Common Solution of Nonlinear Functional Equations via Iterations

Muhammad Arshad, Akbar Azam and Pasquale Vetro

***Abstract*— We obtain common fixed points and points of coincidence of a pair of mappings satisfying a generalized contractive type condition in cone metric spaces. Our results generalize some well-known recent results in the literature.**

Index Terms-- We obtain common fixed points and points of coincidence of a pair of mappings satisfying a generalized contractive type condition in cone metric spaces. Our results generalize some well-known recent results in the literature.

I. INTRODUCTION AND PRELIMINARIES

A large variety of the problems of analysis and applied mathematics reduce to finding solutions of non-linear functional equations which can be formulated in terms of finding the fixed points of a nonlinear mapping. In fact, fixed point theorems are very important tools for proving the existence and uniqueness of the solutions to various mathematical models (differential, integral and partial differential equations and variational inequalities etc.) representing phenomena arising in different fields, such as steady state temperature distribution, chemical equations, neutron transport theory, economic theories, financial analysis, epidemics, biomedical research and flow of fluids. They are also used to study the problems of optimal control related to these systems [11]. Fixed point theory concerned with ordered Banach spaces helps us in finding exact or approximate solutions of boundary value problems [2]. In 1963, S. Ghaler, generalized the idea of metric space and introduced 2-metric space which was followed by a number of papers dealing with this generalized space. A lot of materials are available in other generalized metric spaces, such as, semi metric spaces, quasi semi metric spaces and D-metric spaces. Huang and Zhang [6] introduced the concept of cone metric space and established some fixed point theorems for contractive type mappings in a cone metric space. Subsequently, some other authors [1, 3, 4, 5, 7, 8, 10, 13] studied the existence of fixed points, points of coincidence and common fixed points of mappings

Manuscript received March 15, 2011, accepted March 31, 2011. This work was supported by Higher Education Commission (HEC), Pakistan.
M. Arshad is with the Department of Maths and Stats, International Islamic University, H-10, Islamabad, 44000, Pakistan (Phone: +92-3335103984, Fax: +92-51-226877,
e-mail: marshad_zia@yahoo.com).
A. Azam is with the Department of Mathematics, COMSATS Institute of Information Technology, Chak Shahzad, Islamabad, 44000, Pakistan (e-mail: akbarazam@yahoo.com).
P. Vetro is with the Università degli Studi di Palermo,Dipartimento di Matematica ed Applicazioni,Via Archirafi 34, 90123 Palermo, Italy, (e-mail: vetro@math.unipa.it).

satisfying a contractive type condition in cone metric spaces. In this paper, we obtain points of coincidence and common fixed points for a pair of mappings satisfying a more general contractive type condition. Our results improve and generalize some significant recent results.

A subset P of a real Banach space E is called a *cone* if it has the following properties:

(i) P is non-empty closed and $P \neq \{\mathbf{0}\}$;

(ii) $0 \le a, b \in \mathbb{R}$ and

$x, y \in P \Rightarrow ax + by \in P$;

(iii) $P \cap (-P) = \{\mathbf{0}\}$.

For a given cone $P \subseteq E$, we can define a partial ordering $\le$ on E with respect to P by $x \le y$ if and only if $y - x \in P$. We shall write $x < y$ if $x \le y$ and $x \neq y$, while $x << y$ will stands for $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P. The cone P is called *normal* if there is a number $\kappa \ge 1$ such that for all $x, y, \in E$,

$$\mathbf{0} \le x \le y \Rightarrow \|x\| \le \kappa \|y\|. \tag{1}$$

The least number $\kappa \ge 1$ satisfying (1) is called the *normal constant* of P.

In the following we always suppose that E is a real Banach space and P is a cone in E with $\operatorname{int} P \neq \emptyset$ and $\le$ is a partial ordering with respect to P.

Definition 1 *Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:*

(i) $\mathbf{0} \le d(x, y)$ for all $x, y \in X$ and $d(x, y) = \mathbf{0}$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \le d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a *cone metric* on X and (X, d) is called a *cone metric space*.

