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A Method for Complete Decomposition of a Multiple Tone Signal using a DFT

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***Abstract*— A de novo approach to Discrete Spectral Analysis is presented. A pure tone in a Discrete Fourier Transform (DFT) is mathematically analyzed. A concise, exact bin value equation is developed for so called "leakage". The necessary special treatment this equation requires at integer frequencies is provided. The equation is then inverted across three DFT bins to form an exact frequency determination equation. A method borrowed from Linear Algebra is then used to calculate the magnitude and phase. Finally, an algorithm is described which applies the pure tone equations to separate a well behaved multiple tone signal into its constituent tones. This is a fresh approach on the fundamentals; noise mitigation techniques and closely spaced frequencies are not addressed.**

***Index Terms*—Discrete Spectral Analysis, Discrete Fourier Transforms, Frequency Estimation, Spectral Leakage**

I. Introduction

All points on the unit circle in the complex plane can be located with Euler's magnificent equation.

$$e^{i\theta} = \cos\theta + i \cdot \sin\theta \tag{1}$$

Where θ is the radian measure along the circumference and i is the square root of negative one.

The projection of any point onto the horizontal axis can be accomplished by taking the midpoint of the point and its mirror, known as a conjugate, in the negative direction.

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \tag{2}$$

This is the foundation. Everything rests on this.

II. Setting Up The Pure Tone Signal

A pure tone is a single sinusoidal signal. Either the sine or cosine function can be used to model one mathematically, the cosine being slightly more convenient. Three parameters are needed to define a pure tone sequence (S_k). They are the maximum amplitude (*M*), an angle representing phase (ϕ), and a frequency component (α).

$$S_k = M\cos(\phi + \alpha k) \tag{3}$$

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A signal frame of *N* points is considered. The subscript *k* ranges from *0* to *N-1*. The values of the signal points outside the frame are expected to be the values according to (3), not from a repeat of the frame.

If the frequency of the tone is *f* with units of cycles per frame, then α is defined as:

$$\alpha = f \cdot \frac{2\pi}{N} = \frac{f}{N} \cdot 2\pi \tag{4}$$

The units of α are radians per sample.

III. The Discrete Fourier Transform

The Discrete Fourier Transform (DFT) [1] is a well known linear transform which can (unconventionally) be defined by:

$$Z_n = \frac{1}{N} \sum_{k=0}^{N-1} S_k e^{i\beta_n k} \tag{5}$$

Where:

$$\beta_n = n \cdot \frac{2\pi}{N} = \frac{n}{N} \cdot 2\pi \tag{6}$$

Each Z_n, a complex number, is commonly called a bin, a frequency bin, or a bin of frequency *n*. The subscript *n*, called the bin number, also ranges from 0 to N-1. Therefore, the $e^{i\beta_n}$ expression represents a set of N evenly spaced points around the unit circle. These points are also known as the Roots of Unity because they satisfy the equation:

$$r^N = 1 \tag{7}$$

This is easily proved using (1) by:

$$\left(e^{i\beta_n}\right)^N = e^{i\left(n \cdot \frac{2\pi}{N}\right)N} = \left(e^{i2\pi}\right)^n = 1^n = 1 \tag{8}$$

Each bin value of the DFT can be thought of as the weighted average of a set of roots of unity where the weighting factor is the signal to be analyzed wrapped around the unit circle the number of times of the bin subscript. For instance, Z_7 is the weighted average of the signal sequence stretched and wrapped around the unit circle seven times with a step size of seven ($e^{i\beta_7}$). Easiest to understand is Z_1 with a real valued signal where the weighted wrapping is equivalent to graphing the signal in polar coordinates and the bin value corresponds to the centroid. When viewed this way, it makes sense to apply a 1/N scaling factor and use a positive exponent to wrap in the

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positive direction.

