

Statistical Inferences for Future Outcomes with Applications to Maintenance and Reliability

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Abstract—This paper describes technique for using censored life data from extreme value distributions to construct prediction limits or intervals for future outcomes. In particular, new-sample prediction based on a previous sample (i.e., when for predicting the future failure time of an unit in a new sample there are available the failure data only from a previous sample), within-sample prediction based on the early-failure data from a current experiment (i.e., when for predicting the future failure time of an unit in a sample there are available the early-failure data only from that sample), and new-within-sample prediction based on both the early-failure data from that sample and the data from a previous sample (i.e., when for predicting the future failure time of an unit in a new sample there are available both the early-failure data from that sample and the data from a previous sample) are considered. In order to construct prediction limits or intervals for future outcomes, the invariant embedding technique representing the exact pivotal-based method is used. Numerical examples are given to illustrate applications of the results obtained in this paper to k -out-of- n systems and planning in-service inspections of fatigued structures.

Index Terms— Extreme value distribution, type II censored data, pivotal quantities, k -out-of- n system, predictive inferences

I. INTRODUCTION

THE problem of modeling extreme or rare events arises in many areas where such events can have very negative consequences. Some examples of rare events include extreme floods and snowfalls, high wind speeds, extreme temperatures, large fluctuations in exchange rates, and market crashes. To develop appropriate probabilistic models and assess the risks caused by these events, business analysts and engineers frequently use the extreme value distributions (EVD).

Extreme value distributions are usually considered to comprise the following three families:

Type 1, (Gumbel distribution):

$$\Pr\{X > x\} = \exp\left[-\exp\left(\frac{x-\mu}{\sigma}\right)\right], \quad -\infty < x < \infty, \quad (1)$$

where μ is the location parameter, and σ is the scale

parameter ($\sigma > 0$). The shape of the Gumbel model does not depend on the distribution parameters.

Type 2, (Frechet distribution):

$$\Pr\{X > x\} = 1 - \exp\left[-\left(\frac{x}{\beta}\right)^{-\delta}\right], \quad x \geq 0, \quad (2)$$

where δ is the shape parameter ($\delta > 0$), and β is the scale parameter ($\beta > 0$).

Type 3, (Weibull distribution):

$$\Pr\{X > x\} = \exp\left[-\left(\frac{x}{\beta}\right)^{\delta}\right], \quad x \geq 0, \quad (3)$$

where both distribution parameters (δ - shape, β - scale) are positive. The two-parameter Weibull distribution (3) can be generalized by adding the location (shift) parameter μ :

$$\Pr\{X > x\} = \exp\left[-\left(\frac{x-\mu}{\beta}\right)^{\delta}\right], \quad x \geq \mu. \quad (4)$$

In this model, the location parameter μ can take on any real value, and the distribution is defined for $x \geq \mu$.

It will be noted that the logarithm of a Weibull variable from the distribution (3) follows the Gumbel distribution (1), where $\mu = \ln\beta$ and $\sigma = \delta^{-1}$.

The above models are widely used in risk management, finance, insurance, economics, hydrology, material sciences, telecommunications, and many other industries dealing with extreme events.

Practical problems often require the computation of predictions and prediction limits for future values of random quantities. Consider the following examples: 1) A consumer purchasing a refrigerator would like to have a lower limit for the failure time of the unit to be purchased (with less interest in distribution of the population of units purchased by other consumers); 2) Financial managers in manufacturing companies need upper prediction limits on future warranty costs; 3) When planning life tests, engineers may need to predict the number of failures that will occur by the end of the test or to predict the amount of time that it will take for a specified number of units to fail.

Some applications require a two-sided prediction interval that will, with a specified high degree of confidence, contain the future random variable of interest, say Z . In many applications, however, interest is focused on either an upper prediction limit or a lower prediction limit (e.g., the maximum warranty cost is more important than the minimum, and the time of the early failures in a product population is more important than the last ones).

Conceptually, it is useful to distinguish between “new-sample” prediction, “within-sample” prediction, and “new-within-sample” prediction.

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For new-sample prediction, data from a past sample are used to make predictions on a future unit or sample of units from the same process or population. For example, based on previous (possibly censored) life test data, one could be interested in predicting the time to failure of a new item, time until l failures in a future sample of m units, or number of failures by time t_* in a future sample of m units.

For within-sample prediction, the problem is to predict future events in a sample or process based on early data from that sample or process. For example, if m units are followed until t_k and there are k observed failures, t_1, \dots, t_k , one could be interested in predicting the time of the next failure t_{k+1} ; time until l additional failures, t_{k+l} ; number of additional failures in a future interval.

For new-within-sample prediction, the problem is to predict future events in a sample or process based on early data from that sample or process as well as on a past data sample from the same process or population.

Various solutions have been proposed for the prediction problem, that is, the problem of making inferences on a random sample $\{Y_j; j = 1, \dots, m\}$ given independent observations $\{X_i; i=1, \dots, n\}$ drawn from the same distribution. The Y_j 's and the X_i 's are commonly featured as "future outcomes" and "past outcomes" respectively. Inferences usually bear on some reduction Z of the Y_j 's – possibly a minimal sufficient statistic – and consist of either prediction intervals or likelihood or predictive distribution for Z , depending on different authors.

