Reliability Measures in Circulant Network

*Indra Rajasingh, Bharati Rajan, and R. Sundara Rajan

***Abstract*—Reliability and efficiency are important criteria in the design of interconnection networks. Connectivity is a widely used measurement for network fault-tolerance capacities, while diameter determines routing efficiency along individual paths. In practice, we are interested in high-connectivity, small-diameter networks. Recently, the w-wide diameter $d_w(G)$, $(w-1)$-fault diameter $D_w(G)$ and the w-Rabin number $r_w(G)$, for $w \leq k(G)$ have been used to measure network reliability and efficiency. In this paper, in addition to these parameters we introduce $(w-1)$-fault wide diameter $\rho_w(G)$ and study these parameters for the undirected circulant network.**

***Index Terms*—wide diameter, fault diameter, fault wide diameter, Rabin number, circulant network.**

I. Introduction

RELIABILITY is an important concept in the design of networks. Connectivity and edge connectivity are widely used as reliability measures. In practice, we are often interested in the collection of multipaths between a pair of nodes [6]. The distance $d_G(x,y)$ from a vertex x to another vertex y in a network G is the minimum number of edges of a path from x to y. The diameter $d(G)$ of a network G is the maximum distance from one vertex to another. The connectivity $k(G)$ of a network G is the minimum number of vertices whose removal results in a disconnected or trivial network. According to Menger's theorem, there are atleast k (internally) vertex-disjoint paths from a vertex x to another vertex y in a network of connectivity k [17].

The classical approach to study routing in interconnection networks is to try to find the shortest path between the sending station and the receiving station. Whenever some stations are faulty on the path between the sending station and the receiving station, the management protocol has to find a way to bypass those faulty stations and set up a new path between them. Similarly, if this new path is disconnected again, a third path needs to be set up, if it is possible [4]. In this context, diameter is the measurement for maximum transmission delay and connectivity is a good parameter to study the tolerance of the network on occasions when nodes fail. Fault tolerant interconnection networks can be found in Hsu [17].

For a graph (network) G with connectivity $k(G)$, the parameters w-wide diameter $d_w(G)$, $(w-1)$-fault diameter $D_w(G)$ and the Rabin number $r_w(G)$ for any $w \leq k(G)$ arise from the study of parallel routing, fault-tolerant systems and randomized routing respectively [6], [9], [10], [14]. Due to the widespread use of reliable, efficient and fault-tolerant networks, these three parameters have been the subject of extensive study over the past decade [6]. In the sequal, $(x_1, x_2, ..., x_n)$ denotes a path from x_1 to x_n.

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Indra Rajasingh, Bharati Rajan, and R. Sundara Rajan are with the Department of Mathematics, Loyola College, Chennai 600 034, Tamilnadu, India (e-mail: hellosundar@rediffmail.com).

II. Preliminaries

Definition 1. [4] A container $C(x,y)$ between two distinct nodes x and y in a network G is a set of node-disjoint paths between x and y. The number of paths in $C(x,y)$ is called the width of $C(x,y)$. A $C(x,y)$ container with width w is denoted by $C_w(x,y)$. The length of $C_w(x,y)$, written as $l(C_w(x,y))$, is the length of a longest path in $C_w(x,y)$.

Definition 2. [5] For $w \leq k(G)$, the w-wide distance from x to y in a network G is defined as

$$d_w(x,y) = min\{l(C_w(x,y))/C_w(x,y) \text{ is a container with width } w \text{ between } x \text{ and } y\}$$

The w-wide diameter of G is defined as

$$d_w(G) = max_{x,y \in V(G)} \{d_w(x,y)\}.$$

In other words, for $w \leq k(G)$, the w-wide diameter $d_w(G)$ of a network G is the minimum l such that for any two distinct vertices x and y there exist w vertex-disjoint paths of length at most l from x to y.

The notion of w-wide diameter was introduced by Hsu [6] to unify the concepts of diameter and connectivity. It is desirous that an ideal interconnection network G should be one with connectivity $k(G)$ as large as possible and diameter $d(G)$ as small as possible. The wide-diameter $d_w(G)$ combines connectivity $k(G)$ and diameter $d(G)$, where $1 \leq w \leq k(G)$. Hence $d_w(G)$ is a more suitable parameter than $d(G)$ to measure fault-tolerance and efficiency of parallel processing computer networks. Thus, determining the value of $d_w(G)$ is of significance for a given graph G and an integer w. Hsu [6] proved that this problem is NP-complete [18].

