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Spatial Disorder of Soliton Solutions for $2D$ Nonlinear Schrödinger Lattices

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***Abstract*—In this paper, we employ the construction of topological horseshoes to study the pattern of the soliton solutions to the discrete nonlinear Schrödinger (DNLS) equations in a two-dimensional lattice. The spatial disorder of the DNLS equations is the result of the strong amplitudes and stiffness of the nonlinearities. The complexity of this disorder is determined by the oscillations (number of turning points) of the nonlinearities. Nonnegative soliton solutions of the DNLS equations with a cubic nonlinearity is also discussed.**

***Index Terms*—discrete nonlinear Schrödinger equation, horseshoe, soliton solution, spatial disorder**

I. Introduction

OUR principal focus in this paper is to study the soliton solutions of the time-dependent discrete nonlinear Schrödinger (DNLS) equation with the cubic nonlinearity

$$\begin{cases} \iota \frac{d}{dt}\phi_{m,n} = \\ -\phi_{m,n+1} - \phi_{m,n-1} + 4\phi_{m,n} - \phi_{m+1,n} - \phi_{m-1,n} \\ +\nu|\phi_{m,n}|^2\phi_{m,n}, \\ m,n \in \mathbb{Z}, \end{cases} \tag{1}$$

where $\iota = \sqrt{-1}$. Equation (1) is a discretization of the nonlinear Schrödinger (NLS) equation

$$\iota\frac{\partial}{\partial t}\phi = -\nabla^2\phi + \nu|\phi|^2\phi,$$

where $\phi = \phi(t, \mathbf{x})$, $t \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^2$. The connection with the NLS equations is clearer from the alternative form of (1):

$$\begin{cases} \iota \frac{d}{dt}\phi_{m,n} = \\ -\frac{1}{h^2}(\phi_{m,n+1} + \phi_{m,n-1} - 4\phi_{m,n} + \phi_{m+1,n} + \phi_{m-1,n}) \\ +\nu|\phi_{m,n}|^2\phi_{m,n}, \\ m,n \in \mathbb{Z}. \end{cases}$$

Systems of NLS equations arise in many fields of physics, including condensed matter, hydrodynamics, optics, plasmas, and Bose-Einstein condensates (BECs) (see e.g. [1], [4], [12], [23]). In the presence of strong periodic trapped potentials, a NLS equation can be approximated by a DNLS equation by using the "tight binding approximation" [24]. Equation (1) describes a large class of discrete nonlinear systems such as optical fibers [5], [6], small molecules such as benzene [7], and, more recently, dilute BECs trapped in a multiwell potential [2], [3], [25], [24].

The interplay between disorder and nonlinearity is a central topic of nonlinear science. This raises a number of mathematical questions related to the behavior of many physical systems. In the context of ultracold atomic gases, disorder may result from the roughness of a magnetic trap [9] or a magnetic microtrap [27]. This motivates us to study the spatial disorder of the soliton solutions of the DNLS equation (1). To obtain such soliton solutions, we set $\phi_{m,n}(t) = e^{-\iota\lambda t}u_{m,n}$, $\lambda, u_{m,n} \in \mathbb{R}$, and transform (1) into the time-independent discrete nonlinear Schrödinger equation

$$\begin{aligned} -u_{m,n+1} - u_{m,n-1} + 4u_{m,n} - u_{m+1,n} \\ - u_{m-1,n} + \nu u_{m,n}^3 = \lambda u_{m,n}. \end{aligned} \tag{2}$$

Here, $(\phi_{m,n})$ is called a "soliton solution" if $u_{m,n} \to 0$ exponentially as $\max\{|m|, |n|\} \to \infty$. By nature, discrete solitons represent self-trapped wavepackets in nonlinear periodic structures and result from the interplay between lattice diffraction (or dispersion) and material nonlinearity. Discrete solitons in one-dimensional lattices were first experimentally observed in a nonlinear AlGaAs array by the groups of Silberberg and Aitchison [8]. In subsequent investigations, discrete soliton transport dynamics were studied by [18] in such arrays and it was observed in [21] that the nonlinearly induced escape from a waveguide defect. Optical discrete solitons in two-dimensional nonlinear waveguide arrays were first observed in biased photorefractive crystals by Segev's and Christodoulides's groups [11], [10]. Please refer to the survey article [17] for more details in the developments in the observation of discrete solitons.

Arising from the abundance of physical experiments on discrete solitons, three relevant mathematical issues are proposed: (i) the existence of soliton solutions to (2), (ii) patterns of these soliton solutions and (iii) their complexity. To study the patterns of soliton solutions, the formulation of five-point difference in (2) enable us to study a more general form of the second order elliptic partial difference equation (PdE)

$$\begin{aligned} -\alpha u_{m,n+1} - \beta u_{m,n-1} - u_{m+1,n} \\ - \gamma u_{m-1,n} + f(u_{m,n}) = 0, \end{aligned} \tag{3}$$

where $f \in C^1([a, b])$ and α, β, $\gamma \in \mathbb{R}$ with $\gamma \neq 0$. We further assume the nonlinearity in f in (3) satisfies the following:

($\mathcal{A}1$) Denote $c_1 < c_2 < \cdots < c_N$ the turning points of f in the interval $[a, b]$. Let $c_0 = a$, $c_{N+1} = b$; δ_1 and δ_2, respectively, be the minimal and maximal value of $\{\alpha x + \beta y + \gamma z|\ a \le x, y, z \le b\}$; and δ_3 and δ_4, respectively, be the minimal and maximal value of $\{(\alpha x + \beta y + z)/\gamma|\ a \le x, y, z \le b\}$. Assume there exist closed intervals $I_i \subset [c_i, c_{i+1}]$, for $i = 0, \ldots, N$, such that

$$f(I_i) \supseteq [a + \delta_1, b + \delta_2] \text{ and } f(I_i)/\gamma \supseteq [a + \delta_3, b + \delta_4].$$

By $f(I_i)/\gamma$ we mean the closed interval $\{f(u)/\gamma|\ u \in I_i\}$.

