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Stability of Numerical Schemes for the Delay Blowflies Equation

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***Abstract*—In this paper the delay Nicholson's blowflies equation has been solved by the θ-mathod. The purpose is to analyse stability of the numerical schemes using the linearisation method. Our obtained results show sufficient conditions in which the numerical solutions are stable. Moreover, we also show some nonlinear stability in the case that $\theta = 1$, and show that it gives a well-defined discrete dynamical system on the stability.**

***Index Terms*—delay differential equation, Nicholson's blowflies equation, θ-methods, stability analysis.**

I. Introduction

THE θ-method is an iterative numerical method widely used to approximate solutions of ODEs. In this paper, we use it to find numerical solutions of the delay differential equation:

$$N'(t) = -\delta N(t) + pN(t-\tau)e^{-aN(t-\tau)}, \quad t > 0, \quad (1)$$

where $p > \delta > 0$, and study their numerical stability. Many authors dealt with various kind of stability for numerical methods for DDEs, only a few dealt specifically with the θ-method for (1). We aim to analyse the numerical stability and the long-term behaviour of the numerical solutions of the θ-method for (1). First, we discuss the θ-method for ODEs and how it can be applied to DDEs. Some previous results on the stability analysis of the θ-method are also provided in Section II. Next, in Section III, we apply the θ-method to approximate solutions of (1). Our results are presented in Section IV. The steady-states of the DDE and of the θ-method are the same, but their stability can vary. Under the condition $p > \delta > 0$, we show in Theorem 1 that the zero equilibrium is unstable for the θ-method, as it is for the DDE. For the positive equilibrium, we provide sufficient conditions for the numerical solutions to be asymptotically stable in Theorem 2 and Corollary 3. We show that the purely implicit method is asymptotically stable independently of the numerical step-size h, unlike the Euler method. In addition, Theorem 4 deals with the nonlinear stability analysis of the implicit numerical scheme. Finally, in the last section, we present numerical experiments to support our theorems.

II. The θ-Method for DDEs and Previous Results

We start here by recalling some properties of the θ-method for ODEs. Consider the IVP:

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0, \qquad t > t_0. \quad (2)$$

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The general θ-method solving (2) is in the form

$$y_{n+1} = y_n + h[(1-\theta)f(t_n, y_n) + \theta f(t_{n+1}, y_{n+1})], \quad (3)$$

where $n = 0, 1, 2, \ldots,$ and h is the numerical step-size and $\theta \in [0, 1]$ is a parameter. Here, $y_n = y(t_n)$ where $t_n = t_0 + nh$. The cases $\theta = 0, \theta = 1/2$, and $\theta = 1$ correspond to the *(explicit) Euler* method, the *Trapezoidal* scheme and the *implicit Euler* method, respectively. Moreover, the cases $\theta = 0$ and $\theta = 1$ are called *one-leg θ-methods*, [1]. Throughout this paper, we call the explicit Euler method, the Euler method. In general monograph on numerical analysis, such as [2]-[3], it is shown that the θ-method is convergent for every $\theta \in [0, 1]$. In addition, the method is of order two for $\theta = 1/2$, otherwise it is of order one.

Next, consider the nonlinear DDE:

$$y'(t) = f(t, y(t), y(t-\tau)), \quad t > 0, \quad (4)$$

with the initial data

$$y(t) = \varphi(t), \quad -\tau \le t \le 0, \quad (5)$$

where $f : [0, \infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $y : \mathbb{R} \to \mathbb{R}$, $\tau \ge 0$, and $\varphi(t) \in \mathbb{R}$ is a given initial function. Let $h > 0$ be the numerical step-size, and k be the smallest integer greater than or equal to τ/h. So, the delay τ can be written as

$$\tau = h(k - \xi), \quad (6)$$

where $0 \le \xi < 1$. Then (6) gives the relation

$$t_n - \tau = t_{n-k} + \xi h,$$

which is held for the points $t_n = t_0 + nh$, $n \ge k$. Approximating the delayed argument in (4) with a linear interpolation, the θ-method for (4) is:

$$y_{n+1} = y_n + h(1-\theta)f(t_n, y_n, y_n^\tau) + h\theta f(t_{n+1}, y_{n+1}, y_{n+1}^\tau), \quad (7)$$

where $0 \le \theta \le 1$, $y_n = y(t_n)$, and

$$y_n^\tau = (1-\xi)y_{n-k} + \xi y_{n-k+1} \quad (8)$$

is an approximated value of $y(t_n - \tau)$.

For the simplest case, we set $\xi = 0$. Then $h = \tau/k \in \mathbb{Z}^+$, where $k \in \mathbb{Z}^+$. Hence, y_n^τ in (8) becomes $y_n^\tau = y_{n-k}$. In this case the θ-method (7) becomes

$$\begin{aligned} y_{n+1} \;&=\; y_n + h(1-\theta)f(t_n, y_n, y_{n-k}) \\ &\quad + h\theta f(t_{n+1}, y_{n+1}, y_{n-k+1}), \end{aligned} \quad (9)$$

which is the the *general θ-method* for a DDEs with constant delay. Calvo and Grande [4] proved that the order properties for ODEs extend to the case of DDEs with constant delays. So the θ-method (9) has order two if $\theta = 1/2$, otherwise it is order one. Moreover it is also convergent for every $\theta \in [0, 1]$.