Remark 1. If there exists a container $C^*_w(x,y)$ such that each of the w paths in $C^*_w(x,y)$ is a shortest path between x and y in G, then

$$d_w(x,y) = l(C^*_w(x,y))$$

Definition 3. [11] For $w \leq k(G)$, the $(w-1)$-fault distance from x to y in a network G is

$$D_w(x,y) = max\{d_{G-S}(x,y) : S \subseteq V \text{ with } |S| = w-1 \text{ and } x,y \text{ are not in } S\}$$

where $d_{G-S}(x,y)$ denotes the shortest distance between x and y in $G-S$.

The $(w-1)$-fault diameter of G is

$$D_w(G) = max\{D_w(x,y) : x \text{ and } y \text{ are in } G\}$$

The notion of $D_w(G)$ was defined by Hsu [6] and the special case in which $w = k(G)$ was studied by Krishnamoorthy et al. [9].

Motivated by the definitions of wide diameter and fault diameter in a network, we introduce a new parameter in this paper known as *fault wide diameter* and define it as follows:

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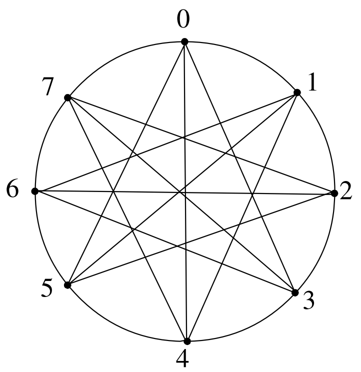

Fig. 1. Circulant graph $G(8; \pm\{1, 3, 4\})$

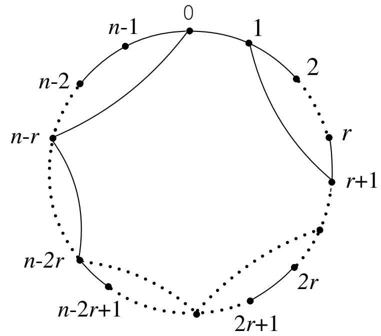

Fig. 2. $2r$-wide diameter of G

Definition 4. For $w \leq k(G)$, the $(w-1)$-fault wide distance from x to y in a network G is

$$\rho_w(x,y) = max\{d_{k(G)-|S|}(x,y) : S \subseteq V \text{ with } |S| = w-1 \text{ and } x,y \text{ are not in } S\}$$

The $(w-1)$-fault wide diameter of G is

$$\rho_w(G) = max\{\rho_w(x,y) : x \text{ and } y \text{ are in } G\}$$

Definition 5. [11] The w-Rabin number $r_w(G)$ of a network G is the minimum l such that, for any $w+1$ distinct vertices $x, y_1, ..., y_w$ there exist w vertex-disjoint (except at x) paths of length at most l from x to y_i, $1 \leq i \leq w$.

This concept was first defined by Hsu [6]. It is clear that when $w = 1$, $d_1(G) = D_1(G) = \rho_k(G) = r_1(G) = d(G)$ for any network G. The following are basic properties and relationships among $d_w(G)$, $D_w(G)$, $\rho_w(G)$ and $r_w(G)$.

Lemma 1. [11] The following statements hold for any network G of connectivity k:
1. $D_1(G) \leq D_2(G) \leq \cdots \leq D_k(G)$
2. $d_1(G) \leq d_2(G) \leq \cdots \leq d_k(G)$
3. $r_1(G) \leq r_2(G) \leq \cdots \leq r_k(G)$
4. $D_w(G) \leq d_w(G)$ and $D_w(G) \leq r_w(G)$ for $1 \leq w \leq k$

It is easy to check that the new parameter $\rho_w(G)$ exhibits the following properties.