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($\mathcal{A}2$) Let $|\gamma'| = \max\{1, |\gamma|\}$. Assume

$$|f'(u)| \geq |\alpha| + |\beta| + \frac{\sqrt{5}+3}{2}|\gamma'|,$$

for all $u \in I_i$, $i = 0, \ldots, N$.

Our first theorem concerns the spatial disorder of PdE (3).

Theorem I.1. *Suppose assumptions* ($\mathcal{A}1$) *and* ($\mathcal{A}2$) *hold. For any* $\mathbf{k} = (k_{m,n})_{m,n\in\mathbb{Z}} \in \{0, \ldots, N\}^{\mathbb{Z}^2}$*, there exists a unique solution* $(u_{m,n})$ *to PdE* (3) *such that*

$$u_{m,n} \in I_{k_{m,n}},$$

for all $m, n \in \mathbb{Z}$.

We see in Theorem I.1 that the strong amplitudes (Assumption ($\mathcal{A}1$)) and stiffness (Assumption ($\mathcal{A}2$)) of the nonlinearities in f lead the PdE (3) to the spatial disorder. The complexity of this disorder is determined by the oscillations (number of turning points) of the nonlinearities. More precisely, the spatial entropy of the PdE (3) equals to $\log(N+1)$. By applying Theorem I.1 to (2), we can prove our second theorem involving the spatial disorder and pattern of soliton solutions to the DNLS equation (2).

Theorem I.2. *Let* ω^* *denote the largest value of real roots of*

$$\frac{2\omega + \tilde{\Delta}}{12} = \sqrt{\frac{3\omega}{\omega - \tilde{\Delta}}} \text{ and } \frac{2\omega - \tilde{\Delta}}{12} = \sqrt{\frac{3\omega}{\omega + \tilde{\Delta}}},$$

where $\tilde{\Delta} = (7+\sqrt{5})/2$. *Suppose* $\nu < 0$ *and* $4-\lambda > \omega^*$. *Then there exist disjoint closed intervals* $I_0, I_1 \subset \mathbb{R}^+$, $0 \in I_0$, *such that for every* $(k_{m,n})_{m,n\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}^2}$*, there exists a unique nonnegative solution* $(u_{m,n})$ *to DNLS equation* (2) *satisfying* $u_{m,n} \in I_{k_{m,n}}$. *In addition, if* $k_{m,n} = 0$ *for* $|m|, |n| > N_0$, *some given positive integer, then*

$u_{m,n} = O(\mu^{\max\{|m|,|n|\}})$ *as* $|m|$ *or* $|n|$ *are sufficiently large.*

Here $0 < \mu < (\sqrt{5}-1)/2$ *is a constant independent of the solution* $(u_{m,n})$.

The solutions in the second assertion of Theorem I.2 are referred to as the so-called "bright solitons". Here both the existence and the variety of solutions to DNLS equation (2) are presented. Specifically, the state at the (m, n)-th site, $u_{m,n}$, can be either dark ($u_{m,n} \in I_0$) or bright ($u_{m,n} \in I_1$) that depends on the configuration $k_{m,n} = 0$ or 1, respectively. Considering only the soliton solutions, the DNLS equation also exhibits the spatial disorder. For the DNLS equations in the one-dimensional lattice (i.e., the case $\alpha = \beta = 0$ and $\gamma = 1$ in (3)), soliton solutions were studied in [22] by the construction of homoclinic/hetronic orbits. In [26], the spatial disorder in the one-dimensional NLS equations equipped with periodic/quasiperiodic trapped potentials was studied, in which a coherent structure ansatz was applied to reduce the NLS equation to a forced Duffing equation. In [20], [13], the soliton solutions of DNLS equations in a two-dimensional lattice was studied in the case $|\nu| \gg 1$ and $\lambda/\nu = O(1)$ by variational techniques. Our result in Theorem I.2 is valid for λ and $\mu = O(1)$. The chaotic behavior of DNLS equations in one-dimensional lattice as well as its synchronization phenomena were studied by [19]. Bifurcation analysis of DNLS equations for the ground state solutions was studied by [15]. Recently, it was reported by [16] the occurrence of the phase separation for the ground state solutions of the DNLS equation in lattices with a general connection topology.

This paper is organized as follows. In Section 2, we prove Theorem I.1 by the construction of a horseshoe in l_∞ for the map $\mathbf{F}$ introduced in (4). We follow the standard process for planner maps and generalize it to an infinite dimensional case. In Section 3, we prove Theorem I.2 by the using of Theorem I.1.

Throughout this paper, we denote $l_\infty = \{\mathbf{u} = (\ldots, u_{-1}, u_0, u_1, \ldots) | \sup_n |u_u| < \infty\}$. For any finite set $\{0, \ldots, N\}$, we denote $\{0, \ldots, N\}^{\mathbb{Z}} = \{\mathbf{k} = (k_n)_{n\in\mathbb{Z}} | k_n \in \{0, \ldots, N\}\}$ and $\{0, \ldots, N\}^{\mathbb{Z}^2} = \{\mathbf{k} = (k_{m,n})_{m,n\in\mathbb{Z}^2} | k_{m,n} \in \{0, \ldots, N\}\}$. We use the boldface alphabets (or symbols) to denote operators (or vectors). We say $\mathbf{u} \leq \mathbf{v}$ if $u_n \leq v_n$ for all $n \in \mathbb{Z}$. We use $\|\cdot\| = \|\cdot\|_\infty$ to denote the infinity norm of an operator or a vector. Note that for any bounded operator $\mathbf{A}$ on l_∞, the infinity norm of $\mathbf{A}$ can be computed by $\|\mathbf{A}\| = \sup_{\|\mathbf{u}\|=1} \|\mathbf{Au}\| = \sup_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} |a_{mn}|$.