Let x_n be a sequence in X and $x \in X$. If for

ISBN: 978-988-18210-6-5
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

each $\mathbf{0} << c$ there is $n_0 \in \mathrm{N}$ such that for all $n \geq n_0$, $d(x_n, x) << c$, then $\{x_n\}$ is said to be *convergent or* $\{x_n\}$ converges to x and x is called the *limit* of $\{x_n\}$. We denote this by $\lim_n x_n = x$, or $x_n \to x$, as $n \to \infty$. If for each $\mathbf{0} << c$ there is $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$, $d(x_n, x_m) << c$, then $\{x_n\}$ is called a *Cauchy sequence* in X. If every Cauchy sequence is convergent in X, then X is called a *complete cone metric space*. Let us recall [5] that if P is a normal cone, then $x_n \in X$, converges to $x \in X$ if and only if $d(x_n, x) \to \mathbf{0}$ as $n \to \infty$. Furthermore, $x_n \in X$ is a Cauchy sequence if and only if $d(x_n, x_m) \to \mathbf{0}$ as $n, m \to \infty$.

Lemma 2 *Let (X, d) be a cone metric space, P be a cone. Let $\{x_n\}$ be a sequence in X and $\{a_n\}$ be a sequence in P converging to* ***0****. If $d(x_n, x_m) \leq a_n$ for every $n \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence.*

Proof. Fix $\mathbf{0} << c$ and choose $I(\mathbf{0}, \delta) = \{x \in E : \|x\| < \delta\}$ such that $c + I(\mathbf{0}, \delta) \subset IntP$. Since $a_n \to \mathbf{0}$, there exists $n_0 \in \mathbb{N}$ be such that $a_n \in I(\mathbf{0}, \delta)$ for every $n \geq n_0$. From $c - a_n \in IntP$, we deduce $d(x_n, x_m) \leq a_n << c$ for every $m, n \geq n_0$ and hence $\{x_n\}$ is a Cauchy sequence.

Remark 3 *Let A, B, C, D, E be non negative real numbers with $A + B + C + D + E < 1$, $B = C$ or $D = E$. If $F = (A + B + D)(1 - C - D)^{-1}$ and $G = (A + C + E)(1 - B - E)^{-1}$, then $FG < 1$. In fact, if $B = C$ then*

$$FG = \frac{A + B + D}{1 - C - D} \cdot \frac{A + C + E}{1 - B - E} = \frac{A + C + D}{1 - B - E} \cdot \frac{A + B + E}{1 - C - D} < 1,$$

and if $D = E$,

$$FG = \frac{A + B + D}{1 - C - D} \cdot \frac{A + C + E}{1 - B - E} = \frac{A + B + E}{1 - C - D} \cdot \frac{A + C + D}{1 - B - E} < 1.$$

A PAIR (f, T) OF SELF-MAPPINGS ON X ARE SAID TO BE WEAKLY COMPATIBLE IF THEY COMMUTE AT THEIR COINCIDENCE POINT (I.E., $fTx = Tfx$ WHENEVER $fx = Tx$). A POINT $y \in X$ IS CALLED POINT OF COINCIDENCE OF T AND f IF THERE EXISTS A POINT $x \in X$ SUCH THAT $y = fx = Tx$.

II. MAIN RESULTS

The following theorem improves/generalizes the results [1, Theorems 2.1, 2.3, 2.4], [4, Theorems 1, 2,3], [6, Theorems 1,3, 4], [7, Theorem 2.8], [10, Theorems 2.3, 2.6, 2.7, 2.8], and [12, Theorems 1, Corollary 2].

Theorem 4 *Let (X, d) be a complete cone metric space, P be a cone and m, n be positive integers. Assume that the mappings $T, f : X \to X$ satisfy:*

$$d(T^m x, T^n y) \leq \alpha d(fx, fy) + \beta[d(fx, T^m x) + d(fy, T^n y)] + \gamma[d(fx, T^n y) + d(fy, T^m x)]$$

for all $x, y \in X$ where α, β, γ are non negative real numbers with $\alpha + 2\beta + 2\gamma < 1$. If $T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of X, then T^m, T^n and f have a unique common point of coincidence. Moreover if (T^m, f) and (T^n, f) are weakly compatible, then T^m, T^n and f have a unique common fixed point.