Lemma 2. Let G be a network of connectivity k. Then
1. $\rho_1(G) \geq \rho_2(G) \geq \cdots \geq \rho_k(G)$
2. $D_w(G) \leq d_w(G) \leq \rho_w(G) \leq r_w(G)$ *for* $1 \leq w \leq k$

In 1994, Chen et al. determined the wide diameter of the cycle prefix network [5]. In 1998, Liaw et al. found fault-tolerant routing in circulant directed graphs and cycle prefix networks [12]. The line connectivity and the fault diameters in pyramid networks were studied by Cao et al. in 1999 [4]. In the same year Liaw et al. determined the Rabin number and wide diameter of butterfly networks [10], [11]. In 2005, Liaw et al. found the wide diameters and Rabin numbers of generalized folded hypercube networks [13]. In 2009, Jia and Zhang found the wide diameter of Cayley graphs of Z_m, the cyclic group of residue classes modulo m and they proved that the k-wide diameter of the Cayley graph Cay(Z_m, A) generated by a k-element set A is $d+1$ for $k = 2$ and is bounded above by $d+1$ for $k = 3$, where d is the diameter of Cay(Z_m, A) [7].

In this paper we compute the w-wide diameter $d_w(G)$, $(w-1)$-fault diameter $D_w(G)$, $(w-1)$-fault wide diameter $\rho_w(G)$, and the w-Rabin number $r_w(G)$ for $w \leq k(G)$ when G is an undirected circulant graph with connectivity k.

III. Main Results

The circulant is a natural generalization of the double loop network and was first considered by Wong and Coppersmith [16]. Circulant graphs have been used for decades in the design of computer and telecommunication networks due to their optimal fault-tolerance and routing capabilities [3]. It is also used in VLSI design and distributed computation [1], [2], [15]. The term circulant comes from the nature of its adjacency matrix. A matrix is circulant if all its rows are periodic rotations of the first one. Circulant matrices have been employed for designing binary codes [8]. Theoretical properties of circulant graphs have been studied extensively and surveyed by Bermond et al. [1]. Every circulant graph is a vertex transitive graph and a Cayley graph [17]. Most of the earlier research concentrated on using the circulant graphs to build interconnection networks for distributed and parallel systems [1], [3].

Definition 6. A circulant undirected graph, denoted by $G(n; \pm\{1, 2, ..., j\})$, $1 \leq j \leq \lfloor n/2 \rfloor$, $n \geq 3$ is defined as an undirected graph consisting of the vertex set $V = \{0, 1, ..., n-1\}$ and the edge set $E = \{(i,j) : |j-i| \equiv s(mod\ n), s \in \{1, 2, ..., j\}\}$.

The circulant graph shown in Fig. 1 is $G(8; \pm\{1, 3, 4\})$. It is clear that $G(n; \pm 1)$ is the undirected cycle C_n and $G(n; \pm\{1, 2, ..., \lfloor n/2 \rfloor\})$ is the complete graph K_n. **Theorem 1.** If G is an undirected circulant graph $G(n; \pm\{1, 2, ..., r\})$, then

$$d_{2r}(G) = \begin{cases} \lfloor \frac{n-2r}{r} \rfloor + 2, & \text{if } n \equiv 1 (mod\ r) \\ \lceil \frac{n-2r}{r} \rceil + 2, & \text{otherwise} \end{cases}$$

for all $r = 1, 2, ..., \lfloor n/2 \rfloor$.

Proof. Since G is $2r$-regular, by Menger's theorem there are exactly $2r$ vertex disjoint paths between any two vertices u and v in G. We begin with the case when u and v are adjacent in C_n. Since G is vertex-transitive, without loss of generality we assume that $u = 0$ and $v = 1$. Since the neighbourhood of 0 namely $N(0)$ is the set $N(0) = \{1, 2, ..., r, n-1, ..., n-r\}$, each path in any container $C_{2r}(0,1)$ contains exactly one member from $N(0)$.

Consider a path P in $C_{2r}(0,1)$ passing through $n-r$. Since $V(P) \cap (N(0) \backslash (n-r)) = \phi$, in order to compute $d_{2r}(G)$ we choose P to be a shortest path between 0 and 1. Thus

$$P = (0, n-r, n-2r, \ldots, 1+2r, 1+r, 1)$$

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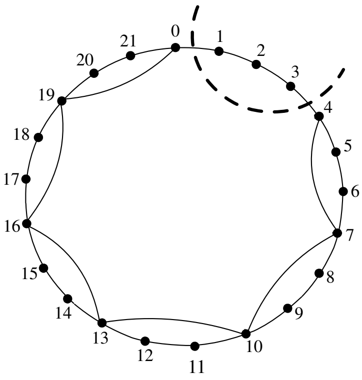