II. Construction of horseshoe and its hyperbolicity

First, we define the map $\mathbf{F} : l_\infty \times l_\infty \to l_\infty \times l_\infty$ by

$$\mathbf{F} : \begin{cases} \bar{\mathbf{u}} = \mathbf{g}(\mathbf{u}) - \gamma \mathbf{v}, \\ \bar{\mathbf{v}} = \mathbf{u}, \end{cases} \tag{4a}$$

where $\mathbf{g} : l_\infty \to l_\infty$ is given by

$$g_n(\mathbf{u}) = -\alpha u_{n+1} + f(u_n) - \beta u_{n-1}, \quad n \in \mathbb{Z}. \tag{4b}$$

Considering a bounded solution $(u_{m,n})_{m,n\in\mathbb{Z}}$ of (3), let $\mathbf{u}^{(m)} = (\ldots, u_{m,-1}, u_{m,0}, u_{m,1}, \ldots)$ for all $m \in \mathbb{Z}$, that is, $\mathbf{u}^{(m)}$ is the m-th row of $(u_{m,n})_{m,n\in\mathbb{Z}}$. Hence, we see that $(\mathbf{u}^{(m+1)}, \mathbf{u}^{(m)}) = \mathbf{F}(\mathbf{u}^{(m)}, \mathbf{u}^{(m-1)})$. This means that $(u_{m,n})_{m,n\in\mathbb{Z}}$ forms an orbit of $\mathbf{F}$ in the m-direction. In this section, we shall construct a horseshoe for $\mathbf{F}$ and prove the hyperbolicity of its invariant Cantor set. To this end, we begin with some basic settings for this construction adopted from [28]. Let $\mathcal{B} \subset l_\infty$ denote the box $\mathcal{B} = \{\mathbf{u} \in l_\infty | a \leq u_n \leq b, \text{ for all } n \in \mathbb{Z}\}$.

Definition II.1. Let μ be a real nonnegative number. A μ-horizontal surface in $\mathcal{B} \times \mathcal{B}$ is the graph of a differentiable function $\mathbf{v} = \mathbf{r}(\mathbf{u})$, $\mathbf{u} \in \mathcal{B}$, satisfying $\|D\mathbf{r}(\mathbf{u})\| \leq \mu$. A μ-horizontal strip in $\mathcal{B} \times \mathcal{B}$ is the set

$$\mathcal{H} = \{(\mathbf{u}, \mathbf{v}) | \mathbf{r}_1(\mathbf{u}) \leq \mathbf{v} \leq \mathbf{r}_2(\mathbf{u}), \ \mathbf{u} \in \mathcal{B}\},$$

where $\mathbf{r}_1 < \mathbf{r}_2$ are μ-horizontal surfaces. Similarly, a μ-vertical surface in $\mathcal{B} \times \mathcal{B}$ is the graph of a differentiable function $\mathbf{u} = \mathbf{s}(\mathbf{v})$, $\mathbf{v} \in \mathcal{B}$, satisfying $\|D\mathbf{s}(\mathbf{v})\| \leq \mu$. A μ-vertical strip in $\mathcal{B} \times \mathcal{B}$ is the set

$$\mathcal{V} = \{(\mathbf{u}, \mathbf{v}) | \mathbf{s}_1(\mathbf{v}) \leq \mathbf{u} \leq \mathbf{s}_2(\mathbf{v}), \ \mathbf{v} \in \mathcal{B}\},$$

where $\mathbf{s}_1 < \mathbf{s}_2$ are μ-horizontal surfaces. The widths of the horizontal and the vertical strips are defined, respectively, as

$$d(\mathcal{H}) = \sup_{\mathbf{u}\in\mathcal{B}} ||\mathbf{r}_1(\mathbf{u}) - \mathbf{r}_2(\mathbf{u})|| , \quad d(\mathcal{V}) = \sup_{\mathbf{v}\in\mathcal{B}} ||\mathbf{s}_1(\mathbf{v}) - \mathbf{s}_2(\mathbf{v})||.$$

Let $\mathbb{E} = \{0, \ldots, N\}^{\mathbb{Z}}$. For a given $\mathbf{k} \in \mathbb{E}$, let

$$\mathcal{B}_{\mathbf{k}} = \{\mathbf{u} \in \mathcal{B} | \ u_n \in I_{k_n}, \ n \in \mathbb{Z}\}.$$

Here N is the number of turning points of f and I_{k_n} are the closed intervals given in $(\mathcal{A}1)$. We define the horizontal and vertical strips

$$\mathcal{H}_{\mathbf{k}} = \mathcal{B} \times \mathcal{B}_{\mathbf{k}} = \{(\mathbf{u}, \mathbf{v}) \in (\mathcal{B} \times \mathcal{B}) | \ \mathbf{v} \in \mathcal{B}_{\mathbf{k}}\},$$
$$\mathcal{V}_{\mathbf{k}} = \mathcal{B}_{\mathbf{k}} \times \mathcal{B} = \{(\mathbf{u}, \mathbf{v}) \in (\mathcal{B} \times \mathcal{B}) | \ \mathbf{u} \in \mathcal{B}_{\mathbf{k}}\}.$$

Now we are ready to construct a horseshoe for $\mathbf{F}$. Before giving any proof, we note from (4) that the inverse of $\mathbf{F}$ is given by

$$\mathbf{F}^{-1} : \begin{cases} \mathbf{u} = \bar{\mathbf{v}}, \\ \mathbf{v} = (\mathbf{g}(\bar{\mathbf{v}}) - \bar{\mathbf{u}})/\gamma. \end{cases}$$

From Theorem II.2 to Lemma II.5, each result is associated with a horizontal and a vertical case. Due to the symmetry of $\mathbf{F}$ and $\mathbf{F}^{-1}$, we shall only give the proofs of the horizontal cases by using the map $\mathbf{F}$. The proofs for the vertical cases can be similarly verified by using $\mathbf{F}^{-1}$.