Methods presenting frequentist prediction intervals basically stem from the theory of similar tests. See for instance Fisher [1], Faulkenberry [2], and Cox [3], who gives an approximate and more general asymptotic solution. Predictive distributions are found in the Bayesian framework (see Aitchison and Sculthorpe [4]), and likelihood concepts have been considered by Fisher [1], in a somewhat ungrounded manner, and by Hinkley [5], who establishes links with frequentist and Bayesian views. Lawless [6] applied the conditional method, which was first suggested by Fisher [7] and promoted further by a number of others (Nechval et al. [8]; Murthy et al. [9]), to different problems relating to the Weibull and extreme value distributions. In practice the proposed methods have limited applications and it is the purpose of this paper to obtain predictive inferences concerning Z via the simple invariant embedding technique [10-14]. The obtained results are given below.

II. STATISTICAL INFERENCE FOR THE THREE PREDICTION SITUATIONS

Having defined the three prediction situations, we consider now the determination of prediction limits. The following results hold.

A. New-Sample Prediction

Theorem 1. (Lower (upper) one-sided conditional prediction limit h on the smallest observation Y_1 from a new (future) sample of m observations from the Gumbel distribution on the basis of the previous data sample). Let $X_1 \leq \dots \leq X_r$ be the first r ordered past observations from a previous sample of size n from the Gumbel distribution (1).

Then a lower one-sided conditional $(1-\alpha)$ prediction limit h on the smallest observation Y_1 from a set of m future ordered observations $Y_1 \leq \dots \leq Y_m$ also from the distribution (1) is given by

$$h = \arg[\Pr\{Y_1 > h | \mathbf{z}\} = 1 - \alpha]$$

$$= \arg \left[\frac{\int_0^\infty v^{r-2} e^{-v \sum_{i=1}^r z_i} \left(m e^{v(h-\hat{\mu})/\hat{\sigma}} + \sum_{i=1}^r e^{vz_i} + (n-r)e^{vz_r} \right)^{-r} dv}{\int_0^\infty v^{r-2} e^{-v \sum_{i=1}^r z_i} \left(\sum_{i=1}^r e^{vz_i} + (n-r)e^{vz_r} \right)^{-r} dv} \right], \quad (5)$$

where

$$\mathbf{z} = (z_1, z_2, \dots, z_r), \quad Z_i = \frac{X_i - \hat{\mu}}{\hat{\sigma}}, \quad i = 1, \dots, r, \quad (6)$$

$\hat{\mu}$ and $\hat{\sigma}$ are the maximum likelihood estimators of μ and σ based on the first r ordered past observations $(X_1 \leq \dots \leq X_r)$ from a sample of size n from the Gumbel distribution, which can be found from solution of

$$\hat{\mu} = \hat{\sigma} \ln \left(\left[\sum_{i=1}^r e^{x_i/\hat{\sigma}} + (n-r)e^{x_r/\hat{\sigma}} \right] / r \right), \quad (7)$$

and

$$\hat{\sigma} = \left(\sum_{i=1}^r x_i e^{x_i/\hat{\sigma}} + (n-r)x_r e^{x_r/\hat{\sigma}} \right) \times \left(\sum_{i=1}^r e^{x_i/\hat{\sigma}} + (n-r)e^{x_r/\hat{\sigma}} \right)^{-1} - \frac{1}{r} \sum_{i=1}^r x_i, \quad (8)$$

(Observe that an upper one-sided conditional α prediction limit h on the smallest observation Y_1 from a set of m future ordered observations $Y_1 \leq \dots \leq Y_m$ may be obtained from a lower one-sided conditional $(1-\alpha)$ prediction limit by replacing $1-\alpha$ by α .)

Proof. The joint density of $X_1 \leq \dots \leq X_r$ is given by

$$f(x_1, \dots, x_r | \mu, \sigma) = \frac{n!}{(n-r)!} \prod_{i=1}^r \frac{1}{\sigma} \exp \left(\frac{x_i - \mu}{\sigma} - \exp \left(\frac{x_i - \mu}{\sigma} \right) \right) \times \exp \left(- (n-r) \exp \left(\frac{x_r - \mu}{\sigma} \right) \right). \quad (9)$$

Let $\hat{\mu}$, $\hat{\sigma}$ be the maximum likelihood estimates of μ , σ , respectively, based on $X_1 \leq \dots \leq X_r$ from a complete sample of size n , and let

$$V_1 = \frac{\hat{\mu} - \mu}{\hat{\sigma}}, \quad V = \frac{\hat{\sigma}}{\sigma}, \quad \text{and} \quad Z_i = \frac{X_i - \hat{\mu}}{\hat{\sigma}}, \quad i = 1(1)r. \quad (10)$$

Parameters μ and σ in (9) are location and scale parameters, respectively, and it is well known that if $\hat{\mu}$ and $\hat{\sigma}$ are estimates of μ and σ , possessing certain invariance properties, then V_1 and V are the pivotal quantities whose distributions depend only on n . Most, if not all, proposed

estimates of μ and σ possess the necessary properties; these include the maximum likelihood estimates and various linear estimates. $Z_i, i=1(1)r$, are ancillary statistics, any $r-2$ of which form a functionally independent set. For notational convenience we include all of z_1, \dots, z_r in (6); z_{r-1} and z_r can be expressed as function of z_1, \dots, z_{r-2} only.