IV. Applying the Fourier Transform

Plugging the pure tone sequence (3), also using (2), into the Fourier Transform definition (5) is straightforward.

$$Z_n = \frac{M}{N}\sum_{k=0}^{N-1}\left(\frac{e^{i(\phi+\alpha k)}+e^{-i(\phi+\alpha k)}}{2}\right)e^{i\beta_n k} \tag{9}$$

This series formula can be split into a sum of two phase rotated geometric series:

$$Z_n = \frac{M}{2N}\left(e^{i\phi}\sum_{k=0}^{N-1}e^{i(\alpha+\beta_n)k} + e^{-i\phi}\sum_{k=0}^{N-1}e^{i(-\alpha+\beta_n)k}\right) \tag{10}$$

V. Eliminating the Summation

The big advantage of arranging the formula using geometric series is that the summation formula for a finite geometric series can be applied.

$$\sum_{k=0}^{N-1}x^k = \frac{1-x^N}{1-x} \tag{11}$$

When it is, with (8), the results look like this:

$$Z_n = \frac{M}{2N}\left(e^{i\phi}\cdot\frac{1-e^{i\alpha N}}{1-e^{i(\alpha+\beta_n)}} + e^{-i\phi}\cdot\frac{1-e^{-i\alpha N}}{1-e^{i(-\alpha+\beta_n)}}\right) \tag{12}$$

There is one big caveat: The frequency *f* cannot be an integer value. This case is handled more carefully later.

It is well known that the DFT of a non-integer frequency pure tone will have nonzero bin values. These bin values are traditionally called “leakage” as if they are some kind of flaw. Equation (12) describes this leakage exactly in closed form without the need of a summation. Eliminating the summation saves an incredible number of calculations.

However, (12) can be put in a much more useful form. Adding the two fractions inside the big parentheses produces a single fraction having a numerator with eight terms and a denominator with four.

The eight terms in the numerator can be paired up and factored using (2) as follows:

$$e^{i\phi}+e^{-i\phi} = 2\cos\phi \tag{13}$$

$$e^{i(\phi+\alpha N)}+e^{-i(\phi+\alpha N)} = 2\cos(\phi+\alpha N) \tag{14}$$

$$e^{i(\phi-\alpha+\beta_n)}+e^{-i(\phi-\alpha-\beta_n)} = 2\cos(\phi-\alpha)e^{i\beta_n} \tag{15}$$

$$e^{i(\phi+\alpha N-\alpha+\beta_n)}+e^{-i(\phi+\alpha N-\alpha-\beta_n)} = 2\cos(\phi+\alpha N-\alpha)e^{i\beta_n} \tag{16}$$

In a similar manner, the four terms in the denominator can be paired up and factored:

$$e^{i(\alpha+\beta_n)}+e^{-i(\alpha-\beta_n)} = 2\cos(\alpha)e^{i\beta_n} \tag{17}$$

$$1+e^{i2\beta_n} = \left(e^{-i\beta_n}+e^{i\beta_n}\right)e^{i\beta_n} = 2\cos(\beta_n)e^{i\beta_n} \tag{18}$$

Putting it all together, and factoring out $-2e^{i\beta_n}$ from the numerator and denominator, results in:

$$Z_n = \frac{M}{2N}\left(\frac{\left[\cos(\phi+\alpha N)-\cos\phi\right]e^{-i\beta_n} - \left[\cos(\phi+\alpha N-\alpha)-\cos(\phi-\alpha)\right]}{\cos\alpha-\cos\beta_n}\right) \tag{19}$$

VI. Naming Expressions

By giving specific names to certain expressions within (19), a much more revealing version can be made:

$$U = \cos(\phi+\alpha N)-\cos\phi = (S_N - S_0)/M \tag{20}$$

$$\hat{U} = \cos(\phi+\alpha N-\alpha)-\cos(\phi-\alpha) = (S_{N-1}-S_{-1})/M \tag{21}$$

$$D_n = 2(\cos\alpha-\cos\beta_n) = 2\cos\alpha-(e^{i\beta_n}+e^{-i\beta_n}) \tag{22}$$

$$Z_n = \frac{M}{N}\cdot\frac{(Ue^{-i\beta_n}-\hat{U})}{D_n} = \frac{(S_{N-1}-S_{-1})-(S_N-S_0)e^{-i\beta_n}}{N(e^{i\beta_n}-2\cos\alpha+e^{-i\beta_n})} \tag{23}$$

This shows that the bin values of a DFT of a pure tone are a fairly simple real valued nonlinear transform of the Roots of Unity.