Proposition II.1. *Let* $\mathbf{A}$ *be a bounded operator on* l_∞*. Suppose* $\mathbf{A}$ *is diagonal dominant, i.e., there exists* $\epsilon > 0$ *such that* $|a_{mm}| > \sum_{n=-\infty, n\neq m}^{\infty} |a_{mn}| + \epsilon$ *for all* $m \in \mathbb{Z}$*. Then* $\mathbf{A}$ *is invertible. In addition, if* $\mathbf{D}$ *is an invertible diagonal bounded operator on* l_∞ *such that* $\|\mathbf{D}^{-1}(\mathbf{A}-\mathbf{D})\| < 1$*, then*

$$\|\mathbf{A}^{-1}\| \leq \frac{\|\mathbf{D}^{-1}\|}{1 - \|\mathbf{D}^{-1}(\mathbf{A} - \mathbf{D})\|}. \tag{5}$$

Proof: Suppose $\mathbf{A}\mathbf{u} = 0$ for some $\mathbf{u} \neq \mathbf{0}$. Then we have $a_{mm}u_m = \sum_{n=-\infty, n\neq m}^{\infty} a_{mn}u_n$ for all $m \in \mathbb{Z}$. Taking absolute on both sides of the equation and applying the triangular inequality to the resulting equation, we obtain $|a_{mm}||u_m| \leq \sum_{n=-\infty, n\neq m}^{\infty} |a_{mn}|\|\mathbf{u}\| < (|a_{mm}| - \epsilon)\|\mathbf{u}\|$. This is a contradiction since $\|\mathbf{u}\| = \sup_{n\in\mathbb{Z}} |u_n|$. The proof of the first assertion is complete. For the second assertion, note that $\mathbf{A}^{-1} = \|(\mathbf{I} + \mathbf{D}^{-1}(\mathbf{A} - \mathbf{D}))^{-1}\mathbf{D}^{-1}\| \leq \|(\mathbf{I} + \mathbf{D}^{-1}(\mathbf{A} - \mathbf{D}))^{-1}\|\|\mathbf{D}^{-1}\|$. On the other hand, since $\|\mathbf{D}^{-1}(\mathbf{A} - \mathbf{D})\| < 1$, we have $\|(\mathbf{I} + \mathbf{D}^{-1}(\mathbf{A} - \mathbf{D}))^{-1}\| \leq \sum_{n=0}^{\infty} \|\mathbf{D}^{-1}(\mathbf{A}-\mathbf{D}))^{-1}\|^n = 1/(1-\|\mathbf{D}^{-1}(\mathbf{A}-\mathbf{D})\|)$. This gives (5). ■

Theorem II.2. *Let* $\Delta = \min_{1\leq i\leq N}\{|f'(u)| | \ u \in I_i\} - (|\alpha| + |\beta| + |\gamma'|)$ *where* $|\gamma'|$ *is defined in* $(\mathcal{A}2)$*. Suppose* μ *is a constant satisfying*

$$\frac{|\gamma'|}{\Delta} < \mu < \frac{\sqrt{5} - 1}{2}.$$

Let $\mathbf{k} \in \mathbb{E}$ *be given. If* $\mathcal{S}$ *is a* μ*-horizontal surface, then* $\mathbf{F}(\mathcal{S} \cap \mathcal{V}_{\mathbf{k}}) \cap (\mathcal{B} \times \mathcal{B})$ *is a* μ*-horizontal surface contained in* $\mathcal{H}_{\mathbf{k}}$*. If* $\mathcal{S}$ *is a* μ*-vertical surface, then* $\mathbf{F}^{-1}(\mathcal{S} \cap \mathcal{H}_{\mathbf{k}}) \cap (\mathcal{B} \times \mathcal{B})$ *is a* μ*-vertical surface contained in* $\mathcal{V}_{\mathbf{k}}$*.*

Here we remark from $(\mathcal{A}2)$ that $\Delta \geq (\sqrt{5} + 1)|\gamma'|/2$. Hence the constant μ in Theorem II.2 is well defined. Moreover, μ is between 0 and 1.

From Theorem II.2, we see that $\mathbf{F}(\mathcal{V}_{\mathbf{k}}) \cap (\mathcal{B} \times \mathcal{B}) \subset \mathcal{H}_{\mathbf{k}}$ and $\mathbf{F}^{-1}(\mathcal{H}_{\mathbf{k}}) \cap (\mathcal{B} \times \mathcal{B}) \subset \mathcal{V}_{\mathbf{k}}$ form a μ-horizontal strip and a μ-vertical strip, respectively. Let

$$\mathcal{H}^*_{\mathbf{k}} = \mathbf{F}(\mathcal{V}_{\mathbf{k}}) \cap (\mathcal{B} \times \mathcal{B}) \ , \quad \mathcal{V}^*_{\mathbf{k}} = \mathbf{F}^{-1}(\mathcal{H}_{\mathbf{k}}) \cap (\mathcal{B} \times \mathcal{B}). \tag{6}$$

Thus the resulting surfaces in Theorem II.2, $\mathbf{F}(\mathcal{S} \cap \mathcal{V}_{\mathbf{k}}) \cap (\mathcal{B} \times \mathcal{B})$ and $\mathbf{F}^{-1}(\mathcal{S} \cap \mathcal{H}_{\mathbf{k}}) \cap (\mathcal{B} \times \mathcal{B})$, can be accordingly rewritten as $\mathbf{F}(\mathcal{S}) \cap \mathcal{H}^*_{\mathbf{k}}$ and $\mathbf{F}(\mathcal{S}) \cap \mathcal{V}^*_{\mathbf{k}}$, respectively. We have the following immediate consequence of Theorem II.2.

Corollary II.3. *Let* μ *be the constant given in Theorem II.2 and* $\mathbf{k} \in \mathbb{E}$ *be given. If* $\mathcal{H}$ *is a* μ*-horizontal strip, then* $\mathbf{F}(\mathcal{H}) \cap \mathcal{H}^*_{\mathbf{k}}$ *is also a* μ*-horizontal strip. If* $\mathcal{V}$ *is a* μ*-vertical strip, then* $\mathbf{F}^{-1}(\mathcal{V}) \cap \mathcal{V}^*_{\mathbf{k}}$ *is also a* μ*-vertical strip.*

In Corollary II.3, we see that $\mathbf{F}$ (resp., $\mathbf{F}^{-1}$) maps a μ-horizontal strip (resp., μ-vertical strip) to an uncountable number of μ-horizontal strips (resp., μ-vertical strips), in which exactly one strip is included in $\mathcal{H}^*_{\mathbf{k}}$ (resp., $\mathcal{V}^*_{\mathbf{k}}$) for each $\mathbf{k} \in \mathbb{E}$. In the next theorem, we will see that every strip becomes thinner under the mapping by a factor less than 1.

