Nondifferentiable Convex Optimization: An Algorithm Using Moreau-Yosida Regularization

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Abstract—**In this paper we present an algorithm for minimization of a nondifferentiable proper closed convex function. Using the second order Dini upper directional derivative of the Moreau-Yosida regularization of the objective function we make a quadratic approximation. The purpose of the paper is to establish that the sequence of points generated by the algorithm has an accumulation point which satisfies the first order necessary and sufficient conditions. A convergence proof is given, as well as an estimate of the rate of convergence.**

Index Terms— Moreau-Yosida regularization, non-smooth convex optimization, second order Dini upper directional derivative.

I. Introduction

The following minimization problem is considered:

$$\min_{x\in R^n} f(x), \qquad (1)$$

where $f: R^n \to R \cup \{+\infty\}$ is a convex and not necessary differentiable function with a nonempty set X^* of minima.

For nonsmooth programs, many approaches have been presented so far and they are often restricted to the convex unconstrained case. It is reasonable because a constrained problem can be easily transformed to an unconstrained problem using a distance function. In general, the various approaches are based on combinations of the following methods: subgradient methods; bundle techniques and the Moreau-Yosida regularization.

For a function f it is very important that its Moreau-Yosida regularization is a new function which has the same set of minima as f and is differentiable with Lipschitz continuous gradient, even when f is not differentiable. In [12, 13, 17] the second order properties of the Moreau-Yosida regularization of a given function f are considered.

Having in mind that the Moreau-Yosida regularization of a proper closed convex function is an LC^1 function, we present an optimization algorithm (using the second order Dini upper directional derivative (described in [1,2]) based on the results from [3]. That is the main idea of this paper.

We shall present an iterative algorithm for finding an optimal solution of problem (1) by generating the sequence of points $\{x_k\}$ of the following form:

$$x_{k+1} = x_k + \alpha_k s_k + \alpha_k{}^2 d_k \quad k = 0,1,\ldots, d_k \neq 0 \qquad (2)$$

where the step-size α_k and the directional vectors s_k and d_k are defined by the particular algorithms.

Paper is organized as follows: in the second section some basic theoretical preliminaries are given; in the third section the Moreau-Yosida regularization and its properties are described; in the fourth section the definition of the second order Dini upper directional derivative and the basic properties are given; in the fifth section the semi-smooth functions and conditions for their minimization are described. Finally in the sixth section a model algorithm is suggested and its convergence is proved, and an estimate rate of its convergence is given, too.

II. Theoretical Preliminaries

Throughout the paper we will use the following notation. A vector s refers to a column vector, and ∇ denotes the gradient operator $\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right)^T$. The Euclidean product is denoted by $\langle \cdot,\cdot \rangle$ and $\|\cdot\|$ is the associated norm. For a given symmetric positive definite linear operator M we set $\langle \cdot,\cdot \rangle_M := \langle M \cdot,\cdot \rangle$; hence it is shortly denoted by $\|x\|_M^2 := \langle x, x \rangle_M$. The smallest and the largest eigenvalue of M we denote by λ and Λ respectively.

The *domain* of a given function $f: R^n \to R \cup \{+\infty\}$ is the set $dom(f) = \{x \in R^n \mid f(x) < +\infty\}$. We say f is proper if its domain is nonempty. The point $x^* = \arg\min_{x\in R^n} f(x)$ refers to the minimum point of a given function $f: R^n \to R \cup \{+\infty\}$.

A vector $g \in R^n$ is said to be a *subgradient* of a given proper convex function $f: R^n \to R \cup \{+\infty\}$ at a point

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$x \in R^n$ if the inequality $f(z) \geq f(x) + g^T \cdot (z - x)$ holds for all $z \in R^n$. The set of all subgradients of $f(x)$ at the point x, called the *subdifferential* at the point x, is denoted by $\partial f(x)$. The subdifferential $\partial f(x)$ is a nonempty set if and only if $x \in dom(f)$. The condition $0 \in \partial f(x)$ is a first order necessary and sufficient condition for a global minimizer for the convex function f at the point $x \in R^n$ (see in [14,15]).