Huang and Zhang [6] proved the above result by restricting that (a) P is normal (b) $f = I_X$ (c) $m = n = 1$

(d) one of the following is satisfied：

i. $\alpha < 1, \beta = \gamma = 0$ ([6, Theorems 1]),

ii. $\beta < \frac{1}{2}, \alpha = \gamma = 0$ ([6, Theorems 3]),

iii. $\gamma < \frac{1}{2}, \alpha = \beta = 0$ ([6, Theorems 3]).

Abbas and Jungck [1] extended the results of Huang and Zhang [6] by removing restriction (b) and obtain common fixed points and points of coincidence of mappings f, T. Meanwhile Rezapour and Hamlbarani [10] improved the results of [6] by omitting the assumption (a). Vetro [12] removed restriction (b) and replaced (d) by combining (i) and (ii). Azam, Arshad and Beg [4] and Jungck et al [7]

ISBN: 978-988-18210-6-5
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

extended these results to a generalized contractive condition by omitting the restrictions (a), (b). The following theorem is a further generalization of Theorem 4 which removes restrictions (a), (b), (c), and replaces (d) with a more generalized contractive condition.

Theorem 5 *Let* (X,d) *be a complete cone metric space,* P *be a cone and* m,n *be positive integers. If the mappings* $T,\ f\ :\ X \to X$ *satisfy:*

$$d(T^m x,T^n y) \le A\,d(fx,fy) + B\,d(fx,T^m x) + Cd(fy,T^n y) + D\,d(fx,T^n y) + E\,d(fy,T^m x)$$

for all $x,y \in X$, *where* A,B,C,D,E *are non negative real numbers with* $A+B+C+D+E<1,\ B=C$ *or* $D=E.$ *If* $T(X) \subseteq f(X)$ *and* $f(X)$ *or* $T(X)$ *is a complete subspace of* X, *then* T^m, T^n *and* f *have a unique common point of coincidence. Moreover if* (T^m, f) *and* (T^n, f) *are weakly compatible, then* T^m, T^n *and* f *have a unique common fixed point.*

Proof. Let x_0 be an arbitrary point in X. Choose a point x_1 in X such that $fx_1 = T^m x_0$. This can be done since $T(X) \subseteq f(X)$. Similarly, choose a point x_2 in X such that $fx_2 = T^n x_1$. Continuing this process having chosen x_n in X, we obtain x_{n+1} in X such that

$$fx_{2k+1} = T^m x_{2k}$$
$$fx_{2k+2} = T^n x_{2k+1}, \quad k = 0,1,2,\dots .$$

Then,

$$\begin{aligned} d(fx_{2k+1}, fx_{2k+2}) &= d(T^m x_{2k}, T^n x_{2k+1}) \\ &\le Ad(fx_{2k}, fx_{2k+1}) \\ &\quad + Bd(fx_{2k}, T^m x_{2k}) \\ &\quad + Cd(fx_{2k+1}, T^n x_{2k+1}) \\ &\quad + Dd(fx_{2k}, T^n x_{2k+1}) \\ &\quad + Ed(fx_{2k+1}, T^m x_{2k}) \\ &\le [A+B]\,d(fx_{2k}, fx_{2k+1}) \\ &\quad + Cd(fx_{2k+1}, fx_{2k+2}) \\ &\quad + D\,d(fx_{2k}, fx_{2k+2}) \\ &\le [A+B+D]\,d(fx_{2k}, fx_{2k+1}) \\ &\quad + [C+D]\,d(fx_{2k+1}, fx_{2k+2}). \end{aligned}$$

It implies that

$$\begin{aligned} &[1-C-D]d(fx_{2k+1}, fx_{2k+2}) \\ &\le [A+B+D]\,d(fx_{2k}, fx_{2k+1}). \end{aligned}$$

That is,

$$d(fx_{2k+1}, fx_{2k+2}) \leqslant F\,d(fx_{2k}, fx_{2k+1}),$$

where $F = \dfrac{A+B+D}{1-C-D}$.