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The stability analysis of the θ-method for DDEs has been investigated by many authors. Barwell [5] studied the numerical method for the linear DDE:

$$y'(t) = \alpha y(t) + \beta y(t-\tau), \quad t > 0, \tag{10}$$

where α and β are complex numbers. He also introduced the concepts of P-stability and GP-stability for DDEs, linked to the concept of A-stability in ODEs, to explain the asymptotically stability regions for the numerical methods. Later, papers on the stability of the θ-method for (10) have been published, for examples [1], [4], [6], [7], dealing with its linear stability. In addition, Torelli [8] has introduced the PN-stability and GPN-stability for the test equation:

$$y'(t) = \alpha(t)y(t) + \beta(t)y(t-\tau), \quad t > t_0, \tag{11}$$

which is a linear nonautonomous equation. His results in [8] and [9] give sufficient conditions for the θ-method to be PN and GPN-stable and provided the definitions of RN-stability and GRN-stability in [8] for the general nonlinear DDE:

$$y'(t) = f(t, y(t), y(t-\tau)),\ t > t_0, \quad y(t) = \varphi(t),\ t \le t_0, \tag{12}$$

Tian [10] also studied the RN-stability and GRN-stability of the θ-method for a class of (12).

III. General Concepts of θ-Methods for the Nicholson's Blowflies Equation

According to (9), the θ-method for the blowflies equation (1) is

$$N_{n+1} = N_n + h(1-\theta)f(N_n, N_{n-k}) + h\theta f(N_{n+1}, N_{n-k+1}), \tag{13}$$

where $\theta \in [0,1]$ and $f(u,v) = -\delta u + pve^{-av}$. Then, after rearranging, the numerical scheme for (1) can be written in explicit form as

$$\begin{aligned} N_{n+1} &= \frac{1-\delta h(1-\theta)}{1+\delta h\theta}N_n + \frac{hp(1-\theta)}{1+\delta h\theta}N_{n-k}e^{-aN_{n-k}} \\ &\quad + \frac{hp\theta}{1+\delta h\theta}N_{n-k+1}e^{-aN_{n-k+1}}. \end{aligned} \tag{14}$$

The implicit numerical scheme is actually an explicit difference equation. In case $\theta = 0$, (14) is the Euler method:

$$N_{n+1} = N_n + hpN_{n-k}e^{-aN_{n-k}}. \tag{15}$$

The Euler method (15) is sometimes called the *purely explicit method* for the blowflies equation. Next, if $\theta = 1/2$, (14) is the Trapezoidal scheme:

$$\begin{aligned} N_{n+1} &= N_n + \frac{h}{2}\left(-\delta N_n + pN_{n-k}e^{-aN_{n-k}}\right) \\ &\quad + \frac{h}{2}\left(-\delta N_{n+1} + pN_{n-k+1}e^{-aN_{n-k+1}}\right). \end{aligned} \tag{16}$$

Finally, if $\theta = 1$, (14) gives the (implicit) Euler method:

$$N_{n+1} = N_n + h(-\delta N_{n+1} + pe^{-aN_{n-k+1}}N_{n-k+1}), \tag{17}$$

or, the so called, the *purely implicit method* for the blowflies equation [1].

Note that, after rearranging, (16) and (17) are actually explicit schemes

$$\begin{aligned} N_{n+1} &= \frac{1-\delta h/2}{1+\delta h/2}N_n + \frac{hp/2}{1+\delta h/2}N_{n-k}e^{-aN_{n-k}} \\ &\quad + \frac{hp/2}{1+\delta h/2}N_{n-k+1}e^{-aN_{n-k+1}} \end{aligned}$$

and

$$N_{n+1} = \frac{1}{1+\delta h}N_n + \frac{hp}{1+\delta h}N_{n-k+1}e^{-aN_{n-k+1}},$$

respectively.

IV. Stability Analysis of the θ-Method for the Nicholson's Blowflies Equation

This section contains our main contributions on the analysis of the stability of the θ-method for the blowflies model (1). We analyse the asymptotic stability of the numerical scheme (14) when $p > \delta > 0$. Our main results provide sufficient conditions for the equilibria of (14) to be asymptotically stable using the linearisation method. In addition, the second part in this section, we give a proof for the nonlinear stability analysis of the implicit Euler scheme (17).

