} \tag{7}$$

where

$$G^x := \frac{(L^x \delta(u) - D_u L^x)\partial_x S_T^x}{D_u S_T^x}, \tag{8}$$

and $L^x$ is given by (4). And

$$D_u G^x = \frac{D_u((L^x \delta(u) - D_u L^x)\partial_x S_T^x) - D_u D_u S_T^x}{(D_u S_T^x)^2}. \tag{9}$$

From equations (3-9), in order to compute the greeks, we need to find $D_u S_T$, $D_u D_u S_T$ and $D_u D_u D_u S_T$, for this we use mainly Proposition 2, so for example for $D_u S_T$, $D_u D_u S_T$ we have

$$\begin{aligned}D_u S_T &= \int_0^T u_t D_t S_T dt \\ D_u D_u S_T &= D_u\left(\int_0^T u_t D_t S_T dt\right) \\ &= \int_0^T u_s D_s\left(\int_0^T u_t D_t S_T dt\right) ds \\ &= \int_0^T u_s \int_s^T D_s(u_t D_t S_T) dt ds \\ &= \int_0^T u_s \int_s^T (u_t D_s D_t S_T + D_t S_T D_s u_t) dt ds.\end{aligned}$$

The next Proposition gives the first, second and third order derivatives of $S_T$ with respect to $D$, needed for the computations of the different Greeks. It gives also the derivative of $S_T$ with respect to $S_0 = x$ needed for the computation of the Delta and Gamma (for the Rho, Vega and Theta, the first derivatives can be computed by the same way.)

*Proposition 6:* For $0 \le t \le T$, we let

$$\xi_t = \exp\left[\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t\right].$$

We have

$$\begin{aligned}\partial_x S_T &= \xi_T \\ D_t S_T &= (\sigma S_t + \gamma g(t))\xi_{T-t} \\ D_s D_t S_T &= \sigma\{(\sigma S_s + \gamma g(s))\xi_{T-s} 1_{s \le t} + (\sigma S_t \\ &\quad + \gamma g(t))\xi_{T-t} 1_{s \le T-t}\} \\ D_l D_s D_t S_T &= \sigma\{\xi_{T-s} 1_{s \le t} \sigma D_l S_s + (\sigma S_s + \gamma g(s)) 1_{s \le t} \\ &\quad D_l \xi_{T-s} + \xi_{T-t} 1_{s \le T-t} \sigma D_l S_t \\ &\quad + (\sigma S_t + \gamma g(t)) 1_{s \le T-t} D_l \xi_{T-t}\}.\end{aligned}$$

*Proof:* By the chain rule of $D_t$ and thanks to Proposition 2 and Proposition 3 we obtain

$$\begin{aligned}\partial_x S_t &= 1 + r\int_0^t \partial_x S_\tau d\tau + \sigma \int_0^t \partial_x S_\tau dW_\tau. \\ D_t S_T &= D_t x + D_t \int_0^T (aS_s + bg(s))\, ds + D_t \int_0^T (\sigma S_s \\ &\quad + \gamma g(s))\, dW_s = \int_t^T D_t (aS_s + bg(s))\, ds \\ &\quad + \int_t^T D_t (\sigma S_s + \gamma g(s))\, dW_s \\ &= a\int_t^T D_t S_s ds + \sigma \int_t^T D_t S_s dW_s + \sigma S_t + \gamma g(t).\end{aligned}$$

Using Itô Lemma, the processes $(\partial_x S_t)_{0 \le t \le T}$ and $(D_t S_T)_{0 \le t \le T}$ can be written as $\partial_x S_t = \xi_t$ and

$$D_t S_T = (\sigma S_t + \gamma g(t))\xi_{T-t}.$$

For the second Malliavin derivative of $S_T$, we have for $0 \le s \le T$

$$\begin{aligned}D_s D_t S_T &= D_s\left((\sigma S_t + \gamma g(t))\xi_{T-t}\right) \\ &= \xi_{T-t} D_s S_t + (\sigma S_t + \gamma g(t)) D_s \xi_{T-t} \\ &= \xi_{T-t}(\sigma S_s + \gamma g(s))\xi_{t-s} 1_{s \le t} \\ &\quad + (\sigma S_t + \gamma g(t)) D_s \xi_{T-t}.\end{aligned}$$

