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Strict Stability Criteria for Impulsive Functional Differential Equations

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***Abstract*—Strict stability is the kind of stability that can give us some information about the rate of decay of the solution. There are some results about strict stability of functional differential equations. In this paper, we shall extend strict stability to Impulsive functional differential equations in which the state variables on the impulses are related to time delay. By using Lyapunov functions and Razumikhin technique, some criteria for strict stability for functional differential equations, in which the state variables on the impulses are related to the time delay are provided, and we can see that impulses do contribute to the system's strict stability behavior.**

***Index Terms*—Impulsive functional differential equation, Strict stability, Lyapunov function, Razumikhin technique, Time delay.**

I. INTRODUCTION

Impulses can make unstable systems stable, so it has been widely used in many fields such as physics, chemistry, biology, population dynamics, industrial robotics and so on. The impulsive differential equations represent a more natural framework for mathematical modeling of many real world phenomena than differential equations. In recent years, significant progress has been made in the theory of impulsive differential equations[3-10] and references therein. In addition to that, functional differential equations have a wide application in our society. So it is important to study them. There are some results on functional differential equations.

We can easily see that in the previous works about impulsive functional differential equations the authors always suppose that the state variables on the impulses are only related to the present state. But in most cases, it is more applicable that the state variables on the impulses the we add are also related to the former state. But there are rare results about impulsive functional differential equations in which state variable on the impulses are related to the time delay.

Strict stability is analogous to Lyapunovs uniform asymptotic stability. It gives us some information about the rate of decay of the solutions. In[1], the authors have explored further the definitions of strict stability of differential equations and have gotten some results. In [9] authors have gotten some results about the strict stability of impulsive functional differential equations in which the state variables on the impulses are not related to the time delay. In this paper, we will consider the strict stability of impulsive functional differential equations in which the state variables on the impulses are related to the time delay.

This paper is organized as follows. In Section II, we introduce some basic definitions and notations.

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In Section III, some criteria in the form of theorem for strict stability of impulsive functional differential equations is obtained in which state variables on the impulses are related to the time delay. Finally, concluding remarks are given in Section IV.

II. PRELIMINARIES

consider the following Impulsive functional differential equation in which the state variables on the impulses are related to time delay.

$$x'(t) = f(t, x_t), \quad t \geq t_0 \quad t \neq \tau_k$$

$$x(\tau_k) = I_k(x(\tau_k^-)) + J_k(x(\tau_k^- - \tau)), \quad k = 1, 2, 3, ..., \tag{1}$$

where x $\in R^n$, f$\in C[R^+ \times D, R^n]$, $I_k, \ J_k \in C[R^n, R^n]$, D is an open set in $PC([-\tau, 0], R^n)$, where $\tau =$ constant > 0, $PC([-\tau, 0, R^n]) = \{\phi : [-\tau, 0] \to R^n, \ \phi(t)$ is continuous everywhere except a finite number of points $\hat{t}$ at which $\phi(\hat{t}^+)$ and $\phi(\hat{t}^-)$ exist and $\phi(\hat{t}^+) = \phi(\hat{t}^-)\}$. $f(t, 0) = 0$, for all $t \in R, \ I_k(0) \equiv 0, \ J_k(0) \equiv 0, \ 0 = \tau_0 < \tau_1 < \tau_2 < \tau_3 < ... < \tau_k < ..., \tau_k \to \infty$, for $\quad k \to \infty \quad and \quad x(t^+) = \lim_{s \to t^+} x(s)$, $x(t^-) = \lim_{s \to t^-} x(s)$.

For each $t \geq t_0, \ x_t \in D$ is defined by $x_t(s) = x(t + s), \quad -\tau \leq s \leq 0$. For $\phi \in PC([-\tau, 0, R^n]), \ |\phi|_1$ is defined by $|\phi|_1 = sup_{-\tau \leq s \leq 0}||\phi||, \ |\phi|_2$ is defined by $|\phi|_2 = inf_{-\tau \leq s \leq 0}||\phi||$, where $||.||$ denotes the norm of a vector R^n. We can see that $x(t) \equiv 0$ is a solution of (1) which we call the zero solution.