A. Asymptotic stability of the θ-method

Consider the nonlinear DDE:

$$y'(t) = f(t, y(t), y(t-\tau)),\ t > t_0, \quad y(t) = \varphi(t),\ t \le t_0, \tag{18}$$

where $f : [0,\infty) \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ and satisfies the conditions

$$\begin{aligned} \mathrm{Re}\langle f(t,y_1,u) - f(t,y_2,u), y_1 - y_2\rangle &\le \sigma(t)\,|y_1 - y_2|^2, \\ |f(t,y,u_1) - f(t,y,u_2)| &\le \gamma(t)\,|u_1 - u_2|. \end{aligned} \tag{19}$$

Note that when f satisfies the conditions (19), and $\sigma(t),\ \gamma(t)$ satisfy the conditions

$$\sigma(t) \le -\beta < 0, \quad \gamma(t) \le -q\sigma(t),\ 0 \le q < 1, \tag{20}$$

then any two solutions $y_1 = u(t)$ and $y_2 = v(t)$ of (18) with different initial functions satisfy

$$\lim_{t\to\infty} \|u(t) - v(t)\| = 0.$$

We expect that the numerical method has a similar behaviour.

A numerical method for DDEs is called an *asymptotically stable method* if, when applied to (18) satisfying (19) and (20), any two numerical solutions $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ satisfy

$$\lim_{n\to\infty} \|u_n - v_n\| = 0,$$

where $t_n = nh, kh = \tau, \tau > 0$ is a constant delay, and $k \in \mathbb{Z}^+$.

In this part, we focus instead on the asymptotic stability of the numerical solutions for the Nicholson's blowflies equation using the linearisation methods. Our results give sufficient conditions for the numerical solutions to be asymptotically stable.

Consider the numerical scheme (14):

$$\begin{aligned} N_{n+1} &= \frac{1-\delta h(1-\theta)}{1+\delta h\theta}N_n + \frac{hp\theta}{1+\delta h\theta}N_{n-k+1}e^{-aN_{n-k+1}} \\ &\quad + \frac{hp(1-\theta)}{1+\delta h\theta}y_{n-k}e^{-aN_{n-k}}. \end{aligned}$$

It is not difficult to see that if $p > \delta$, there exist two equilibria: the zero equilibrium $\bar{N}_0 = 0$, and the positive equilibrium $\bar{N}_+ = \frac{1}{a}\ln\frac{p}{\delta}$. Next, we investigate the asymptotic stability of the equilibria of (14). Let us first consider the linearisation of (14) about an equilibrium $\bar{N}$. We get the linearised equation

$$\begin{aligned} x_{n+1} &= \frac{1-\delta h(1-\theta)}{1+\delta h\theta}x_n + \frac{hp\theta(1-a\bar{N})e^{-a\bar{N}}}{1+\delta h\theta}x_{n-k+1} \\ &\quad + \frac{hp(1-\theta)(1-a\bar{N})}{1+\delta h\theta}x_{n-k}. \end{aligned} \tag{21}$$

We start with the stability analysis of the zero equilibrium $\bar{N}_0$.

Theorem 1. *Assume that $p > \delta > 0$, then $\bar{N}_0 = 0$ is unstable for all $h > 0$.*

Proof: At the zero equilibrium $\bar{N}_0 = 0$, (21) becomes

$$x_{n+1} = \frac{1-\delta h(1-\theta)}{1+\delta h\theta}x_n + \frac{hp\theta}{1+\delta h\theta}x_{n-k+1} + \frac{hp(1-\theta)}{1+\delta h\theta}x_{n-k}. \tag{22}$$

Its characteristic equation is

$$\lambda^{k+1} - \frac{1-\delta h(1-\theta)}{1+\delta h\theta}\lambda^k - \frac{hp\theta}{1+\delta h\theta}\lambda - \frac{hp(1-\theta)}{1+\delta h\theta} = 0. \tag{23}$$

When $\lambda = 1$, the left-hand side of (23) is negative, equal to $\frac{h(\delta-p)}{1+\delta h\theta}$. Because the left-hand side blows-up as $\lambda \to \infty$, there is a real root larger than 1. Hence the zero equilibrium $\bar{N}_0$ is unstable. ■

Next, we consider the stability properties of the positive equilibrium $\bar{N}_+$.

Theorem 2. *Let $p, \delta, a > 0$, and $1 < p/\delta \le e^2$.*

1) *If $\theta \neq 1$ and h is sufficiently small, i.e.*
$$h\delta(1-\theta) \le 1, \tag{24}$$
the positive equilibrium $\bar{N}_+$ of (14) is asymptotically stable.
2) *If $1/2 \le \theta \le 1$ and the condition*
$$h\delta(1-\theta)\left|1-\ln\frac{p}{\delta}\right| \le 1 \tag{25}$$
holds, the positive equilibrium $\bar{N}_+$ of (14) is asymptotically stable.