Using the invariant embedding technique [9]-[14], we then find in a straightforward manner, that the probability element of the joint density of V_1, V , conditional on fixed $\mathbf{z}=(z_1, z_2, \dots, z_r)$, is

$$f(v_1, v | \mathbf{z})dv_1dv = \vartheta(\mathbf{z})v^{r-2} \exp\left(v \sum_{i=1}^r z_i\right) e^{v_1} \times \exp\left[-e^{v_1} \left[\sum_{i=1}^r \exp(z_i v) + (n-r) \exp(z_r v)\right]\right] dv_1 dv, \quad v_1 \in (-\infty, \infty), \quad v \in (0, \infty), \quad (11)$$

where

$$\vartheta(\mathbf{z}) = \left[\int_0^\infty \Gamma(r) v^{r-2} \exp\left(v \sum_{i=1}^r z_i\right) \times \left[\sum_{i=1}^r \exp(z_i v) + (n-r) \exp(z_r v)\right]^{-r} dv \right]^{-1} \quad (12)$$

is the normalizing constant.

Writing

$$\Pr\{Y_1 > h | \mu, \sigma\} = \prod_{j=1}^m \Pr\{Y_j > h | \mu, \sigma\} = \exp\left\{-m \exp\left(\frac{h-\mu}{\sigma}\right)\right\} = \exp\{-m \exp(u_h v + v_1)\} = \Pr\{Y_1 > h | v_1, v\}, \quad (13)$$

where

$$u_h = \frac{h-\hat{\mu}}{\hat{\sigma}}, \quad (14)$$

we have that

$$\Pr\{Y_1 > h | \mathbf{z}\} = \int_0^\infty \int_{-\infty}^\infty \Pr\{Y_1 > h | v_1, v\} f(v_1, v | \mathbf{z}) dv_1 dv. \quad (15)$$

Now v_1 can be integrated out of (15) in a straightforward way to give

$$\Pr\{Y_1 > h | \mathbf{z}\} = \frac{\int_0^\infty v^{r-2} e^{v \sum_{i=1}^r z_i} \left(m e^{v u_h} + \sum_{i=1}^r e^{v z_i} + (n-r) e^{v z_r} \right)^{-r} dv}{\int_0^\infty v^{r-2} e^{v \sum_{i=1}^r z_i} \left(\sum_{i=1}^r e^{v z_i} + (n-r) e^{v z_r} \right)^{-r} dv}. \quad (16)$$

This completes the proof. \square

Corollary 1.1. It follows from (5) that a lower one-sided conditional $(1-\alpha)$ prediction limit h on the minimum Y_1 of a set of m future ordered observations $Y_1 \leq \dots \leq Y_m$ from the two-parameter Weibull distribution (3) is given by

$$h = \arg[\Pr\{Y_1 \geq h | \mathbf{x}\} = 1 - \alpha]$$

$$= \arg \left[\frac{\int_0^\infty v^{r-2} e^{v \sum_{i=1}^r \ln(x_i / \hat{\beta})} \left(m e^{v \hat{\delta} \ln(h / \hat{\beta})} + \sum_{i=1}^r e^{v \hat{\delta} \ln(x_i / \hat{\beta})} + (n-r) e^{v \hat{\delta} \ln(x_r / \hat{\beta})} \right)^{-r} dv}{\int_0^\infty v^{r-2} e^{v \sum_{i=1}^r \ln(x_i / \hat{\beta})} \left(\sum_{i=1}^r e^{v \hat{\delta} \ln(x_i / \hat{\beta})} + (n-r) e^{v \hat{\delta} \ln(x_r / \hat{\beta})} \right)^{-r} dv} \right]^{-1} = 1 - \alpha \quad (17)$$

where $\hat{\beta}$ and $\hat{\delta}$ are the maximum likelihood estimators of β and δ based on the first r ordered past observations ($X_1 \leq \dots \leq X_r$) from a sample of size n from the two-parameter Weibull distribution (3), which can be found from solution of

$$\hat{\beta} = \left[\frac{\sum_{i=1}^r x_i^{\hat{\delta}} + (n-r) x_r^{\hat{\delta}}}{r} \right]^{1/\hat{\delta}}, \quad (18)$$

and

$$\hat{\delta} = \left[\frac{\sum_{i=1}^r x_i^{\hat{\delta}} \ln x_i + (n-r) x_r^{\hat{\delta}} \ln x_r}{\sum_{i=1}^r x_i^{\hat{\delta}} + (n-r) x_r^{\hat{\delta}}} - \frac{1}{r} \sum_{i=1}^r \ln x_i \right]^{-1}, \quad (19)$$