A function $x(t)$ is called a solution of (1) with the initial condition

$x_\sigma = \varphi$

where $\rho \geq t_0$ and $\varphi \in PC([-\tau, 0, R^n])$, the initial value problem of equation (1) is

$$x'(t) = f(t, x_t), \quad t \geq t_0 \quad t \neq \tau_k$$
$$x(\tau_k) = I_k(x(\tau_k^-)) + J_k(x(\tau_k^- - \tau)), \quad k = 1, 2, 3, ...,$$
$$x_\sigma = \varphi \tag{2}$$

Throughout this paper we let the following hypotheses hold.

(H_1) For $t \in [\sigma - \tau, \sigma]$, the solution $x(t; \sigma, \varphi)$ coincides with the function $\varphi(t - \sigma)$

(H_2) For each function $x(s) : [\sigma - \tau, \infty] \to R^n$, which is continuous everywhere except at the point $\{\tau_k\}$ at which $x(\tau_k^+), \ x(\tau_k^-)$ exist and $x(\tau_k^+) = x(\tau_k), \ f(t, x_t)$ is continuous for almost all $t \in [\sigma, \infty)$ and at the discontinuous points f is right continuous.

(H_3) $f(t, \phi)$ is Lipschitzian in ϕ in each compact set in $PC([-\tau, 0], R^n)$.

(H_4) The functions $I_k, \ J_k, \ k = 1, 2, ...,$ are such that if $x \in D, \ I_k \neq 0$ and $J_k \neq 0$, then $I_k(x) + J_k(x(t - \tau)) \in D$.

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Under the hypothesis $(H_1) - (H_4)$, there is a unique solution of problem(2) throughout (σ, φ).
we using the following notation:
$S(\rho) = \{x \in R^n : ||x|| < \rho\}$,
$K = \{a \in C[R^+, R^+] : a(t)$ is monotone strictly increasing and $a(0) = 0\}$,
$K_1 = \{w \in C[R^+, R^+] : w(t) \in K$ and $0 < w(s) < s,\ s > 0\}$,
$PC_1(\rho) = \{\phi \in PC([-\tau, 0],\ R^n) : |\phi|_1 < \rho\}$,
$PC_2(\theta) = \{\phi \in PC([-\tau, 0],\ R^n) : |\phi|_2 > \theta\}$.
We have the following definitions.
Definition 2.1. The trivial solution of (1) is said to be
(1) strictly stable, if for any $\sigma \geq t_0$ and $\epsilon_1 > 0$, there exist a $\delta_1 = \delta_1(\sigma, \epsilon_1)$ such that $\varphi \in PC_1(\delta_1)$ implies $||x(t; \sigma, \varphi)|| < \epsilon_1$ for $t \geq \sigma$, and for every $0 < \delta_2 \leq \delta_1$, there exist an $0 < \epsilon_2 < \delta_2$ such that $\varphi \in PC_2(\delta_2)$ implies $\epsilon_2 < ||x(t; \sigma, \varphi)||,\ t \geq \varphi$.
(2) strictly uniformly stable, if $\delta_1,\ \delta_2$ and ϵ_2 in (SS) are independent of σ.
Definition 2.2. The function $V(x) : [t_0, \infty] \times S(\rho) \to R^+$ belongs to class V_0 if:
(1) the function V is continuous on each of the sets $[\tau_{k-1}, \tau_k) \times S(\rho)$ and for all $t \geq t_0$, $V(t, 0) \equiv 0$;
(2) $V(t, x)$ is locally Lipschitzian in $x \in S(\rho)$;
(3) For each $k = 1, 2, ...$, there exist finite limits
$\lim_{(t,y)\to(\tau_k^-)} V(t, y) = V(\tau_k^-, x)$
$\lim_{(t,y)\to(\tau_k^+)} V(t, y) = V(\tau_k^+, x)$
with $V(\tau_k^+, x) = V(t, x)$ satisfied.
Definition 2.3 Let $V \in V_0$, for $(t, x) \in [\tau_{k-1}, \tau_k) \times S(\rho),\ D^+V$ is defined as
$D^+V(t, x(t)) = lim_{x\to\delta} sup \frac{1}{\delta}\{V(t + \delta, x(t + \delta)) - V(t, x(t))\}$