Proof: At the positive equilibrium $\bar{N}_+ = \frac{1}{a}\ln(p/\delta)$, the linearised equation of (21) is

$$\begin{aligned} x_{n+1} &= \frac{1-\delta h(1-\theta)}{1+\delta h\theta}x_n + \frac{\delta h\theta(1-\ln(p/\delta))}{1+\delta h\theta}x_{n-k+1} \\ &\quad + \frac{h\delta(1-\theta)(1-\ln(p/\delta))}{1+\delta h\theta}x_{n-k}. \end{aligned}$$

The characteristic equation is

$$\begin{aligned} \lambda^{k+1} - \frac{1-\delta h(1-\theta)}{1+\delta h\theta}\lambda^k - \frac{\delta h\theta(1-\ln(p/\delta))}{1+\delta h\theta}\lambda \\ - \frac{h\delta(1-\theta)(1-\ln(p/\delta))}{1+\delta h\theta} &= 0. \end{aligned}$$

When

$$\begin{aligned} \left|\frac{1-\delta h(1-\theta)}{1+\delta h\theta}\right| + \left|\frac{\delta h\theta(1-\ln(p/\delta))}{1+\delta h\theta}\right| \\ + \left|\frac{h\delta(1-\theta)(1-\ln(p/\delta))}{1+\delta h\theta}\right| &\le 1, \end{aligned} \tag{26}$$

the numerical method (14) is stable. We will next consider the conditions in which (26) holds. The condition $0 < p/\delta \le e^2$ is equivalent to the inequality $|1-\ln(p/\delta)| \le 1$. We discuss (26) by breaking it into two cases: $\delta h(1-\theta) > 1$ and $0 \le \delta h(1-\theta) \le 1$.

Case I: $\delta h(1-\theta) > 1$.
From (26), it yields

$$\begin{aligned} &\left|\frac{1-\delta h(1-\theta)}{1+\delta h\theta}\right| + \left|\frac{\delta h\theta(1-\ln(p/\delta))}{1+\delta h\theta}\right| + \left|\frac{h\delta(1-\theta)(1-\ln(p/\delta))}{1+\delta h\theta}\right| \\ &= \frac{1-\delta h(1-\theta) + (\delta h\theta + h\delta(1-\theta))\,|1-\ln(p/\delta)|}{1+\delta h\theta} \\ &\le \frac{1-\delta h(1-\theta) + \delta h\theta + \delta h(1-\theta)}{1+\delta h\theta} = 1. \end{aligned}$$

So it can be seen that when h is sufficiently small, i.e. $h\delta \le 1/(1-\theta)$, the numerical method (14) is asymptotically stable. The proof of the statement *(1)* is now complete.

Case II: $0 \le \delta h(1-\theta) \le 1$.
We assume that

$$\frac{1}{2} \le \theta \le 1 \quad \text{and} \quad \delta h(1-\theta)\left|1-\ln\frac{p}{\delta}\right| \le 1.$$

Then we have

$$\begin{aligned} &\left|\frac{1-\delta h(1-\theta)}{1+\delta h\theta}\right| + \left|\frac{\delta h\theta(1-\ln(p/\delta))}{1+\delta h\theta}\right| + \left|\frac{h\delta(1-\theta)(1-\ln(p/\delta))}{1+\delta h\theta}\right| \\ &= \frac{-1+\delta h(1-\theta)+\delta h|1-\ln(p/\delta)|}{1+\delta h\theta} \\ &= \frac{\delta h + \delta h|1-\ln(p/\delta)|}{1+\delta h\theta} - 1 \le 1. \end{aligned} \tag{27}$$

We will next prove that (27) holds under the condition (25). Let $1/2 \le \theta \le 1$, and consider the inequality $\delta h|1-\ln(p/\delta)|(1-\theta) \le 1$, so

$$\delta h\left|1-\ln\frac{p}{\delta}\right|(2-2\theta) \le 2,$$

then

$$\delta h\left|1-\ln\frac{p}{\delta}\right| \le 2 + (2\theta-1)\delta h\left|1-\ln\frac{p}{\delta}\right|.$$

For $\theta \ge 1/2$, and $|1-\ln(p/\delta)| \le 1$, we have

$$\delta h\left|1-\ln\frac{p}{\delta}\right| \le 2 + (2\theta-1)\delta h.$$

Thus, it is easy to verify that

$$\frac{\delta h + \delta h\,|1-\ln(p/\delta)|}{1+\delta h\theta} \le 2.$$

As a result, we can conclude that if $0 \le p/\delta \le e^2$ and (25) also holds, then (27) holds. This finishes the proof of the statement *(2)*. ■

Clearly, (25) is an improvement of (24), because $|1-\ln(p/\delta)| < 1$ when $1 < p/\delta < e^2$ for $1/2 \le \theta < 1$. In addition, in Theorem 2, it can be seen that if $\theta = 1$, (25) obviously holds independently from the step-size h. Therefore we can conclude about the stability of the implicit Euler method.

Corollary 3. *If $1 < p/\delta \le e^2$, then the numerical solution of the implicit Euler method (17) is asymptotically stable for all $h > 0$.*

Note that Corollary 3 means that when $1 < p/\delta \le e^2$, the positive equilibrium $\bar{N}_+$ attracts all numerical solutions

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even though the step-size h is big. On the contrary, from Theorem 2, the other methods ($0 \leq \theta < 1$) have a limitation on the numerical step-size h. For example, a sufficient condition of h for the Euler method ($\theta = 0$) to be asymptotically stable is that $h < 1/\delta$. For the Trapezoidal method ($\theta = 1/2$), a sufficient condition is $h < 2/\delta$. Hence, when θ is bigger, the value of the step-size h for stability is also bigger. However, in practice, since we use the numerical step size $h = \tau/k$, for $k \in \mathbb{Z}^+$, the maximum value of h for the numerical simulation is τ ($k = 1$).