$$\mathbf{x} = (x_1, \dots, x_r). \quad (20)$$

Theorem 2. (Lower (upper) one-sided prediction limit h on the l th order statistic Y_l from a new (future) sample of m observations from the two-parameter Weibull distribution, where the parameter $\delta=1$, on the basis of the previous data sample). Let $X_1 \leq \dots \leq X_r$ be the first r ordered observations from a previous sample of size n from the two-parameter Weibull distribution (3), where the parameter $\delta=1$. Thus, we deal with the exponential distribution. Then a lower one-sided conditional $(1-\alpha)$ prediction limit h on the l th order statistic Y_l from a set of m future ordered observations $Y_1 \leq \dots \leq Y_m$ also from the above distribution is given by

$$h = \arg[\Pr\{Y_l > h | s\} = 1 - \alpha]$$

$$= \arg \left[\frac{1}{\text{B}(l, m-l+1)} \sum_{j=0}^{l-1} \binom{l-1}{j} (-1)^j \times \frac{1}{(m-l+1+j)[1+(m-l+1+j)h/s]^r} \right] = 1 - \alpha \quad (21)$$

where

$$S = \sum_{i=1}^r X_i + (m-r) X_r. \quad (22)$$

Proof. For the proof we refer to Corollary 2.1 [15]. \square

B. Within-Sample Prediction

Theorem 3 (Lower (upper) one-sided prediction limit h on the l th order statistic Y_l in a sample of m observations from the Gumbel distribution on the basis of the early-failure data $Y_1 \leq \dots \leq Y_k$ from the same sample). Let $Y_1 \leq \dots \leq Y_k$ be the first k ordered early-failure observations from a sample of size m from the Gumbel distribution (1). Then a lower

one-sided conditional $(1-\alpha)$ prediction limit h on the l th order statistic Y_l ($l > k$) from the same sample is given by

$$h = \arg[\Pr\{Y_l > h | \mathbf{u}\} = 1 - \alpha]$$

$$= \arg \left[\frac{\int_0^\infty v^{k-2} e^{-\sum_{i=1}^k u_i} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \frac{(-1)^{l-k-1-j}}{m-k-j} \times \left((m-k-j)e^{v[(h-y_k)/\hat{\sigma}+u_k]} + je^{vu_k} + \sum_{i=1}^k e^{vu_i} \right)^{-k} dv}{\int_0^\infty v^{k-2} e^{-\sum_{i=1}^k u_i} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \times \frac{(-1)^{l-k-1-j}}{m-k-j} \left((m-k)e^{vu_k} + \sum_{i=1}^k e^{vu_i} \right)^{-k} dv} \right] = 1 - \alpha, \quad (23)$$

where

$$\mathbf{u} = (u_1, \dots, u_k), \quad (24)$$

$$U_i = \frac{Y_i - \hat{\mu}}{\hat{\sigma}}, \quad i = 1, \dots, k, \quad (25)$$

$\hat{\mu}$ and $\hat{\sigma}$ are the maximum likelihood estimators of μ and σ based on the first k ordered early-failure observations ($Y_1 \leq \dots \leq Y_k$) from a sample of size m from the Gumbel distribution, which can be found from solution of

$$\hat{\mu} = \hat{\sigma} \ln \left(\frac{\sum_{i=1}^k e^{y_i/\hat{\sigma}} + (m-k)e^{y_k/\hat{\sigma}}}{k} \right), \quad (26)$$

and

$$\hat{\sigma} = \left(\sum_{i=1}^k y_i e^{y_i/\hat{\sigma}} + (m-k)y_k e^{y_k/\hat{\sigma}} \right) \times \left(\sum_{i=1}^k e^{y_i/\hat{\sigma}} + (m-k)e^{y_k/\hat{\sigma}} \right)^{-1} - \frac{1}{k} \sum_{i=1}^k y_i, \quad (27)$$

(Observe that an upper one-sided conditional α prediction limit h on the l th order statistic Y_l based on the first k ordered early-failure observations $Y_1 \leq \dots \leq Y_k$, where $l > k$, from the same sample may be obtained from a lower one-sided conditional $(1-\alpha)$ prediction limit by replacing $1-\alpha$ by α .)

Proof. The joint density of $Y_1 \leq \dots \leq Y_k$ is given by

$$f(y_1, \dots, y_k | \mu, \sigma) = \frac{m!}{(l-k-1)(m-r)!} \prod_{i=1}^k \frac{1}{\sigma} \exp \left(\frac{y_i - \mu}{\sigma} - e^{\frac{y_i - \mu}{\sigma}} \right) \times \left[\exp \left(-e^{\frac{y_k - \mu}{\sigma}} \right) - \exp \left(-e^{\frac{y_l - \mu}{\sigma}} \right) \right]^{l-k-1} \times \frac{1}{\sigma} \exp \left(\frac{y_l - \mu}{\sigma} - e^{\frac{y_l - \mu}{\sigma}} \right) \exp \left(-(m-l)e^{\frac{y_l - \mu}{\sigma}} \right). \quad (28)$$