Fig. 3. 3-fault diameter of $G(22; \pm\{1, 2, 3\})$ is 6

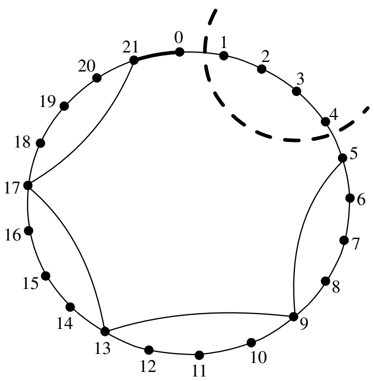

Fig. 4. 5-fault wide diameter of $G(22; \pm\{1, 2, 3\})$ is 5

of length $\lfloor \frac{n-2r}{r} \rfloor + 2$, when $n \equiv 1$ $(mod\ r)$ and

$$P = (0, n-r, n-2r, \ldots, j+r, 1+r, 1), 2 \leq j \leq 1+r$$

of length $\lceil \frac{n-2r}{r} \rceil + 2$ in all other possibilities. Thus every container $C_{2r}(0, 1)$ contains a path of length $\lfloor \frac{n-2r}{r} \rfloor + 2$ if $n \equiv 1$ $(mod\ r)$ and $\lceil \frac{n-2r}{r} \rceil + 2$ if $n \not\equiv 1$ $(mod\ r)$.

We observe that the edge $(0, 1)$ is a path of length 1 and the paths $(0, r-j, 1)$ and $(0, n-r+1+j, 1)$, $0 \leq j \leq r-2$ are paths of length 2 between 0 and 1 passing through vertices in $N(0) \backslash \{n-r\}$. Thus there exist a container $C_{2r}(0, 1)$ in which the longest path is P and all other paths are of length 1 or 2. See Fig. 2.

The case when u and v are not adjacent on C_n naturally yields paths between u and v of length atmost $\lfloor \frac{n-2r}{r} \rfloor + 2$, when $n \equiv 1$ $(mod\ r)$ and atmost $\lceil \frac{n-2r}{r} \rceil + 2$, when $n \not\equiv 1$ $(mod\ r)$.

In the following theorem we obtain the $(r+1)$-fault diameter for the undirected circulant graph $G(n; \pm\{1, 2, ..., r\})$.

Theorem 2. Let G be the circulant graph $G(n; \pm\{1, 2, ..., r\})$. Then

$$D_{r+1}(G) = \left\lfloor \frac{n-r}{r} \right\rfloor$$

for all $r = 2, ..., \lfloor n/2 \rfloor$.

Proof. We begin with r faulty nodes which are consecutive vertices in C_n. Without loss of generality let $1, 2, ..., r$ be the faulty nodes. Let $G^{'} = G \backslash \{1, 2, ..., r\}$. It is easy to see that $d_{G'}(0, r+1) \geq d_{G'}(i, j)$ for all vertices i, j taken modulo n. Now $d_{G'}(0, r+1) = \lfloor \frac{n-r}{r} \rfloor$. On the otherhand, suppose the faulty nodes are not consecutive vertices of C_n, then $d_{G'}(i, j)$ is bounded above by $\lfloor \frac{n-r}{r} \rfloor$. See Fig. 3.

Theorem 3. If G is an undirected circulant graph, then

$$D_2(G(n; \pm\{1, 2, ..., r\})) = \left\lceil \frac{n}{2r} \right\rceil$$

for all $r = 1, 2, ..., \lfloor n/2 \rfloor$.

Proof. Since G is vertex-transitive, let us assume that the vertex labeled r be the faulty node. Let $G' = G \backslash \{r\}$. We assume that $u = 0$ and $v = \lfloor n/2 \rfloor$. In order to compute $D_2(G)$ we choose P to be shortest path between 0 and $\lfloor n/2 \rfloor$. Then

$$P = (0, n-r, n-2r, \ldots, \lfloor n/2 \rfloor)$$

of length $\frac{n}{2r}$ if $n - r \equiv 0$ $(mod\ r)$. Otherwise, we choose the shortest path P between 0 and $\lfloor n/2 \rfloor$ as

$$P = (0, r-1, 2r-1, \ldots, \lfloor n/2 \rfloor)$$

of length $\lceil \frac{n}{2r} \rceil$. It is easy to see that $d_{G'}(0, \lfloor n/2 \rfloor) = \lceil \frac{n}{2r} \rceil \geq d_{G'}(i, j)$ for all vertices i, j taken modulo n.