VII. Bin Oriented Frame of Reference

In cases when f is very close to an integer value F, and n=F, the calculation of Z_n in (23) becomes numerically unstable. This is because the expressions from (20)-(22) are then each the difference of two very close values. In finite numeric representations this limits the precision. By defining a frame of reference relative to the bin number the frequency is near, a much more stable equation can be developed.

$$\sigma = \alpha-\beta_F = (f-F)\frac{2\pi}{N} \approx 0 \tag{24}$$

Substituting $\sigma+\beta_F$ for α in (20)-(22) and recognizing that $\beta_F N = 2\pi F$ is an integer multiple of 2π , yields:

$$U = \cos(\phi+\sigma N)-\cos\phi \tag{25}$$

$$\hat{U} = \cos(\phi+\sigma N-\sigma-\beta_F)-\cos(\phi-\sigma-\beta_F) \tag{26}$$

$$D_f = 2\left[\cos(\sigma+\beta_F)-\cos\beta_F\right] \tag{27}$$

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Each of these equations is the difference of two cosines. Therefore, using the following Trigonometric identity,

$$\cos\theta_1 - \cos\theta_2 = -2\sin(\frac{\theta_1+\theta_2}{2})\sin(\frac{\theta_1-\theta_2}{2}) \quad (28)$$

they can be factored like this:

$$U = -2\sin(\phi+\frac{\sigma N}{2})\sin(\frac{\sigma N}{2}) \quad (29)$$

$$\hat{U} = -2\sin(\phi+\frac{\sigma N}{2}-\sigma-\beta_F)\sin(\frac{\sigma N}{2}) \quad (30)$$

$$D_f = -4\sin(\beta_F+\frac{\sigma}{2})\sin(\frac{\sigma}{2}) \quad (31)$$

Also, for clarity:

$$\rho = \phi + \frac{\sigma N}{2} \quad (32)$$

Recombining these expressions into (23) gives the more complicated, but more computationally accurate version:

$$Z_n = \frac{M}{2}\cdot\frac{\sin(\frac{\sigma N}{2})}{N\sin(\frac{\sigma}{2})}\cdot\frac{\left[\sin(\rho)e^{-i\beta_F}-\sin(\rho-\sigma-\beta_F)\right]}{\sin(\beta_F+\frac{\sigma}{2})} \quad (33)$$

VIII. Solving the Indeterminate Form

When the frequency *f* becomes an integer *F*, the same as σ becoming 0, all the equations for Z_n beyond (10) become indeterminate at *n*=*F* requiring a limit argument to solve. The last bin value equation (33) has the indeterminate portion factored out.

$$\lim_{\sigma\to 0}\frac{\sin\left(\frac{\sigma N}{2}\right)}{N\sin\left(\frac{\sigma}{2}\right)} = \lim_{\sigma\to 0}\frac{\frac{\sin\left(\frac{\sigma N}{2}\right)}{\frac{\sigma N}{2}}}{\frac{\sin\left(\frac{\sigma}{2}\right)}{\frac{\sigma}{2}}} = \frac{1}{1} = 1 \quad (34)$$

Once the indeterminate portion is dealt with, taking the limit of (33) as a whole becomes trivial:

$$Z_F = \lim_{\sigma\to 0} Z_n = \frac{M}{2}\cdot\frac{\left[\sin(\phi)e^{-i\beta_F}-\sin(\phi-\beta_F)\right]}{\sin\beta_F} \quad (35)$$

This complex value can then be split into its real and imaginary parts.

$$\mathrm{Re}(Z_F) = \frac{M}{2}\cdot\frac{\left(\sin\phi\cos\beta_F - \sin\phi\cos\beta_F + \cos\phi\sin\beta_F\right)}{\sin\beta_F} \quad (36)$$

$$\mathrm{Im}(Z_F) = \frac{M}{2}\cdot\frac{\sin(\phi)\left(-\sin\beta_F\right)}{\sin\beta_F} \quad (37)$$

These equations are then simplified further.