Let $\hat{\mu}$, $\hat{\sigma}$ be the maximum likelihood estimates of μ , σ , respectively, based on $Y_1 \leq \dots \leq Y_k$ from a complete sample of size m , and let

$$V_1 = \frac{\hat{\mu} - \mu}{\hat{\sigma}}, \quad V = \frac{\hat{\sigma}}{\sigma}, \quad W = \frac{Y_l - Y_k}{\sigma}, \quad (29)$$

and

$$U_i = \frac{Y_i - \hat{\mu}}{\hat{\sigma}}, \quad i = 1(1)k. \quad (30)$$

Using the invariant embedding technique [9]-[14], we then find in a straightforward manner, that the probability element of the joint density of V_1, V, W , conditional on fixed $\mathbf{u} = (u_1, \dots, u_k)$, is

$$f(v_1, v, w | \mathbf{u}) dv_1 dv dw = \vartheta(\mathbf{u}) v^{k-2} e^{(k+1)v_1} v e^{v(w+u_k)} \exp \left(v \sum_{i=1}^k u_i \right) \times \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} (-1)^{l-k-1-j} \times \exp \left[-e^{v_1} \left((m-k-j)e^{v(w+u_k)} + je^{vu_k} + \sum_{i=1}^k e^{vu_i} \right) \right] dv_1 dv dw, \quad v_1 \in (-\infty, \infty), \quad v \in (0, \infty), \quad w \in (0, \infty), \quad (31)$$

where

$$\vartheta(\mathbf{z}) = \left[\int_0^\infty v^{k-2} e^{-\sum_{i=1}^k u_i} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \frac{(-1)^{l-k-1-j} \Gamma(k)}{m-k-j} \times \left((m-k)e^{vu_k} + \sum_{i=1}^k e^{vu_i} \right)^{-k} dv \right]^{-1} \quad (32)$$

is the normalizing constant.

Using (31), we have that

$$\Pr\{Y_l > h | \mathbf{u}\} = \int_0^\infty \int_{w_h}^\infty \int_{-\infty}^\infty f(v_1, v, w | \mathbf{u}) dv_1 dw dv$$

$$= \left[\frac{\int_0^\infty v^{k-2} e^{-\sum_{i=1}^k u_i} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \frac{(-1)^{l-k-1-j}}{m-k-j} \times \left((m-k-j)e^{v(w_h+u_k)} + je^{vu_k} + \sum_{i=1}^k e^{vu_i} \right)^{-k} dv}{\int_0^\infty v^{k-2} e^{-\sum_{i=1}^k u_i} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \frac{(-1)^{l-k-1-j}}{m-k-j} \times \left((m-k)e^{vu_k} + \sum_{i=1}^k e^{vu_i} \right)^{-k} dv} \right], \quad (33)$$

where

$$w_h = \frac{h - y_k}{\hat{\sigma}}, \quad (34)$$

and the proof is complete. \square

Corollary 3.1. Let $Y_1 \leq \dots \leq Y_k$ be the first k ordered past observations from a sample of size m from the two-parameter Weibull distribution (3). Then (by using (23)) a lower one-sided conditional $(1-\alpha)$ prediction limit h on the l th order statistic Y_l ($l > k$) from the same sample is found as

$$h = \arg[\Pr\{Y_l > h | \mathbf{y}\} = 1 - \alpha]$$

$$= \arg \left[\frac{\int_0^\infty v^{k-2} e^{-v \delta \sum_{i=1}^k \ln \left(\frac{y_i}{\hat{\beta}} \right)} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \frac{(-1)^{l-k-1-j}}{m-k-j} \times \left((m-k-j) e^{v \delta \left[\ln \left(\frac{h}{\hat{\beta}} \right) + \ln \left(\frac{y_k}{\hat{\beta}} \right) \right]} + j e^{v \delta \ln \left(\frac{y_k}{\hat{\beta}} \right)} + \sum_{i=1}^k e^{v \delta \ln \left(\frac{y_i}{\hat{\beta}} \right)} \right)^{-k} dv}{\int_0^\infty v^{k-2} e^{-v \delta \sum_{i=1}^k \ln \left(\frac{y_i}{\hat{\beta}} \right)} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \frac{(-1)^{l-k-1-j}}{m-k-j} \times \left((m-k) e^{v \delta \ln \left(\frac{y_k}{\hat{\beta}} \right)} + \sum_{i=1}^k e^{v \delta \ln \left(\frac{y_i}{\hat{\beta}} \right)} \right)^{-k} dv} \right], \quad (35)$$

where $\hat{\beta}$ and $\hat{\delta}$ are the maximum likelihood estimates of β and δ based on the first r ordered past observations $Y_1 \leq \dots \leq Y_k$ from a sample of size m from the two-parameter Weibull distribution (3), which can be found from solution of

$$\hat{\beta} = \left(\left[\sum_{i=1}^k y_i^{\hat{\delta}} + (m-k) y_k^{\hat{\delta}} \right] / k \right)^{1/\hat{\delta}}, \quad (36)$$

and

$$\hat{\delta} = \left[\frac{\left(\sum_{i=1}^k y_i^{\hat{\delta}} \ln y_i + (m-k) y_k^{\hat{\delta}} \ln y_k \right)}{\left(\sum_{i=1}^k y_i^{\hat{\delta}} + (m-k) y_k^{\hat{\delta}} \right)^{-1} - \frac{1}{k} \sum_{i=1}^k \ln y_i} \right]^{-1}, \quad (37)$$