Similarly we obtain,

$$\begin{aligned} d(fx_{2k+2}, fx_{2k+3}) &= d(T^m x_{2k+2}, T^n x_{2k+1}) \\ &\le [A+C+E]\,d(fx_{2k+1}, fx_{2k+2}) \\ &\quad + [B+E]\,d(fx_{2k+2}, fx_{2k+3}), \end{aligned}$$

which implies

$$d(fx_{2k+2}, fx_{2k+3}) \le G\,d(fx_{2k+1}, fx_{2k+2})$$

with $G = \dfrac{A+C+E}{1-B-E}$.

Now by induction, we obtain for each $k = 0,1,2,\dots$

$$\begin{aligned} d(fx_{2k+1}, fx_{2k+2}) &\le F\,d(fx_{2k}, fx_{2k+1}) \\ &\le (FG)\,d(fx_{2k-1}, fx_{2k}) \\ &\le F(FG)\,d(fx_{2k-2}, fx_{2k-1}) \\ &\le \cdots \le F(FG)^k\,d(fx_0, fx_1) \end{aligned}$$

and

$$\begin{aligned} d(fx_{2k+2}, fx_{2k+3}) &\le G\,d(fx_{2k+1}, fx_{2k+2}) \\ &\le \cdots \le (FG)^{k+1} d(fx_0, fx_1). \end{aligned}$$

By Remark 3 $p<q$ we have

$$\begin{aligned} d(fx_{2p+1}, fx_{2q+1}) &\le d(fx_{2p+1}, fx_{2p+2}) \\ &\quad + d(fx_{2p+2}, fx_{2p+3}) \\ &\quad + d(fx_{2p+3}, fx_{2p+4}) \\ &\quad + \cdots + d(fx_{2q}, fx_{2q+1}) \\ &\le \left[F\sum_{i=p}^{q-1}(FG)^i + \sum_{i=p+1}^{q}(FG)^i \right] \times \\ &\quad d(fx_0, fx_1) \\ &\le \left[\frac{F(FG)^p}{1-FG} + \frac{(FG)^{p+1}}{1-FG} \right] \times \\ &\quad d(fx_0, fx_1) \\ &\le (1+F)\left[\frac{(FG)^p}{1-FG} \right] d(fx_0, fx_1). \end{aligned}$$

ISBN: 978-988-18210-6-5
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

In analogous way, we deduce

$$d(fx_{2p},\ fx_{2q+1}) \le (1+F)\left[\frac{(FG)^p}{1-FG}\right] d(fx_0, fx_1),$$

$$d(fx_{2p},\ fx_{2q}) \le (1+F)\left[\frac{(FG)^p}{1-FG}\right] d(fx_0, fx_1)$$

and

$$d(fx_{2p+1},\ fx_{2q}) \le (1+F)\left[\frac{(FG)^p}{1-FG}\right] d(fx_0, fx_1).$$

Hence, for $0 < n < m$

$$d(fx_n, fx_m) \le a_n,$$

where $a_n = (1+F)\left[\frac{(FG)^p}{1-FG}\right] d(fx_0, fx_1)$ with p the integer part of $n/2$.

Fix $\mathbf{0} << c$ and choose $I(\mathbf{0},\delta) = \{x \in E\ : \|x\| x < \delta\}$ such that $c + I(\mathbf{0},\delta) \subset \mathbf{Int}P$. Since $a_n \to \mathbf{0}$ as $n \to \infty$, by Lemma 2, we deduce that $\{fx_n\}$ is a Cauchy sequence.