$$\mathrm{Re}(Z_F) = \frac{M}{2}\cdot\cos\phi \quad (38)$$

$$\mathrm{Im}(Z_F) = -\frac{M}{2}\cdot\sin\phi \quad (39)$$

Using (1), recombining them into a complex form is also easy.

$$Z_F = \frac{M}{2}e^{-i\phi} \quad (40)$$

This result can be confirmed by inspection using (12) and (10). If *f* is an integer value *F*, both numerators in (12) are always zero because the exponents are integer multiples of $i2\pi$. The denominator on the left is always nonzero (assuming $0 < \alpha < \pi$), and the denominator on the right is nonzero for all values of *n* except *F*. Thus the Z_n values are all zero except for when *n*=*F* where the left summation is zero and the right one is an indeterminate form. For this case, glance at (10), the left summation equals zero and the right one is just the sum of ones equaling *N*. The *N*s cancel and the result is (40).

IX. An Alternate Form of the Pure Tone Bin Value Equation

There is yet another form the bin value equation for a pure tone can be put into that gives a different perspective on the behavior of a DFT.

Substituting (24) into (33) brings back α as the reference variable for the frequency. Also, breaking the bracketed section out as a named variable will be helpful. Note that there is a subtle pair of factors of $(-1)^n$ in the numerators that cancel each other out when this operation is undertaken.

$$Z_n = \frac{M}{2N}\cdot\frac{\sin\left(\frac{\alpha N}{2}\right)}{\sin\left(\frac{\alpha-\beta_n}{2}\right)}\cdot\frac{Y_n}{\sin\left(\frac{\alpha+\beta_n}{2}\right)} \quad (41)$$

$$Y_n = \sin(\tau)e^{-i\beta_n} - \sin(\tau-\alpha) \quad (42)$$

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$$\tau = \phi + \frac{\alpha N}{2} \tag{43}$$

The source of the imaginary part of Y_n (and Z_n) can be changed from being the roots of unity ($e^{-i\beta_n}$) to being a bin invariant characteristic complex value ($e^{-i\tau}$) with the application of (1) and some algebra.

$$Y_n = \sin(\tau)\left(\cos\beta_n - i \cdot \sin\beta_n\right) - \sin(\tau - \alpha) \tag{44}$$

$$Y_n = \sin(\tau)\cos\beta_n + \sin(\beta_n)(e^{-i\tau} - \cos\tau) - \sin(\tau - \alpha) \tag{45}$$

$$Y_n = \sin(\beta_n)e^{-i\tau} + \sin(\tau - \beta_n) - \sin(\tau - \alpha) \tag{46}$$

The trigonometric identity for the difference of two sines is:

$$\sin\theta_1 - \sin\theta_2 = 2\cos\left(\frac{\theta_1 + \theta_2}{2}\right)\sin\left(\frac{\theta_1 - \theta_2}{2}\right) \tag{47}$$

Applying (47) to the latter part of (46) puts it into a more reducible form.

$$Y_n = \sin(\beta_n)e^{-i\tau} + 2\cos(\tau - \frac{\alpha + \beta_n}{2})\sin(\frac{\alpha - \beta_n}{2}) \tag{48}$$

Now, putting Y_n from (48) back into (41), applying an angle addition formula, and simplifying leads to the following succinct alternative form of (23), the equation for pure tone leakage for non-integer frequencies.

$$Z_n = \frac{M}{N} \cdot \sin\left(\frac{\alpha N}{2}\right) \cdot (H_n e^{-i\tau} + \hat{H}_n) \tag{49}$$

$$H_n = \frac{-\sin\beta_n}{\cos\alpha - \cos\beta_n} \tag{50}$$

$$\hat{H}_n = \frac{\cos\tau}{\tan\left(\frac{\alpha + \beta_n}{2}\right)} + \sin\tau \tag{51}$$

This form of the equation gives some more insights into the nature of the DFT on a pure tone. The first thing to notice is that the $\sin(\alpha N / 2)$ term applies to all the bins equally. Also notice that it approaches zero as the frequency approaches an integer value and has its extreme values halfway between integer frequencies. Conversely, the H_n value gets extremely big when the frequency is near *n*. This explains why peaks are narrow when frequencies are near integer values and broader in between.