$$\mathbf{y} = (y_1, \dots, y_k). \quad (38)$$

C. New-Within Sample Prediction

Theorem 4 (Lower (upper) one-sided prediction limit h on the l th order statistic Y_l in a sample of observations from the Gumbel distribution on the basis of both the early-failure data $Y_1 \leq \dots \leq Y_k$ from the same sample and the data $X_1 \leq \dots \leq X_r$ from a previous sample). Let $X_1 \leq \dots \leq X_r$ be the first r ordered observations from a previous sample of size n from the Gumbel distribution (1) and $Y_1 \leq \dots \leq Y_k$ be the first k ordered early-failure observations from a new sample of size m also from the distribution (1). Then a lower one-sided conditional $(1-\alpha)$ prediction limit h on the l th order statistic Y_l ($l > k$) from the same new sample is given by

$$h = \arg[\Pr\{Y_l > h | \mathbf{u}, \mathbf{z}\} = 1 - \alpha]$$

$$= \arg \left[\frac{\int_0^\infty v^{k+r-2} e^{-v \left(\sum_{i=1}^k u_i + \sum_{i=1}^r z_i \right)} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \frac{(-1)^{l-k-1-j}}{m-k-j} \times \left((m-k-j) e^{v[(h-y_k)/\hat{\sigma} + u_k]} + j e^{v u_k} + \sum_{i=1}^k e^{v u_i} \right)^{-(k+r)} + \sum_{i=1}^r e^{v z_i} + (n-r) e^{v z_r}}{\int_0^\infty v^{k+r-2} e^{-v \left(\sum_{i=1}^k u_i + \sum_{i=1}^r z_i \right)} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \frac{(-1)^{l-k-1-j}}{m-k-j} \times \left((m-k) e^{v u_k} + \sum_{i=1}^k e^{v u_i} + \sum_{i=1}^r e^{v z_i} + (n-r) e^{v z_r} \right)^{-(k+r)} dv} \right], \quad (39)$$

where

$$\mathbf{u} = (u_1, \dots, u_k), \quad U_i = \frac{Y_i - \hat{\mu}}{\hat{\sigma}}, \quad i = 1, \dots, k,$$

$$\mathbf{z} = (z_1, z_2, \dots, z_r), \quad Z_i = \frac{X_i - \hat{\mu}}{\hat{\sigma}}, \quad i = 1, \dots, r, \quad (40)$$

$\hat{\mu}$ and $\hat{\sigma}$ are the maximum likelihood estimates of μ and σ (based on both the first k ordered early-failure observations $Y_1 \leq \dots \leq Y_k$ from a sample of size m and the first r ordered observations $X_1 \leq \dots \leq X_r$ from the previous sample of size n from the Gumbel distribution), which can be found from solution of

$$\hat{\mu} = \hat{\sigma} \ln \left[\frac{\left(\sum_{i=1}^k e^{y_i/\hat{\sigma}} + (m-k) e^{y_k/\hat{\sigma}} \right)}{\left(\sum_{i=1}^r e^{x_i/\hat{\sigma}} + (n-r) e^{x_k/\hat{\sigma}} \right)} \right] / (k+r), \quad (41)$$

and

$$\hat{\sigma} = \frac{\sum_{i=1}^k y_i e^{y_i/\hat{\sigma}} + (m-k) y_k e^{y_k/\hat{\sigma}} + \sum_{i=1}^r x_i e^{x_i/\hat{\sigma}} + (n-r) x_k e^{x_k/\hat{\sigma}}}{\sum_{i=1}^k e^{y_i/\hat{\sigma}} + (m-k) e^{y_k/\hat{\sigma}} + \sum_{i=1}^r e^{x_i/\hat{\sigma}} + (n-r) e^{x_k/\hat{\sigma}} - \frac{1}{k+r} \left(\sum_{i=1}^k y_i + \sum_{i=1}^r x_i \right)}. \quad (42)$$

Proof. For the proof we refer to Theorems 1 and 3. \square

Corollary 4.1. Let $X_1 \leq \dots \leq X_r$ be the first r ordered observations from a previous sample of size n from the two-parameter Weibull distribution (3) and $Y_1 \leq \dots \leq Y_k$ be the first k ordered early-failure observations from a new sample of size m also from the distribution (3). Then a lower one-sided conditional $(1-\alpha)$ prediction limit h on the l th order statistic Y_l ($l > k$) from the same new sample is given by