If $f(X)$ is a complete subspace of X, there exist $u, v \in X$ such that $fx_n \to v = fu$ (this holds also if $T(X)$ is complete with $v \in T(X)$). Fix $\mathbf{0} \ll c$ and choose $n_0 \in \mathbb{N}$ be such that

$$d(v, fx_{2n}) << \frac{c}{3k},\quad d(fx_{2n-1}, fx_{2n}) << \frac{c}{3k},\quad d(v, fx_{2n-1}) << \frac{c}{3k}$$

for all $n \ge n_0$, where

$$k = \max\left\{\frac{1+D}{1-B-E}, \frac{A+E}{1-B-E}, \frac{C}{1-B-E}\right\}.$$

Now,

$$\begin{aligned} d(fu, T^m u) &\le d(fu, fx_{2n}) + d(fx_{2n}, T^m u) \\ &\le (1+D)\, d(fu, fx_{2n}) \\ &\quad + (A+E)\, d(fu, fx_{2n-1}) \\ &\quad + C d(fx_{2n-1}, fx_{2n}) \\ &\quad + (B+E)\, d(fu, T^m u). \end{aligned}$$

So,

$$\begin{aligned} d(fu, T^m u) &\le k\, d(fu, fx_{2n}) + k\, d(fu, fx_{2n-1}) \\ &\quad + k\, d(fx_{2n-1}, fx_{2n}) \\ &<< \frac{c}{3} + \frac{c}{3} + \frac{c}{3} = c \end{aligned}$$

Hence

$$d(fu, T^m u) << \frac{c}{p}$$

for every $p \in \mathbb{N}$. From

$$\frac{c}{p} - d(fu, T^m u) \in \mathbf{Int}P,$$

being P closed, as $p \to \infty$, we deduce $-d(fu, T^m u) \in P$ and so $d(fu, T^m u) = \mathbf{0}$. This implies that $fu = T^m u.$

Similarly, by using the inequality,

$$d(fu, T^n u) \le d(fu, x_{2n+1}) + d(fx_{2n+1}, T^n u),$$

we can show that $fu = T^n u,$ which in turn implies that v is a common point of coincidence of T^m, T^n and f, that is

$$v = fu = T^m u = T^n u.$$

Now we show that f, T^m and T^n have a unique common point of coincidence. For this, assume that there exists another point v^* in X such that $v^* = fu^* = T^m u^* = T^n u^*$ for some u^* in X. From

$$\begin{aligned} d(v, v^*) &= d(T^m u, T^n u^*) \\ &\le Ad(fu, fu^*) + Bd(fu, T^m u) \\ &\quad + Cd(fu^*, T^n u^*) \\ &\quad + D\, d(fu, T^n u^*) + Ed(fu^*, T^m u) \\ &\le (A+D+E) d(v, v^*), \end{aligned}$$

we obtain that $v^* = v.$ Moreover, (T^m, f) and (T^n, f) are weakly compatible, then

$$T^m v = T^m fu = fT^m u = fv \text{ and}$$

$$T^n v = T^n fu = fT^n u = fv,$$

which implies $T^m v = T^n v = fv = w$ (say). Then w is a common point of coincidence of T^m, T^n and f therefore, $v = w,$ by uniqueness. Thus v is a unique common fixed point of T^m, T^n and f.

Example 6 *Let* $X = \{1,2,3\}$, $E = \mathbb{R}^2$ *and* $P = \{(x, y) \in E\ :\ x, y \ge 0\}$.

ISBN: 978-988-18210-6-5
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

Define $d : X \times X \to \mathrm{R}^2$ *as follows:*

$$d(x,y)=\begin{cases}\mathbf{0} & \text{if } x=y\\ \left(\frac{5}{7},\frac{10}{3}\right) & \text{if } x\neq y \text{ and } x,y\in X-\{2\}\\ \left(1,\frac{14}{3}\right) & \text{if } x\neq y \text{ and } x,y\in X-\{3\}\\ \left(\frac{4}{7},\frac{8}{3}\right) & \text{if } x\neq y \text{ and } x,y\in X-\{1\}.\end{cases}$$