It is in between integer frequencies and at bins that are away from the frequency where $\hat{H}_n$ has more of a significant effect. When *n* is near 0, or when *n* is near N/2, the same as the frequency being near being a DC component or at the Nyquist frequency, $\hat{H}_n$ will dominate over H_n.

When *n=0* or *n=N/2,* H_n will equal zero. This shows that the imaginary parts at those DFT bins are always zero.

X. Inverting the Bin Value Function to Determine Frequency

In order to be able to decompose a signal consisting of multiple tones, one first has to be able to handle a signal consisting of a single tone. Given a DFT of a single tone, how do you figure out the frequency, magnitude, and phase? The answer (40) is simple and well known when the frequency is an integer value, but not so easy otherwise.

In non-integer frequency situations, the first step is to determine the frequency. It is possible to do this by considering three consecutive bin values and the corresponding bin value equations as a set of three equations with three unknowns and solving for it.

Introducing a temporary variable will make the process a little clearer.

$$T = e^{-i\frac{2\pi}{N}} = e^{-i\beta_1} = \cos\beta_1 - i \cdot \sin\beta_1 \approx 1 \tag{52}$$

Applying (22), (23), and (52) on three consecutive bins and cross multiplying gives these three equations:

$$2N\left(\cos\alpha - \cos\beta_{n-1}\right)Z_{n-1} = MUe^{-i\beta_n}\frac{1}{T} - M\hat{U} \tag{53}$$

$$2N\left(\cos\alpha - \cos\beta_n\right)Z_n = MUe^{-i\beta_n} - M\hat{U} \tag{54}$$

$$2N\left(\cos\alpha - \cos\beta_{n+1}\right)Z_{n+1} = MUe^{-i\beta_n}T - M\hat{U} \tag{55}$$

Subtracting (53) from (54) and (55) from (54), respectively, gives these two equations:

$$2N\left[\cos(\alpha)\left(Z_n - Z_{n-1}\right) - \left(\cos(\beta_n)Z_n - \cos(\beta_{n-1})Z_{n-1}\right)\right] = MUe^{-i\beta_n}\left(1 - \frac{1}{T}\right) \tag{56}$$

$$2N\left[\cos(\alpha)\left(Z_n - Z_{n+1}\right) - \left(\cos(\beta_n)Z_n - \cos(\beta_{n+1})Z_{n+1}\right)\right] = MUe^{-i\beta_n}\left(1 - T\right) \tag{57}$$

Dividing (57) by (56) gives:

$$\frac{\cos(\alpha)\left(Z_n - Z_{n+1}\right) - \cos(\beta_n)Z_n + \cos(\beta_{n+1})Z_{n+1}}{\cos(\alpha)\left(Z_n - Z_{n-1}\right) - \cos(\beta_n)Z_n + \cos(\beta_{n-1})Z_{n-1}} = -T \tag{58}$$

The $\cos(\alpha)$ term appears only twice and can be readily isolated.

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$$\cos\alpha = \frac{-T\cos(\beta_{n-1})Z_{n-1} + (T+1)\cos(\beta_n)Z_n - \cos(\beta_{n+1})Z_{n+1}}{-TZ_{n-1} + (T+1)Z_n - Z_{n+1}} \quad (59)$$

This formula has no problem with integer frequencies as long as the three bin set covers the nonzero bin. In fact, it can be seen to be true by inspection. If only one of the Z's is nonzero, the right side of the equation reduces to the corresponding cosine implying the frequency is that bin number.

In non-integer pure tone cases, any three bin set will do, even the ones at the Nyquist frequency or the DC component if the DFT is wrapped.

The $\cos\alpha$ term is, of course, a stand in for the frequency. The actual value of *f* can be recovered using (4) and the inverse cosine function.

$$f = \alpha \cdot \frac{N}{2\pi} = \cos^{-1}(\cos\alpha) \cdot \frac{N}{2\pi} \quad (60)$$

The inverse cosine function can return multiple values. Almost always, it is the principal value that is important. The other values, when interpreted as frequencies, are known as aliases.