For a convex function f it follows that $f(x) = \max_{z \in R^n} \{ f(z) + g^T (x - z) \}$ holds, where $g \in \partial f(z)$ (see in [10]).

The concept of the subgradient is a simple generalization of the gradient for nondifferentiable convex functions.

The *directional derivative* of a real function f defined on R^n at the point $x' \in R^n$ in the direction $s \in R^n$, denoted by $f'(x', s)$, is $f'(x', s) = \lim_{t \downarrow 0} \frac{f(x' + t \cdot s) - f(x')}{t}$ when this limit exists. For a real convex function a directional derivative at the point $x' \in R^n$ in the direction s exists in any direction $s \in R^n$ (see in [2]).

At the end of this section we recall the definition of an LC^1 function.

Definition 1. A real function f defined on R^n is an LC^1 function on the open set $D \subseteq R^n$ if it is continuously differentiable and its gradient ∇f is locally Lipschitzian, i.e. $\| \nabla f(x) - \nabla f(y) \| \leq L \| x - y \|$ for $x, y \in D$ holds for some $L > 0$.

III. The Moreau-Yosida Regularization

Definition 2. Let $f : R^n \to R \cup \{+\infty\}$ be a proper closed convex function. The *Moreau-Yosida regularization* of a given function f, associated to the metric defined by M, denoted by F, is defined as

$$F(x) := \min_{y \in R^n} \left\{ f(y) + \frac{1}{2} \| y - x \|_M^2 \right\}. \tag{3}$$

The above function is an *infimal convolution*. In [10] it is proved that the infimal convolution of a convex function is also a convex function. Hence the function defined by (3) is a convex function and has the same set of minima as the function f (see in [6]), so the motivation of the study of Moreau-Yosida regularization is due to the fact that $\min_{x \in R^n} f(x)$ is equal to $\min_{x \in R^n} F(x)$.

The minimum point $p(x)$ of the function (3), i. e.:

$$p(x) := \underset{y \in R^n}{argmin} \left\{ f(y) + \frac{1}{2} \| y - x \|_M^2 \right\} \tag{4}$$

is called the *proximal point* of x.

In [6] it is proved that the function F defined by (3) is always differentiable.

The first order regularity of F is well known: without any further assumptions, F has a Lipschitzian gradient on the whole space R^n. More precisely,

$$\| \nabla F(x_1) - \nabla F(x_2) \|^2 \leq \Lambda \langle \nabla F(x_1) - \nabla F(x_2), x_1 - x_2 \rangle$$

holds for all $x_1, x_2 \in R^n$ (see in [12]), where $\nabla F(x) = M(x - p(x)) \in \partial f(p(x))$ and $p(x)$ is the unique minimum in (4). So, according to above consideration and Definition 1, we conclude that F is an LC^1 function.

Lemma 1. [13]: The following statements are equivalent:
(i) x minimizes f; (ii) $p(x) = x$;
(iii) $\nabla F(x) = 0$; (iv) x minimizes F;
(v) $f(p(x)) = f(x)$; (vi) $F(x) = f(x)$.

IV. Dini Second Upper Directional Derivative

We shall give some preliminaries that will be used in the remainder of the paper.

Definition 3. The *second order Dini upper directional derivative* of the function $f \in LC^1$ at the point $x \in R^n$ in the direction $d \in R^n$ is defined to be

$$f''_D(x, d) = \limsup_{\alpha \downarrow 0} \frac{[\nabla f(x + \alpha d) - \nabla f(x)]^T \cdot d}{\alpha}.$$

If ∇f is directionally differentiable at x_k, we have

$$f''_D(x_k, d) = f''(x_k, d) = \lim_{\alpha \downarrow 0} \frac{[\nabla f(x + \alpha d) - \nabla f(x)]^T \cdot d}{\alpha}$$

for all $d \in R^n$.