In the next result we obtain the $(w-1)$-fault wide diameter for an undirected circulant graph.

Theorem 4. Let G be the circulant graph $G(n; \pm\{1, 2, ..., r\})$. Then

$$\rho_2(G) = \left\lceil \frac{n-(2r+1)}{r} \right\rceil + 2$$

for all $r = 1, 2, ..., \lfloor n/2 \rfloor$.

Proof. Since G is vertex-transitive, let us assume that 1 be the faulty node. Let $G' = G \backslash \{1\}$. Then G' is $(2r-1)$-connected. We choose $u = 0$ and $v = 2$. Since the neighbourhood of 0 is the set $N(0) = \{2, 3, ..., r, n-1, ..., n-r\}$, each path in the container $C_{2r-1}(0, 2)$ contains exactly one member from $N(0)$.

Consider a path P in $C_{2r-1}(0, 2)$ passing through $n-r+1$. Since $V(P) \cap (N(0) \backslash (n-r+1)) = \phi$, in order to compute $\rho_2(G)$ we choose P to be a shortest path between 0 and 2. Thus

$$P = (0, n-r+1, n-2r+1, \ldots, 2)$$

of length $\left\lceil \frac{n-(2r+1)}{r} \right\rceil + 2$. Again by Theorem 1, the length of other paths are less than or equal to $\left\lceil \frac{n-(2r+1)}{r} \right\rceil + 2$. It is also easy to see that $d_{2r-1}(0, 2) = \left\lceil \frac{n-(2r+1)}{r} \right\rceil + 2 \geq d_{2r-1}(i, j)$ for all vertices i, j taken modulo n in $G^{'}$.

Theorem 5. Let G be the circulant graph $G(n; \pm\{1, 2, ..., r\})$. Then

$$\rho_3(G) = \begin{cases} \left\lceil \frac{n-(2r-2)}{r} \right\rceil + 2, \text{if } n-(2r-2) \equiv x \ (mod\ r) \\ \qquad \text{where } x \in \{3, 4, \ldots, r-1\} \\ \left\lfloor \frac{n-(2r-2)}{r} \right\rfloor + 2, \text{ otherwise} \end{cases}$$

for all $r = 4, 5, ..., \lfloor n/2 \rfloor$.

Proof. Since G is vertex-transitive, let us assume that $1, 2$ be the faulty nodes. Let $G' = G \backslash \{1, 2\}$. Then G' is $(2r-2)$-connected. We assume that $u = 0$ and $v = 3$. Since the neighbourhood of 0 is the set $N(0) = \{3, ..., r, n-1, ..., n-r\}$, each path in the container $C_{2r-2}(0, 2)$ contains exactly one member from $N(0)$.

Consider a path P in $C_{2r-2}(0, 2)$ passing through $n-r+2$. Since $V(P) \cap (N(0) \backslash (n-r+2)) = \phi$, in order to compute $\rho_3(G)$ we choose P to be a shortest path between 0 and 3.

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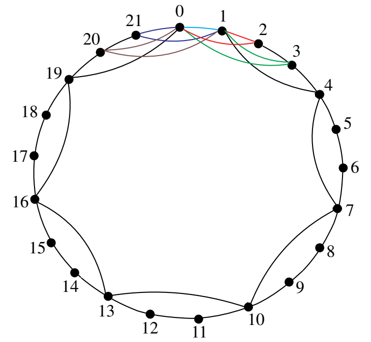

Fig. 5. 6-Rabin number of G

Thus

$$P = (0, n-r+2, n-2r+2, \ldots, 3)$$

of length $\left\lceil \frac{n-(2r-2)}{r} \right\rceil + 2$ if $n-(2r-2) \equiv x(mod\ r)$, where $x \in \{3, 4, ..., r-1\}$. By Theorem 1, the length of other paths are less than or equal to $\left\lceil \frac{n-(2r-2)}{r} \right\rceil + 2$. Also it is easy to see that $d_{2r-2}(0,3) = \left\lceil \frac{n-(2r-2)}{r} \right\rceil + 2 \geq d_{2r-2}(i,j)$ for all vertices i, j taken modulo n in G'.

Remark 2. If $r = 3$, then

$$\rho_3(G) = \left\lceil \frac{n-4}{3} \right\rceil + 1$$

where G be the circulant graph $G(n; \pm\{1, 2, ..., r\})$.