III. Main Result

Now we consider the strict stability of the impulsive functional differential equation (1). We have the following results.
Theorem 3.1: Assume that
(i) There exist $a_1, b_1 \in K, V_1 \in V_0$ such that
$b_1(||x||) \leq V_1(t, x) \leq a_1(||x||)$,
for all $(t, x) \in [t_0 - \tau, \infty) \times S(\rho)$;
(ii) For any solution $x(t)$ of (2) , $V_1(t + s, x(t + s)) \in V_1(t, x(t))$ for $s \in [-\tau, 0]$, implies that $D^+V_1(t, x(t)) \leq 0$.
(iii) For $k \in Z^+$ and $x \in S(\rho)$,
$V_1(\tau_k, I_k(x(\tau_k^-)) + J_k(x(\tau_k^- - \tau))) \leq \frac{1+b_k}{2}[V_1(\tau_k^-, x(\tau_k^-)) + V_1(\tau_k^- - \tau, x(\tau_k^- - \tau))]$, where $d_k \geq 0$ and $\sum_{k=1}^{\infty} d_k < \infty$.
(iv) There exist $a_2,\ b_2 \in K,\ V_2 \in V_0$ such that
$b_2(||x||) \leq V_2(t, x) \leq a_2(||x||)$,
for all $(t, x) \in [t_0 - \tau, \infty) \times S(\rho)$;
(v) For any solution $x(t)$ of (2) , $V_2(t + s, x(t + s)) \in V_2(t, x(t))$ for $s \in [-\tau, 0]$, implies that $D^+V_2(t, x(t)) \leq 0$.
(vi) For $k \in Z^+$ and $x \in S(\rho)$,
$V_2(\tau_k, I_k(x(\tau_k^-)) + J_k(x(\tau_k^- - \tau))) \geq \frac{1-c_k}{2}[V_2(\tau_k^-, x(\tau_k^-)) + V_2(\tau_k^- - \tau, x(\tau_k^- - \tau))]$, where $0 \leq c_k < 1$ and $\sum_{k=1}^{\infty} c_k < \infty$.

Then the trivial solution of (2) is strictly uniformly stable.
Proof : Since $\sum_{k=1}^{\infty} d_k < \infty,\ \sum_{k=1}^{\infty} c_k < \infty$
it follows that $\prod_{k=1}^{\infty}(1 + d_k) = M$ and $\prod_{k=1}^{\infty}(1 - c_k) = N$, obviously $1 \leq M < \infty, 0 < N \leq 1$.
Let $0 < \epsilon_1 < \rho$ and $\sigma \geq t_0$ be given and $\sigma \in [\tau_{k+1}, \tau_k)$ for some $k \in Z^+$. Choose $\delta_1 = \delta_1(\epsilon_1) > 0$ such that $Ma_1(\delta_1) < b_1(\epsilon_1)$
Then we claim that $\varphi \in PC_1(\delta_1)$ implies $||x(t)|| < \epsilon_1,\ t \geq \sigma$
Obviously for any $t \in [\sigma - \tau, \sigma]$, there exists a $\theta \in [-\tau, 0]$ such that
$V_1(t, x(t)) = V_1(\sigma + \theta, x(\sigma + \theta)) \leq a_1(||x(\sigma + \theta)||) = a_1(||x_\sigma(\theta)||) = a_1(||\varphi(\theta)||) \leq a_1(\delta_1)$.
Then we claim that

$$V_1(t, x(t)) \leq a_1(\delta_1), \quad \sigma \leq t < \tau_k. \tag{3}$$

If inequality (3) does not hold, then there is a $\hat{t} \in (\sigma, \tau_k)$ such that

$$V_1(\hat{t}, x(\hat{t})) > a_1(\delta_1) \geq V_1(\sigma, x(\sigma)) \tag{4}$$

which implies that there is a $\check{t} \in (\sigma, \hat{t}]$ such that

$$D^+V_1(\check{t}, x(\check{t})) > 0 \tag{5}$$

and

$$V_1(\check{t} + s, x(\check{t} + s)) \leq V_1(\check{t}, x(\check{t})), \quad s \in [-\tau, 0].$$