Theorem II.4. *Let* μ *be be the constant given in Theorem II.2 and* $\mathbf{k} \in \mathbb{E}$ *be given. Suppose* $\mathcal{H}$ *is a* μ*-horizontal strip and* $\mathcal{V}$ *is a* μ*-vertical strip. If* $\bar{\mathcal{H}} = \mathbf{F}(\mathcal{H}) \cap \mathcal{H}^*_{\mathbf{k}}$ *and* $\tilde{\mathcal{V}} = \mathbf{F}^{-1}(\mathcal{V}) \cap \mathcal{V}^*_{\mathbf{k}}$*, then*

$$d(\bar{\mathcal{H}}) \leq \frac{\mu}{1 - \mu^2} d(\mathcal{H}) \ , \quad d(\tilde{\mathcal{V}}) \leq \frac{\mu}{1 - \mu^2} d(\mathcal{V}).$$

Here we remark that the factor $\mu/(1 - \mu^2) < 1$ by the assumption that $\mu < (\sqrt{5} - 1)/2$. Before proving Theorem II.4, we first prove the following lemma.

Lemma II.5. *Let* μ *be be the constant given in Theorem II.2 and* $\mathbf{k} \in \mathbb{E}$ *be given. Suppose* $(\mathbf{u}, \mathbf{v})$ *is a point in* $\mathcal{B} \times \mathcal{B}$*,* $\boldsymbol{\zeta} = \begin{bmatrix} \boldsymbol{\xi} \\ \boldsymbol{\eta} \end{bmatrix} \in l_\infty \times l_\infty$*,* $\bar{\boldsymbol{\zeta}} = \begin{bmatrix} \bar{\boldsymbol{\xi}} \\ \bar{\boldsymbol{\eta}} \end{bmatrix} := D\mathbf{F}(\mathbf{u}, \mathbf{v})\boldsymbol{\zeta}$*, and* $\tilde{\boldsymbol{\zeta}} = \begin{bmatrix} \tilde{\boldsymbol{\xi}} \\ \tilde{\boldsymbol{\eta}} \end{bmatrix} := D\mathbf{F}^{-1}(\mathbf{u}, \mathbf{v})\boldsymbol{\zeta}$*.*

(a) *Suppose* $(\mathbf{u}, \mathbf{v}) \in \mathcal{V}_{\mathbf{k}}$ *and* $\|\boldsymbol{\eta}\| \leq \mu\|\boldsymbol{\xi}\|$*. If* $\{n_j\}_{j=0}^{\infty} \subset \mathbb{Z}$ *is any sequence such that* $|\xi_{n_j}| \to \|\boldsymbol{\xi}\|$ *as* $j \to \infty$*, then there exists* $N_0 > 0$*, independent of the choice of* $(\mathbf{u}, \mathbf{v})$*, such that*

$$\mu|\bar{\xi}_{n_j}| > \|\boldsymbol{\zeta}\|$$

for $j > N_0$*.*

(b) *Suppose* $(\mathbf{u}, \mathbf{v}) \in \mathcal{H}_{\mathbf{k}}$ *and* $\|\boldsymbol{\xi}\| \leq \mu\|\boldsymbol{\eta}\|$*. If* $\{n_j\}_{j=0}^{\infty} \subset \mathbb{Z}$ *is any sequence such that* $|\eta_{n_j}| \to \|\boldsymbol{\eta}\|$ *as* $j \to \infty$*, then there exists* $N_0 > 0$*, independent of the choice of* $(\mathbf{u}, \mathbf{v})$*, such that*

$$\mu|\tilde{\eta}_{n_j}| > \|\boldsymbol{\zeta}\|$$

for $j > N_0$*.*

Later we will see that Lemma II.5 plays an important role in the proof of the hyperbolicity of $\mathbf{F}$. Now we are in position to prove Theorem II.4.

Now, we are ready to prove Theorem I.1.

Proof of Theorem I.1: Let μ be the constant defined in Theorem II.2. Define

$$\Lambda_{-1} = \bigcup_{\mathbf{k}_{-1}\in\mathbb{E}} \mathcal{H}^*_{\mathbf{k}_{-1}} \ , \quad \Lambda_0 = \bigcup_{\mathbf{k}_0\in\mathbb{E}} \mathcal{V}^*_{\mathbf{k}_0},$$

where $\mathcal{H}^*_{\mathbf{k}_{-1}}$ and $\mathcal{V}^*_{\mathbf{k}_0}$ are defined in (6). Note that $\mathcal{B} \times \mathcal{B}$ is not only a μ-horizontal strip, but also a μ-vertical strip. This implies that each $\mathcal{H}^*_{\mathbf{k}_{-1}}$ and $\mathcal{V}^*_{\mathbf{k}_0}$ are, respectively, a μ-horizontal strip and a μ-vertical strip. Let

$$\Lambda_{-n-1} = \Lambda_{-1} \cap \mathbf{F}(\Lambda_{-1}) \cap \cdots \cap \mathbf{F}^n(\Lambda_{-1}),$$
$$\Lambda_n = \Lambda_0 \cap \mathbf{F}^{-1}(\Lambda_0) \cap \cdots \cap \mathbf{F}^{-n}(\Lambda_0).$$