Since the Moreau-Yosida regularization of a proper closed convex function f is an LC^1 function, we can consider its second order Dini upper directional derivative at the point $x \in R^n$ in the direction $d \in R^n$, i.e.:

$$F''_D(x, d) = \limsup_{\alpha \downarrow 0} \frac{g_1 - g_2}{\alpha} d,$$

where $g_1 \in \partial f(p(x+\alpha d)), g_2 \in \partial f(p(x))$, and $F(x)$ is defined by (3) for $M = I$.

Lemma 2.[2]: Let $f : R^n \to R$ be a closed convex proper function and F is its Moreau –Yosida regularization for $M = I$. Then the next statements are valid.

(i) $F''_D(x_k, kd) = k^2 F''_D(x_k, d)$

(ii) $F''_D(x_k, d_1 + d_2) \le 2(F''_D(x_k, d_1) + F''_D(x_k, d_2))$

(iii) $|F''_D(x_k, d)| \le K \cdot \|d\|^2$, where K is some constant.

V. Semi-Smooth Functions And Optimality Conditions

Definition 4. A function $\nabla F : R^n \to R^n$ is said to be *semi-smooth* at the point $x \in R^n$ if ∇F is locally Lipschitzian at $x \in R^n$ and the limit $\lim\limits_{\substack{h \to d \\ \lambda \downarrow 0}} \{Vh\}$, $V \in \partial^2 F(x + \lambda h)$ exists for any $d \in R^n$.

Note that for a closed convex proper function, the gradient of its Moreau-Yosida regularization is a semi-smooth function.

Lemma 3. *[16]:* If $\nabla F : R^n \to R^n$ is semi-smooth at the point $x \in R^n$ then ∇F is directionally differentiable at $x \in R^n$ and for any $V \in \partial^2 F(x + h), h \to 0$ we have: $Vh - (\nabla F)'(x, h) = o(\|h\|)$. Similarly we have that $h^T V h - F''(x, h) = o(\|h\|^2)$.

Lemma 4. [4]: Let $f : R^n \to R$ be a proper closed convex function and let F be its Moreau-Yosida regularization. So, if $x \in R^n$ is solution of the problem (1) then $F'(x, d) = 0$ and $F''_D(x, d) \ge 0$ for all $d \in R^n$.

Lemma 5. [4]: Let $f : R^n \to R$ be a proper closed convex function, F its Moreau-Yosida regularization, and x a point from R^n. If $F'(x, d) = 0$ and $F''_D(x, d) > 0$ for all $d \in R^n$, then $x \in R^n$ is a strict local minimizer of the problem (1).

VI. A Model Algorithm

In this section an algorithm for solving the problem (1) is introduced. We suppose that at each $x \in R^n$ it is possible to compute $f(x), F(x)$, $\nabla F(x)$ and $F''_D(x, d)$ for a given $d \in R^n$.

At the k-th iteration we consider the following problem

$$\min_{d \in R^n} \Phi_k(d),\ \Phi_k(d) = \nabla F(x_k)^T d + \frac{1}{2} F''_D(x_k, d), \quad (5)$$

where $F''_D(x_k, d)$ stands for the second order Dini upper directional derivative at x_k in the direction d. Note that if Λ is a Lipschitzian constant for F, it is also a Lipschitzian constant for ∇F. The function $\Phi_k(d)$ is called an iteration function. It is easy to see that $\Phi_k(0) = 0$ and $\Phi_k(d)$ is Lipschitzian on R^n. We generate the sequence $\{x_k\}$ of the form

$$x_{k+1} = x_k + \alpha_k s_k + \alpha_k^2 d_k, s_k \ne 0, d_k \ne 0,$$

where the step-size α_k and the direction vectors s_k and d_k are defined by particular algorithms.