$$h = \arg[\Pr\{Y_l > h | \mathbf{y}, \mathbf{x}\} = 1 - \alpha]$$

$$\begin{aligned}
 & \int_0^{\infty} v^{r+k-2} e^{-v\delta \left[\sum_{i=1}^k \ln\left(\frac{y_i}{\beta}\right) + \sum_{i=1}^r \ln\left(\frac{x_i}{\beta}\right) \right]} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \frac{(-1)^{l-k-1-j}}{m-k-j} \\
 & \times \left((m-k-j)e^{-v\delta \left[\ln\left(\frac{h}{\beta}\right) + \ln\left(\frac{y_k}{\beta}\right) \right]} + je^{-v\delta \ln\left(\frac{y_k}{\beta}\right)} \right)^{-(k+r)} \\
 & \times \left(\sum_{i=1}^k e^{-v\delta \ln\left(\frac{y_i}{\beta}\right)} + \sum_{i=1}^r e^{-v\delta \ln\left(\frac{x_i}{\beta}\right)} + (n-r)e^{-v\delta \ln\left(\frac{x_r}{\beta}\right)} \right) dv \\
 = \arg & \int_0^{\infty} v^{r+k-2} e^{-v\delta \left[\sum_{i=1}^k \ln\left(\frac{y_i}{\beta}\right) + \sum_{i=1}^r \ln\left(\frac{x_i}{\beta}\right) \right]} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \frac{(-1)^{l-k-1-j}}{m-k-j} \\
 & \times \left((m-k)e^{-v\delta \ln\left(\frac{y_k}{\beta}\right)} + \sum_{i=1}^k e^{-v\delta \ln\left(\frac{y_i}{\beta}\right)} + \sum_{i=1}^r e^{-v\delta \ln\left(\frac{x_i}{\beta}\right)} \right)^{-(k+r)} \\
 & \times \left(\sum_{i=1}^k e^{-v\delta \ln\left(\frac{y_i}{\beta}\right)} + \sum_{i=1}^r e^{-v\delta \ln\left(\frac{x_i}{\beta}\right)} + (n-r)e^{-v\delta \ln\left(\frac{x_r}{\beta}\right)} \right) dv \\
 & = 1 - \alpha
 \end{aligned} \tag{43}$$

where $\mathbf{y} = (y_1, \dots, y_k)$, $\mathbf{x} = (x_1, \dots, x_r)$, $\hat{\beta}$ and $\hat{\delta}$ are the maximum likelihood estimates of β and δ (based on both the first k ordered early-failure observations $Y_1 \leq \dots \leq Y_k$ from a sample of size m and the first r ordered observations $X_1 \leq \dots \leq X_r$ from the previous sample of size n from the two-parameter Weibull distribution), which can be found from solution of

$$\hat{\beta} = \left[\frac{\sum_{i=1}^k y_i^{\hat{\delta}} + (m-k)y_k^{\hat{\delta}} + \sum_{i=1}^r x_i^{\hat{\delta}} + (n-r)x_r^{\hat{\delta}}}{k+r} \right]^{1/\hat{\delta}}, \tag{44}$$

and

$$\hat{\delta} = \left[\frac{\sum_{i=1}^k y_i^{\hat{\delta}} \ln y_i + (m-k)y_k^{\hat{\delta}} \ln y_k + \sum_{i=1}^r x_i^{\hat{\delta}} \ln x_i + (n-r)x_r^{\hat{\delta}} \ln x_r}{\sum_{i=1}^k y_i^{\hat{\delta}} + (m-k)y_k^{\hat{\delta}} + \sum_{i=1}^r x_i^{\hat{\delta}} + (n-r)x_r^{\hat{\delta}}} - \frac{1}{k+r} \left(\sum_{i=1}^k \ln y_i + \sum_{i=1}^r \ln x_i \right) \right]^{-1}. \tag{45}$$

Proof. For the proof we refer to Corollaries 1.1, 3.1. \square

III. APPLICATION EXAMPLES

A. Warranty Period Prediction for k -out-of- n System

Many technical systems or subsystems have k -out-of- n structure. These so-called k -out-of- n systems consist of n components of the same kind. The entire system is working if at least k of its n components are operating. It fails if $n - k + 1$ or more components fail. Hence, a k -out-of- n system breaks down at the time of the $(n - k + 1)$ th component failure. Since all components start working at the same time, this approach leads to a kind of redundancy called active redundancy of $n-k$ components. An n -component system that works (or is “good”) if and only if at least k of the n components work (or are good) is called a k -out-of- n :G

system. An n component system that fails if and only if at least k of the n components fail is called a k -out-of- n :F system. Based on these two definitions, a k -out-of- n :G system is equivalent to an $(n - k + 1)$ -out-of- n :F system.

The term k -out-of- n system is often used to indicate either a G system or an F system or both. Important particular cases of k -out-of- n systems are parallel and series systems. A series system is equivalent to a 1-out-of- n :F system and to an n -out-of- n :G system while a parallel system is equivalent to an n -out-of- n :F system and to a 1-out-of- n :G system.

The k -out-of- n system structure is a very popular type of redundancy in fault-tolerant systems. It finds wide applications in both industrial and military systems.

Practical examples of k -out-of- n systems are, e.g., an aircraft with four engines which will not crash if at least two out of its four engines remain functioning (2-out-of-4:G system), or a satellite which will have enough power to send signals if not more than four out of its ten batteries are discharged (5-out-of-10:F system). In a communications system with three transmitters, the average message load may be such that at least two transmitters must be operational at all times or critical messages may be lost. Thus, the transmission subsystem functions as a 2-out-of-3:G system. Systems with spares may also be represented by the k -out-of- n system model. In the case of an automobile with four tires, for example, usually one additional spare tire is equipped on the vehicle. Thus, the vehicle can be driven as long as at least 4-out-of-5 tires are in good condition.