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Hence, we may set

$$\Lambda_{-n-1} = \bigcup_{\substack{\mathbf{k}_{-j}\in\mathbb{E} \\ j=1,\ldots,n+1}} \mathcal{H}^*_{\mathbf{k}_{-1},\mathbf{k}_{-2},\ldots,\mathbf{k}_{-n-1}}$$

and

$$\Lambda_n = \bigcup_{\substack{\mathbf{k}_j\in\mathbb{E} \\ j=0,\ldots,n}} \mathcal{V}^*_{\mathbf{k}_0,\mathbf{k}_1,\ldots,\mathbf{k}_n},$$

where

$$\mathcal{H}^*_{\mathbf{k}_{-1},\ldots,\mathbf{k}_{-n-1}} = \{(\mathbf{u},\mathbf{v}) \in \mathcal{B}\times\mathcal{B} |\ \mathbf{F}^{-j}(\mathbf{u},\mathbf{v}) \in \mathcal{H}^*_{\mathbf{k}_{-j-1}}, j=0,\ldots,n\}$$

and

$$\mathcal{V}^*_{\mathbf{k}_0,\mathbf{k}_1,\ldots,\mathbf{k}_n} = \{(\mathbf{u},\mathbf{v}) \in \mathcal{B}\times\mathcal{B} |\ \mathbf{F}^{j}(\mathbf{u},\mathbf{v}) \in \mathcal{V}^*_{\mathbf{k}_j}, j=0,\ldots,n\}.$$

Note that $\mathcal{H}^*_{\mathbf{k}_{-1},\ldots,\mathbf{k}_{-n-1}} = \mathcal{H}^*_{\mathbf{k}_{-1}} \cap F(\mathcal{H}^*_{\mathbf{k}_{-2},\ldots,\mathbf{k}_{-n-1}})$ and $\mathcal{V}^*_{\mathbf{k}_0,\mathbf{k}_1,\ldots,\mathbf{k}_n} = \mathcal{V}^*_{\mathbf{k}_0} \cap F^{-1}(\mathcal{V}^*_{\mathbf{k}_1,\ldots,\mathbf{k}_n})$. By Corollary II.3, an inductive argument shows that each $\mathcal{H}^*_{\mathbf{k}_{-1},\ldots,\mathbf{k}_{-n-1}}$ and $\mathcal{V}^*_{\mathbf{k}_0,\mathbf{k}_1,\ldots,\mathbf{k}_n}$ are, respectively, a μ-horizontal and a μ-vertical strip. In addition, it follows from Theorem II.4 that

$$d\left(\mathcal{H}^*_{\mathbf{k}_{-1},\mathbf{k}_{-2},\ldots,\mathbf{k}_{-n-1}}\right) \le \left(\frac{\mu}{1-\mu^2}\right)^n d(\mathcal{B})$$
$$d\left(\mathcal{V}^*_{\mathbf{k}_0,\mathbf{k}_1,\ldots,\mathbf{k}_n}\right) \le \left(\frac{\mu}{1-\mu^2}\right)^n d(\mathcal{B}).$$

Hence, for any sequences $(\mathbf{k}_{-1},\mathbf{k}_{-2}\ldots)$ and $(\mathbf{k}_0,\mathbf{k}_1,\ldots) \in \mathbb{E}^{\mathbb{N}}$,

$$\begin{array}{l} \bigcap_{n=1}^{\infty} \mathcal{H}^*_{\mathbf{k}_{-1},\mathbf{k}_{-2},\ldots,\mathbf{k}_{-n}} \\ \bigcap_{n=0}^{\infty} \mathcal{V}^*_{\mathbf{k}_0,\mathbf{k}_1,\ldots,\mathbf{k}_n} \end{array} \tag{7}$$

are decreasing to two surfaces, say $\mathcal{H}^*_{\mathbf{k}_{-1},\mathbf{k}_{-2},\ldots} = \{\mathbf{v} = \mathbf{r}(\mathbf{u})\}$ and $\mathcal{V}^*_{\mathbf{k}_0,\mathbf{k}_1,\ldots} = \{\mathbf{u} = \mathbf{s}(\mathbf{v})\}$. Here we note that $\mathbf{r}$ and $\mathbf{s}$ may not be differentiable. However, the uniform convergency of the upper and lower surfaces in (7) implies they satisfy a Lipschitz condition with Lipschitz constant μ; i.e., for any $\mathbf{u}_1,\mathbf{u}_2,\mathbf{v}_1,\mathbf{v}_2 \in \mathcal{B}$,

$$\|\mathbf{r}(\mathbf{u}_1)-\mathbf{r}(\mathbf{u}_2)\| \le \mu\|\mathbf{u}_1-\mathbf{u}_2\|, \quad \|\mathbf{s}(\mathbf{v}_1)-\mathbf{s}(\mathbf{v}_2)\| \le \mu\|\mathbf{v}_1-\mathbf{v}_2\|.$$

Since $|\mu| < 1$, by the contraction mapping theorem, the equation

$$\begin{cases} \mathbf{v} = \mathbf{r}(\mathbf{u}) \\ \mathbf{u} = \mathbf{s}(\mathbf{v}) \end{cases}$$

has a unique solution in $\mathcal{B}\times\mathcal{B}$. This means $\mathcal{H}^*_{\mathbf{k}_{-1},\mathbf{k}_{-2},\ldots} = \{\mathbf{v} = \mathbf{r}(\mathbf{u})\}$ and $\mathcal{V}^*_{\mathbf{k}_0,\mathbf{k}_1,\ldots} = \{\mathbf{u} = \mathbf{s}(\mathbf{v})\}$ have a unique intersection. Hence, the invariant set $\Lambda = \Lambda_{-\infty} \cap \Lambda_\infty$ is a Cantor set. Denote Σ the symbolic space $\Sigma = \{(\ldots,\mathbf{k}_{-1}|\mathbf{k}_0,\mathbf{k}_1,\ldots) |\ \mathbf{k}_n \in \mathbb{E},\ n\in\mathbb{Z}\}$. To see $\mathbf{F}|_\Lambda$ is topological conjugate to the full shift σ on Σ, we define the function

$$\phi(\mathbf{p}) = (\ldots,\mathbf{k}_{-1}|\mathbf{k}_0,\mathbf{k}_1,\ldots),$$

where $\mathbf{p} = \mathcal{H}_{\mathbf{k}_{-1},\mathbf{k}_{-2},\ldots} \cap \mathcal{V}_{\mathbf{k}_0,\mathbf{k}_1,\ldots}$. It is easy to verify that ϕ is a homeomorphism from Λ to Σ. We only need to show that $\phi(\mathbf{F}(\mathbf{p})) = \sigma(\phi(\mathbf{p}))$. From the construction of $\mathcal{V}^*_{\mathbf{k}_0,\mathbf{k}_1,\ldots}$, we have

$$\mathbf{F}(\mathcal{V}^*_{\mathbf{k}_0,\mathbf{k}_1,\ldots}) = \mathcal{V}^*_{\mathbf{k}_1,\mathbf{k}_2,\ldots}. \tag{8}$$