For a given $q \in (0,1)$ the step-size α_k is a number satisfying $\alpha_k = q^{i(k)}$, where $i(k)$ is the smallest integer from $\{0,1,2,...\}$ such that

$$\begin{aligned} &F(x_{k+1}) - F(x_k) \le \\ &\le \sigma\left(q^{i(k)} \nabla F(x_k)^T s_k - \frac{1}{2} q^{4i(k)} (F''_D(x_k, d_k)) \right) \end{aligned} \quad (6)$$

and where $\sigma \in (0,1)$ is a preassigned constant, and $x_0 \in R^n$ is a given point.

We make the following assumptions.

A1. Suppose that $c_1\|d\|^2 \le F''_D(x_k, d) \le c_2\|d\|^2$ hold for some c_1 and c_2 such that $0 < c_1 < c_2$ for every $d \in R^n$.

A2. $\|d_k\| = 1,\ \|s_k\| = 1,\ k = 0,1,2,...$

A3. There exists a value $\beta > 0$ such that $\nabla F(x_k)^T s_k \le -\beta \|\nabla F(x_k)\| \cdot \|s_k\|, k = 0,1,2,...$

Lemma 6. [4]: Under the assumption A1 the function $\Phi_k(\cdot)$ is coercive.

Remark. Coercivity of the function Φ_k assures that the optimal solution of the problem (5) exists (see in [16]). It also means that, under the assumption A1 the direction sequence $\{d_k\}$ is a bounded sequence on R^n (proof is analogous to the proof in [16]).

Proposition 1. [3]: If the Moreau-Yosida regularization $F(\cdot)$ of the proper closed convex function $f(\cdot)$ satisfies assumption A1 then :

(i) the function $F(\cdot)$ is uniformly and, hence, strictly convex;

(ii) the level set $L(x_0)=\{x \in R^n : F(x) \le F(x_0)\}$ is a compact convex set, and

(iii) there exists a unique point x^* such that $F(x^*) = \min_{x \in L(x_0)} F(x)$.

Lemma 7. [4]: The following statements are equivalent:

(i) $d = 0$ is the globally optimal solution of the problem (5)

(ii) 0 is the optimum of the objective function in (5)

(iii) the corresponding x_k is such that $0 \in \partial f(x_k)$

Convergence theorem. Suppose that f is a proper closed convex function and its Moreau-Yosida regularization F satisfies assumptions A1, A2 and A3. Then for any initial point $x_0 \in R^n, x_k \to x_\infty$, as $k \to +\infty$, where x_∞ is a unique minimal point of the function f.

Proof. If $d_k \ne 0$ is a solution of (5), it follows that $\Phi_k(d_k) \le 0 = \Phi_k(0)$. Consequently, we have by assumption A1 that

$$\nabla F(x_k)^T d_k \le -\frac{1}{2} F''_D(x_k, d_k) \le -\frac{1}{2} c_1 \|d_k\|^2 < 0. \qquad (7)$$

From the above inequality it follows that the vector d_k is a descent direction at x_k. By (6) and assumption A1 we get

$$\begin{aligned} F(x_{k+1}) - F(x_k) &\le \sigma\left(q^{i(k)} \nabla F(x_k)^T s_k - \frac{1}{2} q^{4i(k)} F''_D(x_k, d_k) \right) \\ &\le \sigma\left(-\beta \|\nabla F(x_k)\| \cdot \|s_k\| - \frac{1}{2} q^{4i(k)} c_1 \|d_k\|^2 \right) < 0 \end{aligned} \qquad (8)$$

for every $d_k \ne 0$. Hence the sequence $\{F(x_k)\}$ has the descent property, and, consequently, the sequence $\{x_k\} \subset L(x_0)$. Since $L(x_0)$ is by the Proposition 1 a compact convex set, it follows that the sequence $\{x_k\}$ is bounded. Therefore there exist accumulation points of the sequence $\{x_k\}$.

Since ∇F is continuous, then, if $\nabla F(x_k) \to 0, k \to +\infty$ then it follows that every accumulation point x_∞ of the sequence $\{x_k\}$ satisfies $\nabla F(x_\infty) = 0$. Since F is (by the Proposition 1) strictly convex, there exists a unique point $x_\infty \in L(x_0)$ such that $\nabla F(x_\infty) = 0$. Hence, the sequence $\{x_k\}$ has a unique limit point x_∞ and it is a global minimizer of F and by Lemma 1 it is a global minimizer of the function f.