XI. THE DISCRETE VONHANN LIKE WEIGHTING

What is really neat is that (58) can be put in a bilinear matrix product form.

$$\cos\alpha = \frac{\vec{V} \bullet \begin{vmatrix} \cos\beta_{n-1} & 0 & 0 \\ 0 & \cos\beta_n & 0 \\ 0 & 0 & \cos\beta_{n+1} \end{vmatrix} \bullet \begin{pmatrix} Z_{n-1} \\ Z_n \\ Z_{n+1} \end{pmatrix}}{\vec{V} \bullet \begin{pmatrix} Z_{n-1} \\ Z_n \\ Z_{n+1} \end{pmatrix}} \quad (61)$$

Where the complex vector $\vec{V}$ acts as a weighting vector. Since the vector is linear in the numerator and the denominator it can be rescaled arbitrarily. The formula for $\vec{V}$, rescaled for exponential symmetry, is:

$$\vec{V} = \left\langle -\sqrt{T} \quad \sqrt{T} + \frac{1}{\sqrt{T}} \quad \frac{-1}{\sqrt{T}} \right\rangle \quad (62)$$

This vector approaches the continuous VonHann [2] weighting as the number of samples increases.

$$\lim_{N\to\infty} \vec{V} = \langle -1 \quad 2 \quad -1 \rangle \quad (63)$$

The VonHann weighting (rescaled) is an approximation for the denominator in (60) which gets better as the number of samples increases. The exact corresponding window function [3] to (61) is unimportant because the actual DFT values are needed in the numerator. This means the complex weighted average has to be calculated. Fortunately, common intermediary values in the numerator and denominator allow them to be calculated together efficiently.

The limit vector also happens to be an alternating row from Pascal's triangle. [4]

In a similar manner to the previous section, a four bin patch can be used for the simultaneous equations. The four bin case allows a one degree of freedom choice in weighting. A linearly parameterized centered rescaled weighting is

$$\vec{W} = \left\langle -T \quad T+2 \quad -\frac{1}{T} - 2 \quad \frac{1}{T} \right\rangle \quad (64)$$

This vector also approaches a row from Pascal's triangle.

$$\lim_{N\to\infty} \vec{W} = \langle -1 \quad 3 \quad -3 \quad 1 \rangle \quad (65)$$

This pattern can be extended to longer vectors, but that makes little practical sense. The whole point of a DFT is that it concentrates the information about a tone into a narrow set of bins. When a tone is near an integer frequency value, the corresponding bin will be larger than both its neighbors and the three bin equation should be used. When a tone is in between integer frequencies, there will be two bins with significant magnitude, and the four bin equation should be used with the two larger bins in the center.

XII. ESTIMATING THE PHASE VALUE

The significance of the VonHann weighting being an alternating row of Pascal's triangle is that it completely zeroes out any linear sequence.

$$\langle -1 \quad 2 \quad -1 \rangle \begin{pmatrix} m-d \\ m \\ m+d \end{pmatrix} = 0 \quad (66)$$

Recognizing that the sequence of $\widehat{H}_{n-1}, \widehat{H}_n, \widehat{H}_{n+1}$ is nearly linear, applying the VonHann weighted sum to a set of DFT bins using (49) results in the following approximation:

$$\langle -1 \quad 2 \quad -1 \rangle \begin{pmatrix} Z_{n-1} \\ Z_n \\ Z_{n+1} \end{pmatrix} \approx K_n e^{-i\tau} \quad (67)$$

Where:

$$K_n = \frac{M}{N} \cdot \sin\left(\frac{\alpha N}{2}\right) \cdot (-H_{n-1} + 2H_n - H_{n+1}) \quad (68)$$

If $\cos\tau$ is zero the approximation becomes an exact equation. The approximation will also be more accurate when *n* is near *N/4* where H_n has the most significance.

The value of K_n could be used to estimate the value of *M*, but there is a better way detailed in the next section.

The usefulness of (67) is that the angle of the VonHann weighted sum can be used to get a very good estimate of τ.