B. Nonlinear Stability of the Implicit Euler Method

Consider the following IVP:

$$y'(t) = f(t, y(t), y(t-\tau)),\ t > 0, \quad y(t) = \varphi(t),\ t \leq 0, \tag{28}$$

where f satisfies the conditions (19). Here, $\sigma(t)$ and $\gamma(t)$ satisfy

$$\sigma(t) \leq 0, \quad \gamma(t) \leq -\sigma(t), \tag{29}$$

for all $t \geq 0$. Suppose that $y(t)$ and $z(t)$ are solutions of (28) with different initial functions $\varphi_1(t)$ and $\varphi_2(t)$, respectively. If the conditions (19) and (29) hold, then $y(t)$ and $z(t)$ satisfy

$$\|y(t) - z(t)\| \leq \max_{t \leq 0} \|\varphi_1(t) - \varphi_2(t)\|.$$

Here, we expect that any two numerical solutions have a similar bound, i.e.

$$\|y_n - z_n\| \leq \max_{t \leq 0} \|\varphi_1(t) - \varphi_2(t)\|, \tag{30}$$

which is the concept of RN-stability and GRN-stability. The condition (30) makes the numerical solution preserve the contractivity properties of the analytical solution. In other word, we can say that, if the condition (30) is satisfied, the numerical solution is bounded and does not move away from its analytical solutions. In Jiaoxun and Yuhao [3], pp 193, it is stated that the θ-method for DDEs is GRN-stable (and hence RN-stable) if and only if $\theta = 1$. Here, we show that the implicit Euler method ($\theta = 1$) for the Nicholson's blowflies equation (1)

$$N_{n+1} = N_n + h(-\delta N_{n+1} + p e^{-aN_{n-k+1}} N_{n-k+1}) \tag{31}$$

satisfies the condition (30) by using similar techniques than in [3].

Theorem 4. *Let $p > \delta > 0$. Assume that $\{y_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ are two solutions of the implicit Euler method (31) with different initial functions $\varphi_1(t)$ and $\varphi_2(t)$, respectively. Then, for all $h > 0$,*

$$\|y_n - z_n\| \leq \max_{t \leq 0} \|\varphi_1(t) - \varphi_2(t)\|.$$

Proof: Here, we use similar techniques as in [3] to prove the result. Consider the two difference equations

$$y_{n+1} = y_n + (-h\delta y_{n+1} + hpe^{-ay_{n-k+1}} y_{n-k+1}), \tag{32}$$
$$z_{n+1} = z_n + (-h\delta z_{n+1} + hpe^{-az_{n-k+1}} z_{n-k+1}). \tag{33}$$

Let $\varepsilon_n = y_n - z_n$, then, from (32) and (33), we have

$$\varepsilon_{n+1} = \varepsilon_n + (-h\delta\varepsilon_{n+1} + hp\Delta g), \tag{34}$$

where

$$\Delta g = e^{-ay_{n-k+1}} y_{n-k+1} - e^{-az_{n-k+1}} z_{n-k+1}.$$

Take norms both sides of (34), we have

$$\begin{aligned} \|\varepsilon_{n+1}\|^2 &= \|\varepsilon_n + (-h\delta\varepsilon_{n+1} + hp\Delta g)\|^2 \\ &= \|\varepsilon_n\|^2 + 2\mathrm{Re}\,\langle \varepsilon_n, -h\delta\varepsilon_{n+1} + hp\Delta g\rangle \\ &\quad + h^2p^2\| - h\delta\varepsilon_{n+1} + hp\Delta g\|^2 \end{aligned} \tag{35}$$

From (34), $\varepsilon_n = \varepsilon_{n+1} + h\delta\varepsilon_{n+1} - hp\Delta g$, then (35) leads to the inequality

$$\|\varepsilon_{n+1}\|^2 \leq \|\varepsilon_n\|^2 + 2h\|\varepsilon_{n+1}\| \left(p\|\Delta g\| - \delta\|\varepsilon_{n+1}\|\right).$$

For $p > \delta$, we have

$$\|\varepsilon_{n+1}\|^2 \leq \|\varepsilon_n\|^2 + 2hp\|\varepsilon_{n+1}\| \left(\|\Delta g\| - \|\varepsilon_{n+1}\|\right). \tag{36}$$

We shall prove that

$$\|\varepsilon_n\| \leq S, \quad \text{where} \quad S = \max_{t \leq 0} \|\varphi_1(t) - \varphi_2(t)\|.$$