Define the mappings $T, f : X \to X$ *as follows:*

$$f(x)=x,$$

$$T(x)=\begin{cases}1 & \text{if } x\neq 2\\ 3 & \text{if } x=2.\end{cases}$$

Note that $T^2(x)=1$ *for each* $x\in X$,

$$d(T^2(3),T(2))=\left(\frac{5}{7},\frac{10}{3}\right).$$

Then, if $\alpha+2\beta+2\gamma<1$ *we have*

$$\left(\frac{4\alpha+9\beta+7\gamma}{7},\frac{8\alpha+18\beta+14\gamma}{3}\right)$$
$$<\left(\frac{5\alpha+10\beta+10\gamma}{7},\frac{10\alpha+20\beta+20\gamma}{3}\right)$$
$$\leq\left(\frac{5(\alpha+2\beta+2\gamma)}{7},\frac{10}{3}(\alpha+2\beta+2\gamma)\right)$$
$$<\left(\frac{5}{7},\frac{10}{3}\right)=d(T^2(3),T(2)).$$

This implies

$$\alpha d(f(3),f(2))+\beta\begin{bmatrix}d(f(3),T^2(3))\\+d(f(2),T(2))\end{bmatrix}$$
$$+\gamma\left[d(f(3),T(2))+d(f(2),T^2(3))\right]$$
$$=\alpha d(3,2)+\beta\left[d(3,1)+d(2,3)\right]$$
$$+\gamma\left[d(3,3)+d(2,1)\right]$$
$$<\left(\frac{5}{7},\frac{10}{3}\right)=d(T^2(3),T(2)$$

for all $\alpha,\beta,\gamma\in[0,1)$ *with* $\alpha+2\beta+2\gamma<1$.

Therefore, Theorem 4 and its corollaries (, Theorems 2.1, 2.3, 2.4], [4, Theorems 1, 2, 3], [6, Theorem 2.8], [10, Theorems 2.3, 2.6, 2.7, 2.8] and [12, Theorem 1, Corollary 2]) are not applicable. From

$$d(T^2x,Ty)=\begin{cases}\mathbf{0} & \text{if } y\neq 2\\ \left(\frac{5}{7},\frac{10}{3}\right) & \text{if } y=2\end{cases}$$

and

$$A\,d(fx,fy)+B\,d(fx,T^2x)+Cd(fy,Ty)$$
$$+D\,d(fx,Ty)+E\,d(fy,T^2x)=\left(\frac{5}{7},\frac{10}{3}\right)$$

for $y=2$ and $A=B=C=D=0,\ E=\frac{5}{7}$, it follows that all conditions of Theorem 5are satisfied for $A=B=C=D=0,\ E=\frac{5}{7}$ and so T and f have a unique common point of coincidence and a unique common fixed point.

Corollary 7 *Let* (X,d) *be a complete cone metric space,* P *be a cone and* m,n *be positive integers. If a mapping* $T : X\to X$ *satisfies:*

$$d(T^m x,T^n y)\leq A\,d(x,y)+B\,d(x,T^m x)$$
$$+Cd(y,T^n y)+D\,d(x,T^n y)+E\,d(y,T^m x)$$

for all $x,y\in X$ *, where* A,B,C,D,E *are non negative real numbers with* $A+B+C+D+E<1,\ B=C$ *or* $D=E.$ *Then* T *has a unique fixed point.*

Proof. By Theorem 5, we get $x\in X$ such that $T^m x=T^n x=x.$ The result then follows from the fact that

$$d(Tx,x)=d(TT^m x,T^n x)=d(T^m Tx,T^n x)$$
$$\leq Ad(Tx,x)+Bd(Tx,T^m Tx)+Cd(x,T^n x)$$
$$+Dd(Tx,T^n x)+Ed(x,T^m Tx)$$
$$\leq A\,d(Tx,x)+Bd(Tx,Tx)+Cd(x,x)$$
$$+Dd(Tx,x)+Ed(x,Tx)$$
$$=(A+D+E)\,d(Tx,x),$$

which implies $Tx=x.$

Example (Applications) 8 *Let*

$X=C([1,3],\mathbb{R}),\ E=\mathbb{R}^2,\ a>0$ *and*

$$d(x,y)=\left(\sup_{t\in[1,3]}|x(t)-y(t)|,a\sup_{t\in[1,3]}|x(t)-y(t)|\right)$$

for every $x,y\in X$, *and*

$P=\{(u,v)\in\mathbb{R}^2 : u,v\geq 0\}.$ *It is easily seen that* (X,d) *is a complete cone metric space. Define* $T : X\to X$ *by*