Theorem 6. If G is an undirected circulant graph, then

$$\rho_{r+1}(G(n; \pm\{1, 2, ..., r\})) = \left\lceil \frac{n-(r+2)}{r} \right\rceil + 1$$

for all $r = 1, 2, ..., \lfloor n/2 \rfloor$.

Proof. Since G is vertex-transitive, let us assume that $1, 2, ..., r$ be the faulty nodes. Let $G' = G \backslash \{1, 2, ..., r\}$. Then G' is r-connected. We assume that $u = 0$ and $v = r+1$. Since the neighbourhood of 0 is the set $N(0) = \{n-1, ..., n-r\}$, each path in the container $C_r(0, r+1)$ contains exactly one member from $N(0)$.

Consider a path P in $C_r(0, r+1)$ passing through $n-1$. In order to compute $\rho_{r+1}(G)$ we choose P to be a shortest path between 0 and $r+1$. Thus

$$P = (0, n-1, n-1+r, \ldots, r+1)$$

of length $\left\lceil \frac{n-(r+2)}{r} \right\rceil + 1$. By Theorem 1, the length of other paths are less than or equal to $\left\lceil \frac{n-(r+2)}{r} \right\rceil + 1$. It is also easy to see that $d_r(0, r+1) = \left\lceil \frac{n-(r+2)}{r} \right\rceil + 1 \geq d_r(i,j)$ for all vertices i, j taken modulo n in G'. See Fig. 4.

We now proceed to determine the w-Rabin number for an undirected circulant graph.

Theorem 7. If G is an undirected circulant graph, then

$$r_{2m-1}(G(n; \pm\{1, 2, ..., m\})) = \left\lfloor \frac{n-3}{m} \right\rfloor + 1$$

for all $m = 1, 2, ..., \left\lfloor \frac{n}{2} \right\rfloor$.

Proof. By definition of Rabin number, we choose $2m$ points (say $u, u_1, u_2, \ldots, u_{2m-1}$) in G. It is enough to consider the case when these points are consecutive points in C_n labeled $i, i_1, \ldots, i_{2m-1}$ modulo n. We claim that, there are $2m-1$ vertex-disjoint paths from i to i_k, $1 \leq k \leq 2m-1$ of length atmost $\left\lfloor \frac{n-3}{m} \right\rfloor + 1$. We in fact, construct $2m-1$ vertex-disjoint paths from i to i_k, $1 \leq k \leq 2m-1$ as follows.

The edge (i, i_1) is a path of length 1 from i to i_1. Further (i, i_{n-m+l_1}, i_1), $1 \leq l_1 \leq m-1$ and (i, i_{n+l_2}, i_1), $2 \leq l_2 \leq m$ are $2m-2$ number of paths in G of length 2 from i to i_1. The edge (i, i_2) is a path of length 1 from i to i_2. Further (i, i_1, i_2), (i, i_{n+l_1}, i_2), $3 \leq l_1 \leq m$ and (i, i_{n-m+l_2}, i_2), $2 \leq l_2 \leq m-1$ are $2m-3$ number of paths in G of length 2 from i to i_2. Also $(i, i+n-r, i+n-2r, \ldots, i_2+r, i_2)$ is a path of length $\left\lfloor \frac{n-3}{m} \right\rfloor + 1$.

Similarly, the $2m-1$ vertex-disjoint paths from i to i_k, $3 \leq k \leq 2m-1$ are of length less than or equal to $\left\lfloor \frac{n-3}{m} \right\rfloor + 1$.

Theorem 8. If G is an undirected circulant graph $G(n; \pm\{1, 2, ..., m\})$, then

$$r_{2m}(G) = \begin{cases} \left\lfloor \frac{n-2m}{m} \right\rfloor + 2, & \text{if } n \equiv 1(mod\ m) \\ \left\lceil \frac{n-2m}{r} \right\rceil + 2, & \text{otherwise} \end{cases}$$

for all $m = 1, 2, ..., \left\lfloor \frac{n}{2} \right\rfloor$.

Proof. This theorem is an easy consequence of Theorem 7. See Fig. 5.

IV. Conclusion

In this paper we have found the w-wide diameter, w-fault diameter, w-fault wide diameter and w-Rabin number for a circulant undirected graph G. It would be interesting to study reliability measures in interconnection network such as hypercube network, butterfly network, mesh network and so on.

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