By condition (ii) , which implies that $D^+V_1(\check{t}, x(\check{t})) \leq 0$. This contradicts inequality(5), so inequality (3) holds.
In view of inequality (3) and condition (iii) , we have
$V_1(\tau_k, x(\tau_k) = V_1(\tau_k, I_k(x(\tau_k^-)) + J_k(x(\tau_k^- - \tau))) \leq \frac{1+b_k}{2}[V_1(\tau_k^-, x(\tau_k^-)) + V_1(\tau_k^- - \tau, x(\tau_k^- - \tau))] \leq (1 + b_k)a_1(\delta_1)$
Next we prove that

$$V_1(\tau, x(\tau) \leq (1 + b_k)a_1(\delta_1), \quad \tau_k \leq t < \tau_{k+1} \tag{6}$$

If inequality (6) does not hold, then there is an $\hat{s} \in (\tau_k, \tau_{k+1})$ such that
$V_1(\hat{s}, x(\hat{s})) > (1 + b_k)a_1(\delta_1) \geq V_1(\tau_k, x(\tau_k))$
which implies that there is an $\check{s} \in (\tau_k, \hat{s})$ such that

$$D^+V_1(\check{s}, x(\check{s})) > 0 \tag{7}$$

and
$V_1(\check{s} + s, x(\check{s} + s)) \leq V_1(\check{s}, x(\check{s})), \quad s \in [-\tau, 0]$.
By condition (ii), which implies that $D^+V_1(\check{s}, x(\check{s})) \leq 0$. This contradicts inequality (7), so inequality (6) holds.
In view of inequality (6) and condition (iii), we have
$V_1(\tau_{k+1}, x(\tau_{k+1})) = V_1(\tau_{k+1}, I_{k+1}(x(\tau_{k+1}^-)) + J_{k+1}(x(\tau_{k+1}^- - \tau))) \leq \frac{1+b_{k+1}}{2}[V_1(\tau_{k+1}^-, x(\tau_{k+1}^-)) + V_1(\tau_{k+1}^- - \tau, x(\tau_{k+1}^- - \tau))] \leq (1 + b_{k+1})a_1(\delta_1)$

By simple induction, we can prove in general, that for $k = 0, 1, 2, ...$,
$V_1(t, x(t)) \leq (1 + b_{m+k})...(1 + b_k)a_1(\delta_1), \quad \tau_{m+k} \leq t < \tau_{m+k+1}$.
$V_1(\tau_{m+k+1}, x(\tau_{m+k+1})) \leq (1 + b_{m+k+1})(1 + b_{m+k})...(1 + b_k)a_1(\delta_1)$.
This together with inequality (3) yields
$V_1(t, x(t)) \leq Ma_1(\delta_1)$
From this and condition (i) we have
$b_1(||x(t)||) \leq V_1(t, x(t)) \leq Ma_1(\delta_1) < b_1(\epsilon_1), \quad t \geq \sigma$

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Thus, we have
$||x(t)|| < \epsilon_1, \quad t \geq \sigma.$
Now, Let $0 < \delta_2 \leq \delta_1$ and choose $0 < \epsilon_2 < \delta_2$ such that $a_2(\epsilon_2) < Nb_2(\delta_2)$.
Next we claim that $\varphi \in PC_2(\delta_2)$ implies $||x|| > \epsilon_2, \ t \geq \sigma$. If this holds. $\varphi \in PC_1(\delta_1) \cap PC_2(\delta_2)$ implies $\epsilon_2 < ||x||\epsilon_1, \quad t \geq \sigma.$
Obviously for any $t \in [\sigma - \tau, \sigma]$, there exists a $\theta \in [-\tau, 0]$ such that
$V_2(t, x(t)) = V_2(\sigma + \theta, x(\sigma + \theta)) \geq b_2(||x(\sigma + \theta)||) = b_2(||x_\sigma(\theta)||) = b_2(||\varphi(\theta)||) \geq b_2(\delta_2).$
Then we claim that

$$V_2(t, x(t)) \geq b_2(\delta_2), \quad \sigma \leq t < \tau_k. \tag{8}$$

If inequality (8) does not hold, then there is a $\bar{t} \in (\sigma, \tau_k)$ such that

