Singularity Analysis of a Parallel Manipulator with Cylindrical Joints

Zhumadil Zh. Baigunchekov, and Myrzabay B. Izmambetov

***Abstract* — In this paper we are presenting some results of the singularity analysis of the new parallel manipulator with cylindrical joints. We have derived the differential kinematic relations between two vectors: mobile-platform velocity and the active-joint rates. These relations comprise two matrices, the forward - and the inverse - kinematics Jacobians. The analytical approach to the research of these relations, based only on linear algebra, has yielded interesting results on identification of the singular configurations of the parallel manipulator.**

***Index Terms* — Jacobian matrix, mobile platform, parallel manipulator, singularity analysis.**

I. Introduction

Parallel manipulators in comparison with serial manipulators have big carrying capacity, high velocities and positioning accuracy [1]. However, parallel manipulators have many singularity configurations in which they cannot carry out their functional tasks [2]. Gosselin and Angeles [3], Tsai [4] describe three types of singularity, based on the properties of the Jacobian matrices. The Jacobian of a parallel manipulator can be divided into two matrix: one matrix $\mathbf{J_x}$ associated with the direct kinematics and the other matrix $\mathbf{J_q}$ associated with the inverse kinematics.

In this paper some results for identification of the singularity configuration of a parallel manipulator with cylindrical joints (PM 3CCC) based on research of degeneration of Jacobian matrices are presented.

II. Geometry of a PM 3CCC

A PM 3CCC is formed by connection of a mobile platform 3 with a base 0 by three spatial dyads *ABC, DEF, GHI* of kind **CCC**, where **C** - cylindrical joint (Fig. 1). A spatial kinematic chain with two links and three kinematic pairs is called a spatial dyad which has zero degree-of-freedom. As a spatial dyad has zero degree-of-freedom it does not impose restriction on a mobile platform 3 and six degree-of-freedom of a mobile platform is remained. Each cylindrical joint has two degree-of-freedom: one rotation and one translation. Six movements of input cylindrical joints *A, F, I* (three rotations and three translations) are generalized coordinates of a parallel manipulator with cylindrical joints PM 3CCC.

A PM 3CCC is intended for reproduction of movement of a mobile platform 3 or the frame $PX_PY_PZ_P$ attached to it with respect to base frame $U_oV_oW_o$

$$\left.\begin{array}{l} U_{0P}=U_{0P}(\mathbf{q}(t)),\ V_{0P}=V_{0P}(\mathbf{q}(t)),\ W_{0P}=W_{0P}(\mathbf{q}(t)) \\ \gamma_{0P}=\gamma_{0P}(\mathbf{q}(t)),\ \alpha_{0P}=\alpha_{0P}(\mathbf{q}(t)),\ \beta_{0P}=\beta_{0P}(\mathbf{q}(t)) \end{array}\right\}, \quad (1)$$

where $\mathbf{q}(t)=[\theta_7(t), s_7(t), \theta_8(t), s_8(t), \theta_9(t), s_9(t),]^{\mathrm{T}}$ - a vector of the input generalized coordinates; γ_{0P}, α_{0P} и β_{0P} - the components of relative orientation of coordinate systems $PX_PY_PZ_P$ and $OU_oV_oW_o$.

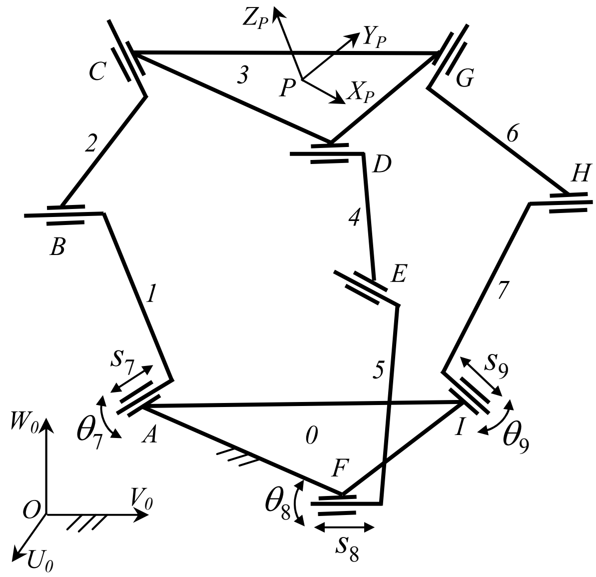

Fig. 1. Parallel manipulator with cylindrical joints PM 3CCC

For definition of constant and variable parameters of parallel manipulator we fixed the right Cartesian coordinate systems *UVW* and *XYZ* to each element of each joint. The axes *W* and *Z* of coordinate systems *UVW* and *XYZ* are directed on a axis of rotation and translation of cylindrical joints.