In reliability theory, the lifetime of a k -out-of- n :G system is usually described by the $(n - k + 1)$ th order statistic Y_{n-k+1} from the sample Y_1, \dots, Y_n , where the random variable Y_i represents the ordered lifetime or failure time of the component of the system, $1 \leq i \leq n$. In the conventional modeling of these structures, the component lifetimes are supposed to be independent and identically distributed random variables. Translating this approach back into the technical sphere, it reflects the assumption that the failure of any component does not affect the remaining ones.

Let $X_1 \leq \dots \leq X_r$ be the first r ordered observations of time to failure for identical structural components of aircraft from a sample of size n from the two-parameter Weibull distribution (3) (as the results of fatigue tests conducted on the components), where the parameter β is unknown, the parameter $\delta=1$. Let us assume that $r=2$, $n=5$, and $s = 16000$ flight hours. There is a k -out-of- m :G system of the same identical structure components of aircraft, operating independently, where $k=2$, $m=4$. Then it follows from (21) that the upper one-sided conditional α prediction limit h^{upper} on the lifetime of this system can be calculated as

$$\begin{aligned}
 h^{\text{upper}} &= \arg[\Pr\{Y_l > h \mid s\} = \alpha] \\
 &= \arg \left[\frac{1}{B(l, m-l+1)} \sum_{j=0}^{l-1} \binom{l-1}{j} (-1)^j \right. \\
 & \quad \left. \times \frac{1}{(m-l+1+j)[1+(m-l+1+j)h/s]^r} = \alpha \right] \\
 &= 53750 \text{ flight hours,} \tag{46}
 \end{aligned}$$

where $l=m-k+1=3$, $\alpha=0.05$.

The warranty period for the lifetime of this system (i.e., the lower one-sided conditional $(1-\alpha)$ prediction limit h_{lower} on the lifetime of this system) is given by

$$h_{\text{lower}} = \arg[\Pr\{Y_l > h | s\} = 1 - \alpha]$$

$$= \arg \left[\frac{1}{B(l, m-l+1)} \sum_{j=0}^{l-1} \binom{l-1}{j} (-1)^j \right]$$

$$\times \frac{1}{(m-l+1+j)[1+(m-l+1+j)h/s]^r} = 1 - \alpha$$

$$= 1860 \text{ flight hours.} \quad (47)$$

B. Inspection Policy for Fatigued Structures

As fatigued (say, aircraft) structures begin to age (that is, as flight hours accumulate), existing subcritical cracks or new cracks can grow in some high-stress points of the structural components. The usual approach is to inspect the structures periodically at certain intervals. Thus, a catastrophic accident during flight can be avoided.

Let us assume that in a fleet of m aircraft there are m of the same individual structure components, operating independently. Let Y_1 be the minimum time to crack initiation in the above components. In other words, let Y_1 be the smallest observation from an independent second sample of m observations from the Weibull distribution (3). Suppose an inspection is carried out at time t , and this shows that initial crack (which may be detected) has not yet occurred. We now have to schedule the next inspection. Let X be the random time to crack initiation. Then we schedule the next inspection at time $\tau > t$, where τ satisfies

$$\Pr\{X > \tau | X > t\} = 1 - \alpha. \quad (48)$$

Equation (48) says that the next inspection is scheduled so that, with probability $1-\alpha$, the aircraft structure component is still working and free of initial crack prior to inspection.

It follows (48) and (17) that (with $\tau_0 = 0$) $\tau_j, j=1, 2, \dots$ can be calculated as

$$\tau_j = \arg[\Pr\{Y_1 \geq \tau_j | \mathbf{x}\} = (1 - \alpha)^j]$$

$$= \arg \left[\frac{\int_0^\infty v^{r-2} e^{-v \sum_{i=1}^r \ln(x_i/\bar{\beta})} \left(m e^{v \delta \ln(\tau_j/\bar{\beta})} + \sum_{i=1}^r e^{v \delta \ln(x_i/\bar{\beta})} + (n-r) e^{v \delta \ln(x_r/\bar{\beta})} \right)^{-r} dv}{\int_0^\infty v^{r-2} e^{-v \sum_{i=1}^r \ln(x_i/\bar{\beta})} \left(\sum_{i=1}^r e^{v \delta \ln(x_i/\bar{\beta})} + (n-r) e^{v \delta \ln(x_r/\bar{\beta})} \right)^{-r} dv} \right]$$

$$= (1 - \alpha)^j \quad (49)$$

IV. CONCLUSIONS AND DIRECTIONS FOR FUTURE RESEARCH

Prepare Prediction of an unobserved random variable is a fundamental problem in statistics. The aim of this paper is to construct lower (upper) prediction limits that are exceeded with probability $1-\alpha$ (α) by future observations or functions of observations. The prediction limits depend upon a

previously complete or type II censored data sample and/or early-failure data of a new sample from the same distribution.

The methodology described here can be extended in several different directions to handle various problems that arise in practice. We have illustrated the prediction methods for log-location-scale distributions (such as the Gumbel or Weibull distribution). Application to other distributions could follow directly.

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