Suppose that for any $n \leq j \ \ (j \geq 0)$, $\|\varepsilon_n\| \leq S$.
Case I: if $\|\varepsilon_{j+1}\| \leq \|\varepsilon_j\|$, then $\|\varepsilon_{j+1}\| \leq S$.
Case II: if $\|\varepsilon_{j+1}\| > \|\varepsilon_j\|$, then (36) produces

$$0 < \|\varepsilon_{j+1}\|^2 - \|\varepsilon_j\|^2 \leq 2hp\|\varepsilon_{j+1}\| \left(\|\Delta g\| - \|\varepsilon_{j+1}\|\right). \tag{37}$$

Consider $\|\Delta g\|$ and use the mean-value theorem. We have

$$\|\Delta g\| = |g'(\xi)| \|y_{j-k+1} - z_{j-k+1}\| = |g'(\xi)| \|\varepsilon_{j-k+1}\|,$$

where ξ is a point between y_{j-k+1} and z_{j-k+1}. Now, (37) can be rewritten as

$$0 < \|\varepsilon_{j+1}\|^2 - \|\varepsilon_j\|^2 \leq 2hpQ\|\varepsilon_{j+1}\| \left(\|\varepsilon_{j-k+1}\| - \|\varepsilon_{j+1}\|\right),$$

where

$$Q = \max\{1, |g'(\xi)|\}.$$

Because $2hpQ > 0$, then $\|\varepsilon_{j-k+1}\| - \|\varepsilon_{j+1}\| > 0$. So, we have

$$\|\varepsilon_{j+1}\| < \|\varepsilon_{j-k+1}\| \leq S,$$

for $k \geq 1$. Now we can conclude that $\|\varepsilon_n\| \leq S$, i.e.

$$\|\varepsilon_n\| = \|y_n - z_n\| \leq S,$$

for all $n \geq 1$. Thus it follows that the implicit Euler method (31) is bounded by $S = \max_{t \leq 0} \|\varphi(t) - \varphi(t)\|$, for all $h > 0$. ∎

Note that Theorem 4 shows that the implicit Euler method is contractive for all numerical step-size $h > 0$, i.e. the numerical solutions will not move away as time tends to infinity. In other word, we can say that the implicit Euler method is bounded by the initial conditions and its stability is independent on the step-size h.

V. Numerical Examples

In this section we present some numerical examples performed using the θ-method. The examples given here are mainly focused on three methods: the explicit Euler method ($\theta = 0$), the Trapezoidal scheme ($\theta = 0.5$) and the implicit Euler method ($\theta = 1$).

First, Figures 1 and 2 compare the numerical solutions from the Euler method, the Trapezoidal scheme, and the implicit Euler method with a small numerical step-size ($h = 0.01$). Here, the step-size h satisfies the conditions on Theorem 1 and Theorem 2. In Figure 1 we use $p = 2.0, \delta = 1.1, a = 2.0, \tau = 1.0$, and the initial data $\varphi(t) = 0.5,\ t \leq 0$.

The results show that the numerical solutions for all methods are similar. They are monotone decreasing and converge to the equilibrium $\bar{N} = 0.2989$. In Figure 2, we use different parameter values $p = e^2, \delta = 1.1, a = 2.0$ and $\tau = 3.5$. The initial function is the same as in Figure 1, i.e. $\varphi(t) = 0.5,\ t \leq 0$. Again, the numerical solutions from all methods are similar but their behaviour is different. They oscillate at the beginning before they converge to the equilibrium $\bar{N} = 0.9523$. The differences here are caused by the ratio p/δ. In Figure 1, the ratio $1 < p/\delta < e$, so the solutions are monotone. In contrast, the ratio in Figure 2 is close to but less than e^2, so the solutions oscillate about the equilibrium and they are asymptotically stable.

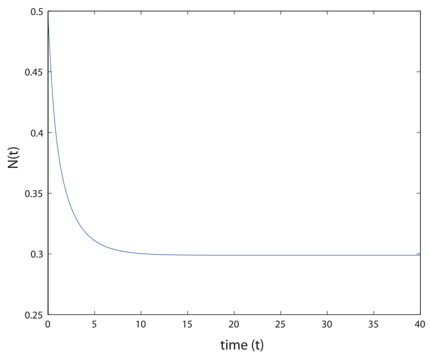

Fig. 1. Numerical examples of the θ-methods for the Nicholson's blowflies equation (1) when $p = 2.0, \delta = 1.1, a = 2.0, \tau = 1.0$ and $h = 0.01$.

Next, we increase the numerical step-size to $h = 3.5$. In this case, h does not satisfy the condition in Theorem 2 for the Euler method, but it satisfies it for the Trapezoidal scheme and the implicit Euler method. As the result, the numerical solutions from the Euler method cannot be performed when h is too large. Figure 3 illustrates the difference on the numerical solutions between the Trapezoidal scheme (Figure 3(a)) and the implicit Euler method (Figure 3(b)) with the same parameter values as in Figure 2. Here, we can see that both solutions are asymptotically stable and converge to the equilibrium $\bar{N} = 0.9523$ as t tends to infinity. However, when we compare the results with the smaller h in Figure 2, the Trapezoidal scheme approximates the exact solution better than the implicit Euler method. This is because the Trapezoidal scheme is order two on the approximation, while the implicit Euler method is only of order one.