Transformation matrix $\mathbf{T}_{jk}$ between coordinate systems $U_jV_jW_j$ and $X_kY_kZ_k$ fixed on the ends of binary link jk has a view (Fig. 2)

$$\mathbf{T}_{jk}=\mathbf{T}_{jk}(a_{jk},b_{jk},c_{jk},\alpha_{jk},\beta_{jk},\gamma_{jk})=$$

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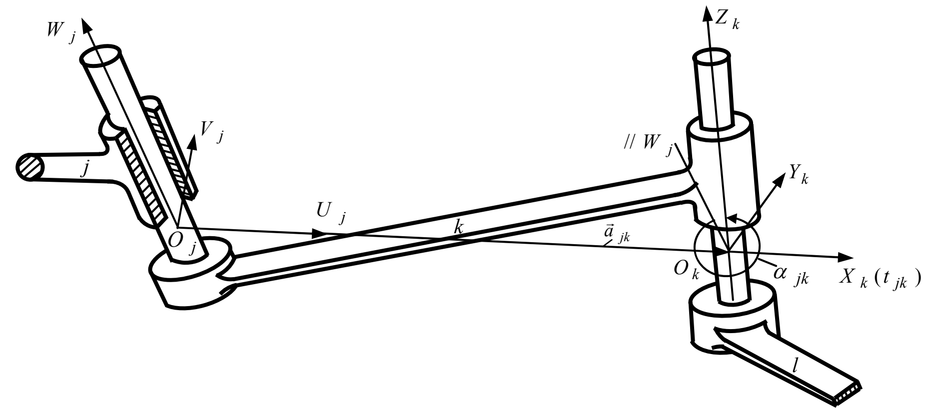

Fig. 2. Binary link *jk* of kind CC

$$=\begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ t_{41} & t_{42} & t_{43} & t_{44} \end{bmatrix} = \left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline {}^{j}\boldsymbol{\tau}_k & {}^{j}\mathbf{R}_k \end{array}\right], \qquad (2)$$

where $t_{11}=1$, $t_{12}=t_{13}=t_{14}=0$,

$t_{21}=a_{jk}\cdot\cos\gamma_{jk}+b_{jk}\cdot\sin\gamma_{jk}\cdot\sin\alpha_{jk}$,

$t_{22}=\cos\gamma_{jk}\cdot\cos\beta_{jk}-\sin\gamma_{jk}\cdot\cos\alpha_{jk}\cdot\sin\beta_{jk}$,

$t_{23}=-\cos\gamma_{jk}\cdot\sin\beta_{jk}-\sin\gamma_{jk}\cdot\cos\alpha_{jk}\cdot\cos\beta_{jk}$,

$t_{24}=\sin\gamma_{jk}\cdot\sin\alpha_{jk}$,

$t_{31}=a_{jk}\cdot\sin\gamma_{jk}-b_{jk}\cdot\cos\gamma_{jk}\cdot\sin\alpha_{jk}$,

$t_{32}=\sin\gamma_{jk}\cdot\cos\beta_{jk}+\cos\gamma_{jk}\cdot\cos\alpha_{jk}\cdot\sin\beta_{jk}$,

$t_{33}=\cos\gamma_{jk}\cdot\cos\alpha_{jk}\cdot\cos\beta_{jk}-\sin\gamma_{jk}\cdot\sin\beta_{jk}$,

$t_{34}=-\cos\gamma_{jk}\cdot\sin\alpha_{jk}$, $t_{41}=c_{jk}+b_{jk}\cdot\cos\alpha_{jk}$,

$t_{42}=\sin\alpha_{jk}\cdot\sin\beta_{jk}$, $t_{43}=\sin\alpha_{jk}\cdot\cos\beta_{jk}$, $t_{44}=\cos\alpha_{jk}$.

The following six parameters define the relative positions of the two coordinate systems $U_jV_jW_j$ and $X_kY_kZ_k$: a_{jk} - a distance from axis W_j to axis Z_k which is measured along the direction of t_{jk}; t_{jk} – a common perpendicular between axes W_j and Z_k; α_{jk} - an angle between positive directions of axes W_j and Z_k which is measured counter clockwise relatively to positive direction of t_{jk}; b_{jk} - a distance from direction of t_{jk} to direction of the axis X_k which is measured along positive direction of an axis Z_k; β_{jk} - an angle between positive directions of t_{jk} and axis X_k which is measured counter clockwise relatively to positive direction of axis Z_k; c_{jk}- a distance from direction of an axis U_j to direction of t_{jk} which is measured along positive direction of an axis W_j; γ_{jk} - an angle between positive directions of axis U_j and t_{jk} which is measured counter clockwise relatively to positive direction of an axis W_j.