$$V_2(\bar{t}, x(\bar{t})) < b_2(\delta_2) \leq V_2(\sigma, x(\sigma)) \tag{9}$$

which implies that there is a $t_1 \in (\sigma, \bar{t}]$such that

$$D^+V_2(t_1, x(t_1)) < 0 \tag{10}$$

and
$V_2(t_1 + s, x(t_1 + s)) \geq V_2(t_1, x(t_1)), \quad s \in [-\tau, 0].$
By condition (v), this implies that $D^+V_2(t_1, x(t_1)) \geq 0$. This contradicts inequality (10), so inequality (9) holds.
In view of inequality (3) and condition (iii) , we have
$V_2(\tau_k, x(\tau_k) = V_2(\tau_k, I_k(x(\tau_k^-)) + J_k(x(\tau_k^- - \tau))) \geq \frac{1-c_k}{2}[V_2(\tau_k^-, x(\tau_k^-)) + V_2(\tau_k^- - \tau, x(\tau_k^- - \tau))] \geq (1 - c_k)b_2(\delta_2)$
Next we prove that

$$V_2(t, x(t)) \geq (1 - c_k)b_2(\delta_2), \quad \tau_k \leq t < \tau_{k+1} \tag{11}$$

If inequality (11) does not hold, then there is an $\hat{s} \in (\tau_k, \tau_{k+1})$ such that
$V_2(\hat{r}, x(\hat{r})) < (1 - c_k)b_2(\delta_2) \leq V_2(\tau_k, x(\tau_k))$
which implies that there is an $\check{r} \in (\tau_k, \hat{r})$ such that

$$D^+V_2(\check{r}, x(\check{r})) < 0 \tag{12}$$

and
$V_2(\check{r} + s, x(\check{r} + s)) \geq V_2(\check{r}, x(\check{r})), \quad s \in [-\tau, 0].$
By condition (v), which implies that $D^+V_2(\check{r}, x(\check{r})) \geq 0$. This contradicts inequality (12), so inequality (11) holds.
In view of inequality (8) and condition (vi), we have
$V_2(\tau_{k+1}, x(\tau_{k+1}) = V_2(\tau_{k+1}, I_{k+1}(x(\tau_{k+1}^-)) + J_{k+1}(x(\tau_{k+1}^- - \tau))) \geq \frac{1-c_{k+1}}{2}[V_2(\tau_{k+1}^-, x(\tau_{k+1}^-)) + V_2(\tau_{k+1}^- - \tau, x(\tau_{k+1}^- - \tau))] \geq (1 - c_{k+1})b_2(\delta_2)$
By simple induction, we can prove in general, that for $k = 0, 1, 2, ...,$
$V_2(t, x(t)) \geq (1 - c_{m+k})...(1 - c_k)b_2(\delta_2), \quad \tau_{m+k} \geq t < \tau_{m+k+1}.$
$V_2(\tau_{m+k+1}, x(\tau_{m+k+1})) \geq (1 - c_{m+k+1})(1 - c_{m+k})...(1 - c_k)b_2(\delta_2).$
This together with inequality (9) yields
$V_2(t, x(t)) \geq Mb_2(\delta_2)$
From this and condition (iv) we have
$a_2(||x(t)||) \geq V_2(t, x(t)) \geq Nb_2(\delta_2) > a_2(\epsilon_1), \quad t \geq \sigma$
Thus, we have
$||x(t)|| > \epsilon_2, \quad t \geq \sigma.$

Thus , the zero solution of (2) is strictly uniformly stable. The proof of Theorem is complete.

IV. Conclusion

In this paper, the strict stability of impulsive functional differential equations in which the state variables on the impulses are related to the time delay is considered. By using Lyapunov functions and Razumikhin technique, we have obtained some results for the strict stability. Strict stability theorem for impulsive functional differential equation has been extended to impulsive functional differential equations in which the state variables on the impulses are related to the time delay. We can see that impulses do contribute to a systems of strict stability behavior. Viewing its scope in future, we will do some further research on impulsive functional differential equations in which the state variables on the impulse are related to the time delay.

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