Therefore we have to prove that $\nabla F(x_k) \to 0, k \to +\infty$. Let K_1 be a set of indices such that $\lim_{k \in K_1} x_k = x_\infty$. Then there are two cases to consider:

a) The set of indices $\{i(k)\}$ for $k \in K_1$, is uniformly bounded above by a number I. From A2, A3 and (6) it follows that:

$$\begin{aligned} &F(x_{k+1}) - F(x_k) \le \\ &\le \sigma\left(q^{i(k)} \nabla F(x_k)^T s_k - \frac{1}{2} q^{4i(k)} F''_D(x_k, d_k) \right) \\ &\le \sigma\left(q^I \nabla F(x_k)^T s_k - \frac{1}{2} q^{4I} F''_D(x_k, d_k) \right) \\ &\le \sigma q^I \beta \|\nabla F(x_k)\| \cdot \|s_k\| - \frac{\sigma}{2} q^{4I} F''_D(x_k, d_k) \\ &\le -\beta \sigma q^I \|\nabla F(x_k)\| - \frac{1}{2} q^{4I} \sigma F''_D(x_k, d_k). \end{aligned}$$

Hence, it follows that

$$\begin{aligned} &F(x_k) - F(x_{k+1}) \ge \\ &\ge \beta \sigma q^I \|\nabla F(x)\| + \frac{1}{2} q^{4I} \sigma F''_D(x_k, d_k). \end{aligned} \qquad (9)$$

Since $\{F(x_k)\}$ is bounded bellow and $F(x_{k+1}) - F(x_k) \to 0$ as $k \to \infty, k \in K_1$, from (9) it follows that $\|\nabla F(x_k)\| \to 0$ and $F''_D(x_k; d_k) \to 0$, $k \to \infty$ $k \in K_1$ i.e. x_∞ is a stationary point of the objective function F, i.e. $\nabla F(x_\infty) = 0$. From Lemma 1 it follows that x_∞ is a unique optimal point of the function f.

b) There is a subset $K_2 \subset K_1$ such that $\lim_{k \to \infty} i(k) = +\infty$. By the definition of $i(k)$, we have for $k \in K_2$ that

$$\begin{aligned} &F(x_{k+1}) - F(x_k) > \\ &> \sigma\left(q^{i(k)-1} \nabla F(x_k)^T s_k - \frac{1}{2} q^{4i(k)-4} F''_D(x_k, d_k) \right). \end{aligned} \qquad (10).$$

By Definitin 3, A1 and Lemma 2 we have

$$\begin{aligned} &F(x_{k+1}) - F(x_k) = \\ &= q^{i(k)-1} \nabla F(x_k)^T s_k + q^{2i(k)-2} \nabla F^T(x_k) d_k + \\ &+ \frac{1}{2} F''_D\left(x_k, q^{i(k)-1} s_k + q^{2i(k)-2} d_k\right) + o\left(q^{2i(k)-2}\right) \end{aligned}$$

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$$\le q^{i(k)-1}\nabla F(x_k)^T s_k + q^{2i(k)-2}\nabla F^T(x_k)d_k + + F''_D(x_k, q^{i(k)-1}s_k) + F''_D(x_k, q^{2i(k)-2}d_k) + o(q^{2i(k)-2})$$

$$= q^{i(k)-1}\nabla F(x_k)^T s_k + q^{2i(k)-2}\nabla F^T(x_k)d_k + + q^{2i(k)-2}F''_D(x_k, s_k) + q^{4i(k)-4}F''_D(x_k, d_k) + o(q^{2i(k)-2})$$

$$\le q^{i(k)-1}\nabla F(x_k)^T s_k + q^{2i(k)-2}\nabla F^T(x_k)d_k + + c_2 q^{2i(k)-2}\|s_k\|^2 + c_2 q^{4i(k)-4}\|d_k\|^2 + o(q^{2i(k)-2}).$$