Finally, Figure 4 and Figure 5 represent the numerical solutions obtained from the implicit Euler method. From Theorem 4, we know that all numerical solutions from the implicit Euler method are bounded by the initial conditions. In addition, Corollary 3 states that if $1 < p/\delta < e^2$, all solutions of the implicit Euler method are asymptotically stable. The results in both Figure 4 and Figure 5 support Corollary 3 and Theorem 4. However, like with general numerical approximations, when the step-size h is bigger, the numerical approximations are less accurate.

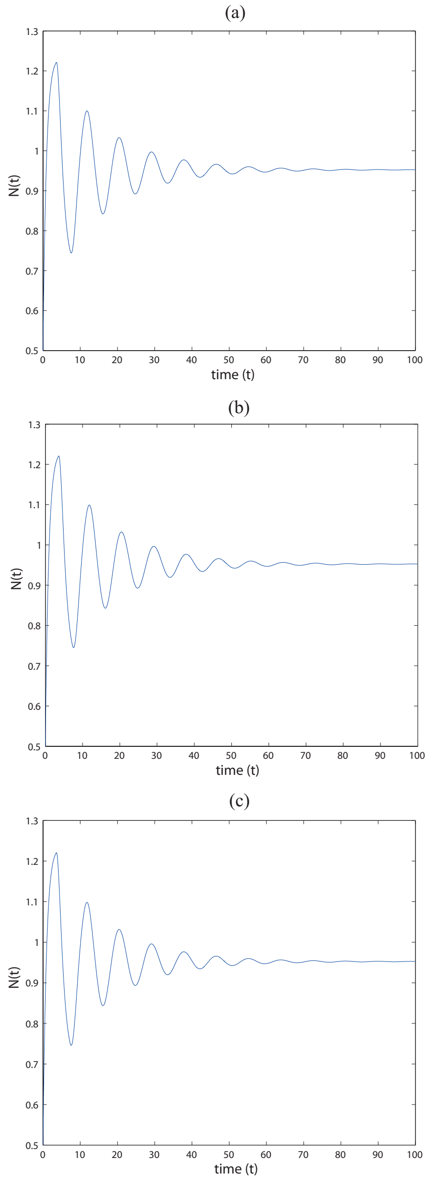

Fig. 2. Numerical examples of the θ-methods for the Nicholson's blowflies equation (1) when $p = e^2, \delta = 1.1, a = 2.0, \tau = 3.5$ and $h = 0.01$; where (a) $\theta = 0$, (b) $\theta = 0.5$ and (c) $\theta = 1$.

References

[1] M. Z. Liu and M. N. Spijker, "The stability of θ-methods in the numerical solutions of delay differential equations," *IMA Journal of Numerical Analysis*, vol. 10, no. 1, pp. 31–48, 1990.

[2] A. Iserles, *A First Course in the Numerical Analysis of Differential Equations*. UK: Cambridge University Press, 2003.

[3] K. Jiaoxun and C. Yuhao, *Stability of Numerical Methods for Delay Differential Equations*. Beijing: Science Press, 2005.

[4] M. Calvo and T. Grande, "On the asymptotic stability of θ-methods for delay differential equations," *Numerische Mathematik*, vol. 54, no. 3, pp. 257–269, 1988.

[5] V. K. Barwell, "Special stability problems for functional differential equations," *BIT*, vol. 15, pp. 130–135, 1975.

[6] N. Guglielmi, "Delay dependent stability regions of θ-methods for delay differential equations," *IMA Journal of Numerical Analysis*, vol. 18, no. 3, pp. 399–418, 1998.

[7] H. J. Tian and J. X. Kuang, "The stability analysis of the θ-methods for delay differential equations," *Journal of Computational Mathematics*, vol. 14, no. 3, pp. 203–212, 1996.

[8] L. Torelli, "Stability of numerical methods for delay differential equations," *Journal of Computational and Applied Mathematics*, vol. 25, no. 1, pp. 15–26, 1989.

[9] ——, "A sufficient condition for GPN-stability for delay differential equations," *Numerische Mathematik*, vol. 59, no. 1, pp. 311–320, 1991.

[10] H. J. Tian, "Dissipativity and exponential stability of θ-methods for singularly perturbed delay differential equations with a bounded lag," *Journal of Computational Mathematics*, vol. 21, no. 6, pp. 715–726, 2003.