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This estimate, along with the calculated frequency, can be used to obtain an estimate of ϕ.

$$\phi_{est} = \left(\tau_{est} - \frac{\alpha N}{2} \right) \mod 2\pi \tag{69}$$

XIII. Calculating Magnitude and Phase

It is possible to solve for the phase and magnitude using the equations in the previous section. However, they are actually approximations. In order to get exact results, a different approach needs to be taken. Thanks to the ability to synthesize a DFT of a pure tone using (23), (33), or (40), Linear Algebra techniques can be employed. The whole key to this approach is considering a three bin set of the DFT as a complex vector in a vector space. A four bin set can be used just as well.

In the integer frequency case, a shift (δ) in the phase represents a rotation in the DFT bin value. This is because

$$e^{-i(\phi-\delta)} = e^{-i\phi} \cdot e^{i\delta} \tag{70}$$

In the non-integer case this is approximately true at the bins nearest the frequency. This can be seen from (49).

The idea is to use sequences of parameters to generate a sequence of single basis vectors that converge to the DFT bin values. Although building a full basis is possible for an exact value, within the context of the algorithm this iterative approach will require fewer calculations overall for just as good numerical values.

The frequency is calculated and remains fixed. The initial value for the magnitude is set at one. The initial phase value can be estimated or set arbitrarily to any value. Then the iterations begin. A guess basis vector *G* for the three bins is generated using (23), (33), or (40).

The next task is to find the complex coefficient *c* for the best fit of vector *G* to the subset of DFT bins, represented by vector *Z*. The best bin to do this at is the one nearest the frequency. It will have the largest magnitude and is known as a peak bin. Let *P* be that bin number.

$$Z = \left\langle \begin{matrix} Z_{P-1} \\ Z_P \\ Z_{P+1} \end{matrix} \right\rangle = cG \tag{71}$$

To find the best fit coefficient, dot both sides by the complex conjugate of G, called $\overline{G}$, and then solve for *c*.

$$c = \frac{\overline{G} \bullet Z}{\overline{G} \bullet G} = \|c\| \cdot e^{i\delta} \tag{72}$$

The value of the coefficient *c* represents the adjustment to be made to the guessed parameter values. The magnitude of *c* is the adjustment to the magnitude. The angle of c (δ), usually $\tan^{-1}\left(\frac{\text{Im}(c)}{\text{Re}(c)}\right)$, is the adjustment to the phase angle and is subtracted from ϕ like in (70). Once the magnitude and phase guesses are adjusted another iteration can be made. A new guess vector is generated, a new coefficient calculated, and new adjustments made. Convergence can be measured by how close *c* gets to unity.

XIV. The Decomposition Algorithm

All the needed tools are now in place for decomposing a signal consisting of multiple tones. This simplified algorithm assumes that the frequencies are well spaced and that the tones are steady through the entire time frame.

The process starts by taking the DFT of the signal and calculating a set of bins. Next, the bins are scanned for the presence of tones. For each tone there will be a peak bin.

At the first peak, the DFT is treated like the single tone case and an initial frequency with one pass of magnitude and phase calculations are done. The calculated values will be close, but not accurate. This is due to the interfering presence of the other tones as well as the approximate nature of the magnitude and phase calculations.

At the second peak, before the parameters are calculated, the effect of the first tone is significantly removed. This is done by generating a small patch of DFT using the first tone's estimated parameters and subtracting it from the DFT bins to be processed.

At the third peak, the effects of the first two tones are removed, or largely removed, and then the tone is processed like the single tone case. At every subsequent peak, the same process occurs. The previous tones' effects are subtracted out and the peak is treated like a single tone case.

When the last peak is reached, and all the other tones' effects subtracted away, it is very close to being a single tone case and a very good answer is obtained.

At this point, the iterations begin. The first tone gets processed again. The DFT bins are copied, but this time the effects of the other tones can be subtracted away. The parameters are recalculated resulting in a much more accurate estimate. Each tone is then processed in a similar manner. The last tone in the second pass is now going to be very accurate.

As many passes as desired can be calculated, but for any practical purposes, three or four should be sufficient. If the peaks are farther apart, the algorithm converges faster. For a signal matching the assumptions and given sufficient iterations, the decomposition will be complete. Otherwise, the result will be a very good best fit.

Appendix I

Pseudo-code for decomposition algorithm.