$$T(x(t))=4+\int_1^t\left(x(u)+u^2\right)e^{u-1}du.$$

For $x,y\in X$

ISBN: 978-988-18210-6-5
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$$d(Tx,Ty)=\begin{pmatrix}\sup_{t\in[1,3]}|Tx(t)-Ty(t)|,\\ a\sup_{t\in[1,3]}|Tx(t)-Ty(t)|\end{pmatrix}$$
$$\leq\begin{pmatrix}\int_1^3\sup_{t\in[1,3]}|(x(u)-y(u))|e^2du,\\ a\int_1^3\sup_{t\in[1,3]}|(x(u)-y(u))|e^2du\end{pmatrix}$$
$$=2e^2d(x,y).$$

Similarly,

$$d(T^nx,T^ny)\leq e^{2n}\frac{2^n}{n!}d(x,y).$$

Note that

$$e^{2n}\frac{2^n}{n!}=\begin{cases}109 & \text{if } n=2\\ 1987 & \text{if } n=4\\ 1.31 & \text{if } n=37\\ 0.53 & \text{if } n=38.\end{cases}$$

Thus for $A=0.53, B=C=D=E=0, m=n=38,$ *all conditions of Corollary 7 are satisfied and so* T *has a unique fixed point, which is the unique solution of the integral equation:*

$$x(t)=4+\int_1^t\left(x(u)+u^2\right)e^{u-1}du.$$

or the differential equation:

$$x'(t)=\left(x+t^2\right)e^{t-1},\quad t\in[1,3],\quad x(1)=4.$$

Hence, the use of Corollary 7 is a delightful way of showing the existence and uniqueness of solutions for the following class of integral equations:

$$b+\int_a^t K(x(u),u)du=x(t)\in C([a,b],\mathbb{R}^n).$$

REFERENCES

[1] M. Abbas and G. Jungck, Common fixed point results for non commuting mappings without continuity in one metric spaces, J. Math. Anal. Appl. 341 (2008) 416--420.

[2] H. Amann, Fixed point Equations and non-linear eigen-value problems in ordered Banach spaces, SIAM Review, 18 (1976) 620--709.

[3] M. Arshad, A. Azam and P. Vetro, Some common fixed point results in cone metric spaces, Fixed Point Theory Appl. 2009 (2009), 11 pp., Article ID 493965.

[4] A. Azam, M. Arshad and I. Beg, Common fixed points of two maps in cone metric spaces, Rend. Circ. Mat. Palermo 57 (2008) 433--441.

[5] C. Di Bari and P. Vetro, Weakly -pairs and common fixed points in cone metric spaces, Rend. Circ. Mat. Palermo 58 (2009) 125--132.

[6] L.-G. Huang and X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 332 (2007) 1468--1476.

[7] G. Jungck, S. Radenovic, S. Radojevic and V. Rakocevic, Common fixed point theorems for weakly compatible pairs on cone metric spaces, Fixed Point Theory Appl. 2009 (2009), 13 pp., Article ID 643840.

[8] D. Klim,. Wardowski, Dynamic processes and fixed points of set-valued nonlinear contractions in cone metric spaces, Nonlinear Analysis (2009) doi:10.1016/j.na.2009.04.001.

[9] V.Popa, A general fixed point theorem for weakly compatible mappings in compact metric spaces,Turk J. Math.25 (2001) 465-474.

[10] S. Rezapour and R. Hamlbarani, Some notes on paper "Cone metric spaces and fixed point theorems of contractive mappings.", J. Math. Anal. Appl., 345 (2008) 719--724.

[11] N. Shahzad, Fixed points for generalized contractions and applications to control theory, Nonlinear Analysis 68 (8) (2008) 867--876.

[12] P. Vetro, Common fixed points in cone metric spaces, Rend. Circ. Mat. Palermo, 56 (2007) 464--468.

[13] D. Wardowski, Endpoints and fixed points of set-valued contractions in cone metric spaces, Nonlinear Analysis 71 (2009) 512--516.

ISBN: 978-988-18210-6-5
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)