Hence, from (10) it follows that:

$$\sigma\left(q^{i(k)-1}\nabla F(x_k)^T s_k - \frac{1}{2}q^{4i(k)-4}F''_D(x_k, d_k)\right)$$
$$\le q^{i(k)-1}\nabla F^T(x_k)s_k + q^{2i(k)-2}\nabla F^T(x_k)d_k + c_2 q^{2i(k)-2}\|s_k\|^2 + c_2 q^{4i(k)-4}\|d_k\|^2 + o(q^{2i(k)-2}).$$

Accumulating all terms of order higer than $o(q^{2i(k)-2})$ into the $o(q^{2i(k)-2})$ (by assumption A2), and using the fact that $\nabla F^T(x_k)d_k \le 0$ from the last inequality it follows that:

$$\sigma q^{i(k)-1}\nabla F(x_k)^T s_k < q^{i(k)-1}\nabla F^T(x_k)s_k + + c_2 q^{2i(k)-2}\|s_k\|^2 + o(q^{2i(k)-2}),$$

i.e.

$$(1-\sigma)q^{i(k)-1}\nabla F^T(x_k)s_k + c_2 q^{2i(k)-2}\|s_k\|^2 + o(q^{2i(k)-2}) > 0.$$

Hence , dividing by $c_2 \cdot q^{i(k)-1}$, by A2 and A3 it follows that

$$q^{i(k)-1} > \frac{\sigma-1}{c_2}\nabla F^T(x_k)s_k + \frac{o(q^{i(k)-1})}{c_2}$$
$$\ge \beta\frac{1-\sigma}{c_2}\|\nabla F(x_k)\| + \frac{o(q^{i(k)-1})}{c_2}.$$

Since $q^{i(k)-1} \to 0$ as $k \to \infty, k \in K_2$, it follows that $\|\nabla F(x_k)\| \to 0$ as $k \to \infty, k \in K_2$.

In order to have a finite value $i(k)$, it is sufficient that s_k and d_k have descent properties, i.e. $\nabla F(x_k)^T s_k < 0$ and $\nabla F(x_k)^T d_k < 0$ whenever $\nabla F(x_k) \ne 0$.

The first relation follows from A3 and the second relation follows from (7).

At a saddle point the relation (6) becomes

$$F(x_{k+1}) - F(x_k) \le -\frac{\sigma}{2}q^{4i(k)-4}F''_D(x_k, d_k). \tag{11}$$

In the case $d_k \ne 0$ by Lemma 7 and hence, by assumption A1 it follows that $F''_D(x_k, d_k) > 0$; so (11) can be clearly satisfied.

Convergence rate theorem. Under the assumptions of the previous theorem we have that the following estimate holds for the sequence $\{x_k\}$ generated by the algorithm.

$$F(x_n) - F(x_\infty) \le \mu_0\left[1 + \mu_0\frac{1}{\eta^2}\sum_{k=0}^{n-1}\frac{F(x_k) - F(x_{k+1})}{\|\nabla F(x_k)\|^2}\right]^{-1}$$

for $n = 1,2,3,\ldots$, where $\mu_0 = F(x_0) - F(x_\infty)$ and $diam L(x_0) = \eta < +\infty$ (since by Proposition 1 it follows that $L(x_0)$ is bounded).

Proof . The proof directly follows from the Theorem 9.2, page 167, in [11].

VII. Conclusion

The Moreau-Yosida regularization is a powerful tool for smoothing nondifferentiable functions. It allows us to transform the solving an NDO problem into the solving an LC^1 optimization problem using the properties of this regularization.

To our knowledge this is a new approach to solving NDO problems, and in some sense it is close to the proximal quasi Newton algorithm.

The algorithm presented in this paper is based on the algorithm from [3]. This method uses minimization along a plane defined by the vectors s_k and d_k to generate a new iterative point at each iteration. Relating to the algorithm in [3], the presented algorithm is defined and converges for noonsmooth convex function.

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