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But

$$
\begin{aligned}
D_s\xi_\nu &= D_s \exp\left[\left(a-\frac{\sigma^2}{2}\right)\nu+\sigma W_\nu\right] \\
&= \exp\left[\left(a-\frac{\sigma^2}{2}\right)\nu+\sigma W_\nu\right]\sigma D_s(W_\nu) \\
&= \exp\left[\left(a-\frac{\sigma^2}{2}\right)\nu+\sigma W_\nu\right]\sigma D_s(\int_0^\nu dW_v) \\
&= \sigma\xi_\nu 1_{s\le\nu}.
\end{aligned}
$$

thus

$$
\begin{aligned}
D_sD_tS_T &= D_r\left((\sigma S_t+\gamma g(t))\xi_{T-t}\right) \\
&= \sigma\xi_{T-t}D_sS_t+(\sigma S_t+\gamma g(t))D_s\xi_{T-t} \\
&= \sigma\xi_{T-t}(\sigma S_s+\gamma g(s))\xi_{t-s}1_{s\le t} \\
&\quad +(\sigma S_t+\gamma g(t))\sigma\xi_{T-t}1_{s\le T-t} \\
&= \sigma\left\{(\sigma S_s+\gamma g(s))\xi_{T-s}1_{s\le t}\right. \\
&\quad \left.+(\sigma S_t+\gamma g(t))\xi_{T-t}1_{s\le T-t}\right\}.
\end{aligned}
$$

The third Malliavin derivative of $S_T$ can be computed as follows, for $0\le l\le T$

$$
\begin{aligned}
D_lD_sD_tS_T &= \sigma D_l\left\{(\sigma S_s+\gamma g(s))\xi_{T-s}1_{s\le t}\right. \\
&\quad \left.+(\sigma S_t+\gamma g(t))\xi_{T-t}1_{s\le T-t}\right\} \\
&= \sigma\left\{\xi_{T-s}1_{s\le t}\sigma D_lS_s+(\sigma S_s+\gamma g(s))\right. \\
&\quad 1_{s\le t}D_l\xi_{T-s}+\xi_{T-t}1_{s\le T-t}\sigma D_lS_t \\
&\quad \left.+(\sigma S_t+\gamma g(t))1_{r\le T-t}D_l\xi_{T-t}\right\}.
\end{aligned}
$$

■

## III. Conclusion

The calculation of the price sensitivities of a financial derivative (like an option or a portfolio of option contracts) is of paramount importance for implementing hedging strategies that are successful to neutralize the underlying risk. This is the case especially during a financial crisis in which the need for dealing with the increased level of risk is urgent. While different approaches have been utilized in the literature to calculate the price sensitivities during normal circumstance, none has focused on this issue during a financial crisis. This paper is the first attempt, to our best knowledge, to deal with this issue by suggesting a formula for computing each of the underlying price sensitivities in a more precise manner during a financial crisis based on the Malliavin calculus. Mathematical proof for each proposition is provided. Thus, the results obtained from this paper are expected to improve on the success of the hedging strategies that must be undertaken by the investor during a financial crisis, a period in which the need for hedging is more imperative than normal circumstances.

## References

[1] F. Black and M. Scholes, "The pricing of options and corporate liabilities," *Journal of Political Economy*, vol. 81, pp. 637–654, 1973.
[2] K. J. Forbes and R. Rigobon, "No contagion, only interdependence: measuring stock market comovements," *Journal of Finance*, vol. LVII, no. 2223-61, 2002.
[3] D. Sornette, *Why Stock Markets Crash: Critical Events in Complex Financial Markets*. Princeton University Press, Princeton, NJ, 2003.
[4] F. Lillo and F. Mantenga, "Power-law relaxation in a complex system: Omori law after a financial market crash," *Phys. Rev. E*, vol. 68, 2003.
[5] R. Savit, "Nonlinearities and chaotic effects in options prices," *J. Futures Mark.*, vol. 9, no. 507, 1989.
[6] E. Deeba, G. Dibeh, and S. Xie, "An algorithm for solving bond pricing problem," *Appl. Math. Comput.*, vol. 128, no. 81, 2002.
[7] J. Tvedt, "Valuation of european futures options in the bifex market," *Physica A*, vol. 18, pp. 167–175, 1998.
[8] G. Dibeh and G. Chahda, "Option pricing in markets with noisy cyclical and crash dynamics," *Finance Lett.*, vol. 3, no. 2, pp. 25–32, 2005.
[9] J. McCauley, *The Dynamics of Markets: Econophysics and Finance*. Cambridge University Press, Cambridge, 2004.
[10] G. Dibeh and H. Harmanani, "Option pricing during post-crash relaxation times," *Physica A*, vol. 380, pp. 357–365, 2007.
[11] D. Nualart, *The Malliavin Calculus and Related Topics*. Springer-Verlag, 1995.
[12] B. Øksendal, "An introduction to malliavin calculus with applications to economics," 1996, working paper 3, Institute of Finance and Management Science, Norwegian School of Economics and Business Administration.
[13] E. Fournié, J. Lasry, J. Lebuchoux, P. Lions, and N. Touzi, "Applications of Malliavin calculus to Monte Carlo methods in finance," *Finance and Stochastics*, vol. 3, no. 4, pp. 391–412, 1999.