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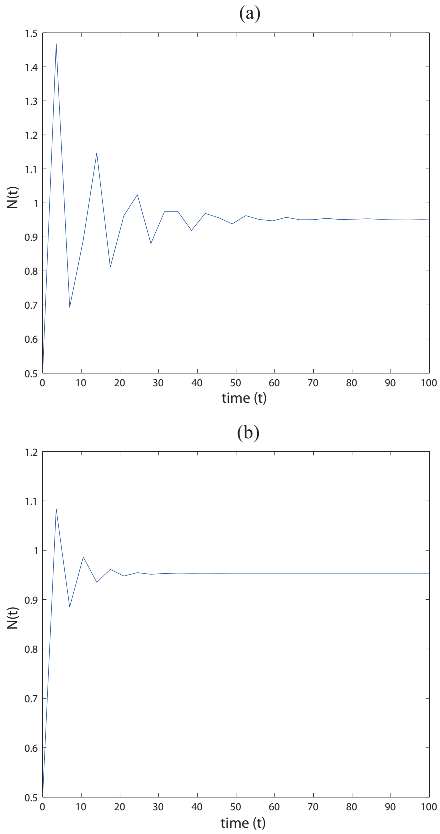

Fig. 3. Numerical examples of the θ-methods for the Nicholson's blowflies equation (1) when $p = e^2, \delta = 1.1, a = 2.0$ and $h = \tau = 3.5$; where (a) $\theta = 0.5$ and (b) $\theta = 1$.

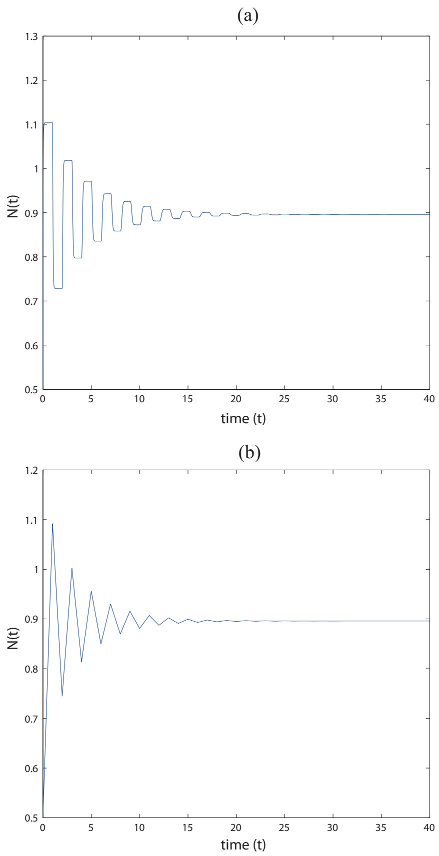

Fig. 4. Numerical examples of the θ-methods for the Nicholson's blowflies equation (1) when $p = 300, \delta = 50, a = 2.0$ and $\tau = 1.0$; where (a) $h = 0.01$, (b) $h = 1.0$.

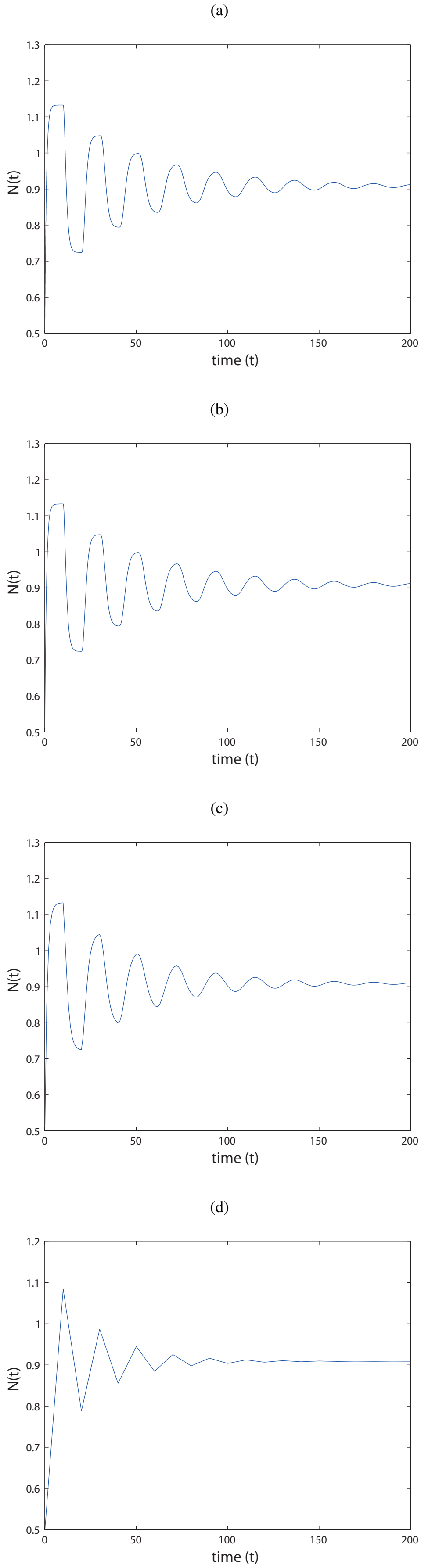

Fig. 5. Numerical examples from the implicit Euler method for(1) with different values of h when $p = e^2, \delta = 1.2, a = 2.0, \tau = 10$; where (a) $h = 0.01$, (b) $h = 0.1$, (c) $h = 1$ and (d) $h = 10$.