Top Level

Step 1. Take the DFT
Step 2. Initial Pass Gathering Tones
Step 3. Iterate Until Converged

============================

```
Initial Pass Gathering Tones

  For Each Bin
     If Bin Is Larger Than Neighbors
        Add Tone to List
        Process Tone
     End If
  Next Bin
==============================
Iterate Until Converged

  Do Until Close Enough
     For Each of the Tones
        Process Tone
     Next Tone
  Loop

==============================
Process Tone

  Step 1. Grab Patch of DFT
  Step 2. Remove Effects of Other Tones
  Step 3. Calculate Frequency From Patch
  Step 4. Synthesize a Patch sized Guess Vector
  Step 5. Calculate Adjustment Coefficient
  Step 6. Apply Magnitude Adjustment
  Step 7. Apply Phase Adjustment

==============================
Remove Effects of Other Tones

  For Each of the Other Tones
     Synthesize a Patch sized Effect Vector
     Subtract Effect Vector from Patch
  Next Other Tone

==============================
```

Appendix II

Source code for DFT synthesis.

```
#include <math.h>
#include <stdio.h>

int main( int argc, char *argv[] )
{
//--- Set the Tone Parameters
    double Mag  = 1.000;
    double Phi  = 1.234;
    double Freq = 5.678;

//--- Set the DFT Parameters
    double Pi = 3.14159265358979323;
    double N  = 32;
    int n_first = 0;
    int n_last  = N/2 - 1;

//--- Calculate the Parameter Dependent Terms
    double Alpha, Gamma, U, U_hat;

    Alpha = Freq * 2*Pi/N;
    Gamma = Phi + Alpha * N;

    U = cos( Gamma )  - cos( Phi );

    U_hat = cos( Gamma - Alpha )
          - cos( Phi   - Alpha );
//--- Synthesize the DFT
    double Beta_n, Sigma, Rho, D_n;
    double Re_Z_n, Im_Z_n, tmp;

    for( int n = n_first; n < n_last; n++ )
    {
      Beta_n = n * 2*Pi/N;

      D_n =  2 * (  cos( Alpha ) - cos( Beta_n )  );

      if( D_n == 0.0 )
      {
         Re_Z_n =  Mag/2 * cos( Phi );
         Im_Z_n = -Mag/2 * sin( Phi );
      }
      else if( fabs( D_n ) < 0.0001 )
      {
         Sigma  = Alpha - Beta_n;
         Rho    = Phi + Sigma * N/2.0;

         tmp = Mag * sin( N * Sigma/2.0 )
       / ( 2.0 * N * sin( Sigma/2.0 )
          * sin( Beta_n + Sigma/2.0 ) );

         Re_Z_n = tmp
        * ( sin( Rho ) * cos( Beta_n )
          - sin( Rho - Sigma - Beta_n ) );

         Im_Z_n = tmp
        * ( sin( Rho ) * -sin( Beta_n ) );
      }
      else
      {
         Re_Z_n = Mag/N/D_n
        * ( U * cos( Beta_n ) - U_hat );

         Im_Z_n = Mag/N/D_n
        * ( -U * sin( Beta_n ) );
      }
      printf( "%3d  %14.11f %14.11f\n",
       n, Re_Z_n, Im_Z_n );
    }

//--- Exit
    return 0;
}
```

References

[1] http://en.wikipedia.org/wiki/Discrete_Fourier_transform. "Note that the normalization factor multiplying the DFT and IDFT (here 1 and 1/N) and the signs of the exponents are merely conventions, and differ in some treatments."

[2] fredric j. harris, "On the Use of Windows for Harmonic Analysis with the Discrete Fourier Transform," in Proceedings of the IEEE, Vol 66 No 1, January 1978, p. 62

[3] $w_k = 2\left[\cos\left(\frac{\pi}{N}\right) - \cos\left((k+\frac{1}{2})\frac{2\pi}{N}\right)\right] = 4\sin\left((k+1)\frac{\pi}{N}\right)\sin\left(k\frac{\pi}{N}\right)$

[4] https://oeis.org/wiki/Pascal_triangle, "The alternating sign sum for the nth row gives the powers of 0"