On the other hand, $\mathbf{p} \in \mathcal{H}^*_{\mathbf{k}_{-1},\mathbf{k}_{-2},\ldots} \cap \mathcal{V}^*_{\mathbf{k}_0} \subset \mathcal{H}^*_{\mathbf{k}_{-1},\mathbf{k}_{-2},\ldots} \cap \mathcal{V}_{\mathbf{k}_0}$. From Theorem II.2, it implies $\mathbf{F}(\mathbf{p}) \in \mathcal{H}^*_{\mathbf{k}_0,\mathbf{k}_{-1},\mathbf{k}_{-2},\ldots}$. Together with (8), this shows

$$\begin{aligned} \phi(\mathbf{F}(\mathbf{p})) &= \phi(\mathcal{H}^*_{\mathbf{k}_0,\mathbf{k}_{-1},\mathbf{k}_{-2},\ldots} \cap \mathcal{V}^*_{\mathbf{k}_1,\mathbf{k}_2,\ldots}) \\ &= (\ldots,\mathbf{k}_{-1},\mathbf{k}_0|\mathbf{k}_1,\ldots) = \sigma(\phi(\mathbf{p})). \end{aligned}$$

■

Before proving the hyperbolicity of Λ, we shall adopt the following theorem in [14, p. 266].

Theorem II.6. *A compact* $\mathbf{F}$*-invariant set* Λ *is hyperbolic if there exists* $\kappa > 1$ *such that for every* $\mathbf{p} \in \Lambda$ *there is a decomposition* $T_{\mathbf{p}}\mathcal{M} = S_{\mathbf{p}} \oplus T_{\mathbf{p}}$ *(in general, not* $D\mathbf{F}$ *invariant), a family of the horizontal cones* $H_{\mathbf{p}} \supset S_{\mathbf{p}}$*, and a family of vertical cones* $V_{\mathbf{p}} \supset T_{\mathbf{p}}$ *associated with the decomposition such that*

$$D\mathbf{F}(\mathbf{p})H_{\mathbf{p}} \subset \mathit{Int}\ H_{\mathbf{F}(\mathbf{p})}, \quad D\mathbf{F}^{-1}(\mathbf{p})V_{\mathbf{p}} \subset \mathit{Int}\ V_{\mathbf{F}(\mathbf{p})}, \tag{9}$$

and

$$\|D\mathbf{F}(\mathbf{p})\boldsymbol{\zeta}\| \ge \kappa\|\boldsymbol{\zeta}\| \textit{ for } \boldsymbol{\zeta} \in H_{\mathbf{p}}, \tag{10}$$
$$\|D\mathbf{F}^{-1}(\mathbf{p})\boldsymbol{\zeta}\| \ge \kappa\|\boldsymbol{\zeta}\| \textit{ for } \boldsymbol{\zeta} \in V_{\mathbf{F}(\mathbf{p})}. \tag{11}$$

Proof of Theorem I.1: The hyperbolicity of Λ*:* We shall prove the hyperbolicity by verifying the conditions in Theorem II.6. First, let

$$S_{\mathbf{p}} = \{\begin{bmatrix} \mathbf{0} \\ \boldsymbol{\eta} \end{bmatrix} \in l_\infty \times l_\infty |\ \boldsymbol{\eta} \in l_\infty\},$$
$$T_{\mathbf{p}} = \{\begin{bmatrix} \boldsymbol{\xi} \\ \mathbf{0} \end{bmatrix} \in l_\infty \times l_\infty |\ \boldsymbol{\xi} \in l_\infty\},$$

and

$$H_{\mathbf{p}} = \{\begin{bmatrix} \boldsymbol{\xi} \\ \boldsymbol{\eta} \end{bmatrix} \in l_\infty \times l_\infty |\ \|\boldsymbol{\eta}\| \le \mu\|\boldsymbol{\xi}\|\},$$
$$V_{\mathbf{p}} = \{\begin{bmatrix} \boldsymbol{\xi} \\ \boldsymbol{\eta} \end{bmatrix} \in l_\infty \times l_\infty |\ \|\boldsymbol{\xi}\| \le \mu\|\boldsymbol{\eta}\|\}.$$

Here we see that $S_{\mathbf{p}} \subset H_{\mathbf{p}}$ and $T_{\mathbf{p}} \subset V_{\mathbf{p}}$. Now, let $\mathbf{p} = (\mathbf{u},\mathbf{v}) \in \Lambda$ and $\boldsymbol{\zeta} = \begin{bmatrix} \boldsymbol{\xi} \\ \boldsymbol{\eta} \end{bmatrix} \in S_{\mathbf{p}}$ be given. Hence, $\mathbf{p} \in \mathcal{V}_{\mathbf{k}}$ for some $\mathbf{k} \in \mathbb{E}$. Moreover, there exists a μ-horizontal surface $\mathcal{S} = \{\mathbf{v} = \mathbf{r}(\mathbf{u})\}$ containing $\mathbf{p}$ such that $\boldsymbol{\zeta}$ is a tangent vector to $\mathcal{S}$ at $\mathbf{p}$, i.e. $\boldsymbol{\eta} = D\mathbf{r}(\mathbf{u})\boldsymbol{\xi}$. Since $\mathbf{F}(\mathbf{p}) \in \Lambda \subset \mathcal{B}\times\mathcal{B}$, it follows from Theorem II.2 that the connected component of $\mathbf{F}(\mathcal{S}) \cap (\mathcal{B}\times\mathcal{B})$ containing $\mathbf{F}(\mathbf{p})$, denoted by $\bar{\mathcal{S}}$, is also a μ-horizontal surface. Suppose $\bar{\mathcal{S}}$ is the graph of $\bar{\mathbf{v}} = \bar{\mathbf{r}}(\bar{\mathbf{u}})$. Consequently, $\bar{\boldsymbol{\zeta}} = \begin{bmatrix} \bar{\boldsymbol{\xi}} \\ \bar{\boldsymbol{\eta}} \end{bmatrix} := D\mathbf{F}(\mathbf{p})\boldsymbol{\zeta}$ is a tangent vector to $\bar{\mathcal{S}}$ at $\mathbf{F}(\mathbf{p})$. Hence $\bar{\boldsymbol{\eta}} = D\bar{\mathbf{r}}(\bar{\mathbf{u}})\bar{\boldsymbol{\xi}}$. From the result of Step 2 in the proof of Theorem II.2, we conclude that

$$\|\bar{\boldsymbol{\eta}}\| = \|D\bar{\mathbf{r}}(\bar{\mathbf{u}})\bar{\boldsymbol{\xi}}\| < \mu\|\bar{\boldsymbol{\xi}}\|.$$

This proves the first invariance condition in (9). The second can be similarly obtained. Letting $\kappa = 1/\mu$, the Contraction and Expansion condition (10) follows from Lemma II.5 directly. This completes the proof. ■

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III. Application to the cubic nonlinearities

In this section, we shall give the proof of Theorem I.2.

Proof of Theorem I.2: Let $\omega = 4-\lambda$ and define $f(u) = \omega u + \nu u^3$. To prove the first assertion of Theorem I.2, it suffices to apply Theorem I.1 by verifying assumptions $(\mathcal{A}1)$ and $(\mathcal{A}2)$ for f. Now let $a = 0$ and $b = \sqrt{-\omega/\nu}$. Here a and b are zeros of f in $\mathbb{R}^+$. Denote $u_1 = \sqrt{(\tilde{\Delta}-\omega)/3\nu}$, $u_2 = \sqrt{-(\tilde{\Delta}+\omega)/3\nu}$ and $\epsilon > 0$ sufficiently small. Hence, the constants δ_1 and δ_2 are given by

$$\delta_1 = 0 \text{ and } \delta_2 = 3b.$$

Let $I_0 = [a, u_1-\epsilon]$ and $I_1 = [u_2+\epsilon, b]$. Since $\alpha = \beta = \gamma = 1$, it is easy to verify that $f'(u_1) = \tilde{\Delta} = (\sqrt{5}+3)|\gamma'|/2 + (|\alpha|+|\beta|)$ and $f'(u_2) = -\tilde{\Delta}$. This implies $f'(u) > \tilde{\Delta}$ for $u \in I_0$ and $f'(u) < -\tilde{\Delta}$ for $u \in I_1$ because $u_1 < c < u_2$ and f' is quadratical, where c denotes the positive turning point of f. Hence, $(\mathcal{A}2)$ is satisfied. To prove the validity of $(\mathcal{A}1)$, we first show that

$$f(I_0) \supseteq [a+\delta_1, b+\delta_2]. \tag{12}$$

It is easily seen that $f(a) \leq a+\delta_1$. Since f is monotonic on I_0, if we can show that

$$f(u_1) > b+\delta_2, \tag{13}$$

then (12) holds true. A calculation leads to that (13) is equivalent to the inequality $2\omega + \tilde{\Delta} > 12\sqrt{3\omega/(\omega-\tilde{\Delta})}$. This is true due to the assumption of this theorem. The conclusion $f(I_1) \supseteq [a+\delta_1, b+\delta_2]$ can also be shown by a similar argument. Hence, by Theorem I.1 the first assertion is proven.

Now, we show the second assertion of Theorem I.2. Let $(k_{m,n}) \in \{0,1\}^{\mathbb{Z}^2}$ satisfying $k_{m,n} = 0$ for $|m|$, $|n| > N$. From the first assertion, there exists a solution $(u_{m,n})$ to DNLS equation (2) such that $u_{m,n} \in I_{k_{m,n}}$ for all $m, n \in \mathbb{Z}$. Let $\mathbf{F}$ be the map given in (4) with $\alpha = \beta = \gamma = 1$. Let $\mathbf{u}^{(m)}$, $\mathbf{u'}^{(m)} \in l_\infty$ and $\mathbf{k}^{(m)}$, $\mathbf{k'}^{(m)} \in \{0,1\}^{\mathbb{Z}}$ be given by

$$\begin{aligned}
\mathbf{u}^{(m)} &= \text{the } m\text{-th row of } (u_{m,n}),\\
\mathbf{u'}^{(n)} &= \text{the } n\text{-th column of } (u_{m,n}),\\
\mathbf{k}^{(m)} &= \text{the } m\text{-th row of } (k_{m,n}),\\
\mathbf{u'}^{(n)} &= \text{the } u\text{-th column of } (u_{m,n}).
\end{aligned}$$

Using (2), we see that $(\mathbf{u}^{(m+1)}, \mathbf{u}^{(m)}) = \mathbf{F}(\mathbf{u}^{(m)}, \mathbf{u}^{(m-1)})$ for all $m \in \mathbb{Z}$. Since $\mathbf{k}^{(m)} \to \mathbf{0}$ as $|m| \to \infty$, the topological conjugacy between $\mathbf{F}$ and the full shift σ leads to $(\mathbf{u}^{(m)}, \mathbf{u}^{(m-1)}) \to (\mathbf{0}, \mathbf{0})$ as $m \to \infty$. However, $\mathbf{F}$ is hyperbolic. We conclude that

$$\|\mathbf{u}^{(m)}\| = O(\mu^{|m|}) \text{ for } |m| \text{ sufficiently large.} \tag{14}$$

A similar argument shows that

$$\|\mathbf{u'}^{(n)}\| = O(\mu^{|n|}) \text{ for } |n| \text{ sufficiently large.} \tag{15}$$

Combination of (14) and (15) leads to the assertion of this theorem. ■

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