III. Generation of Jacobian matrices

3PMCCC has same three legs of kind CCC. Let's consider the i-th leg which has a kinematic chain $OO_{7i}O'_{7i}O_{1i}O'_{1i}O_{2i}O'_{2i}O_{3i}P$ (Fig. 3). A loop-closure equation for the kinematic chain $OO_{7i}O'_{7i}O_{1i}O'_{1i}O_{2i}O'_{2i}O_{3i}PO$ can be written as

$$\mathbf{r}_P=\mathbf{r}_{O_{7i}}+\mathbf{s}_{7i}+\mathbf{a}_{71i}+\mathbf{s}_{1i}+\mathbf{a}_{12i}+\mathbf{s}_{2i}-\mathbf{g}_i,\quad i=1,2,3, \qquad (3)$$

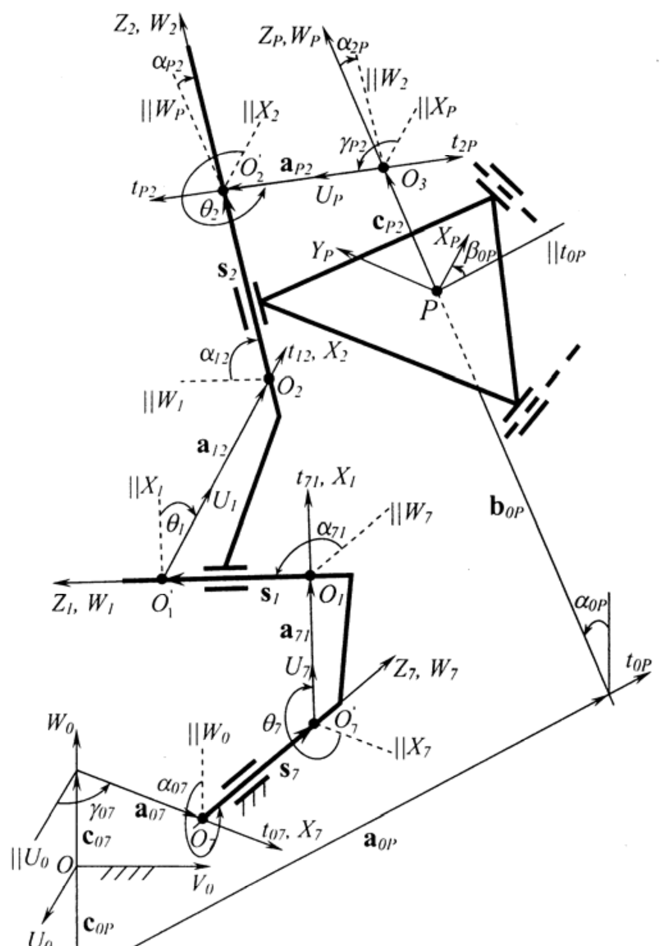

Fig. 3. Geometry of the *i*-th leg of a PM 3CCC

where

$$\begin{bmatrix}1\\ \mathbf{r}_P\end{bmatrix}=\mathbf{T}_{0P}\cdot\begin{bmatrix}1\\ {}^{P}\mathbf{r}_P\end{bmatrix}=\mathbf{T}_{0P}\cdot\begin{bmatrix}1\\0\\0\\0\end{bmatrix}=\begin{bmatrix}1\\ a_{0P}\cos\gamma_{0P}\\ a_{0P}\sin\gamma_{0P}\\ c_{0P}\end{bmatrix} \qquad (4)$$

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$$\mathbf{g}_i = \overline{PO'_{2i}} = \mathbf{c}_{P2i} + \mathbf{a}_{P2i} = \begin{bmatrix} 1 \\ \mathbf{g}_i \end{bmatrix} = \mathbf{T}_{0P} \cdot \begin{bmatrix} 1 \\ {}^{P}\mathbf{g}_i \end{bmatrix} =$$

$$= \mathbf{T}_{0P} \cdot \mathbf{T}_{P2i} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{T}_{0P} \cdot \begin{bmatrix} 1 \\ a_{P2i} \cos \gamma_{P2i} \\ a_{P2i} \sin \gamma_{P2i} \\ c_{P2i} \end{bmatrix} \quad (5)$$

Differentiating (3) with respect to time, we obtain

$$\dot{\mathbf{r}}_P = \dot{s}_{7i}\mathbf{e}_{7i} + \dot{\boldsymbol{\theta}}_{7i} \times \mathbf{r}_{7i} + \dot{s}_{1i}\mathbf{e}_{1i} + \dot{\boldsymbol{\theta}}_{1i} \times \mathbf{a}_{12i} + \\ + \dot{s}_{2i}\mathbf{e}_{2i} - \boldsymbol{\omega}_P \times \mathbf{f}_i, \quad i = 1,\ 2,\ 3 \quad (6)$$

where $\dot{\mathbf{r}}_P = \left[\dot{U}_{0P}, \dot{V}_{0P}, \dot{W}_{0P}\right]^T = \left[\vartheta_{P_{U_0}}, \vartheta_{P_{V_0}}, \vartheta_{P_{W_0}}\right]^T$ - a vector of velocity of a point P; $\dot{\mathbf{s}}_{7i} = \dot{s}_{7i} \cdot \mathbf{e}_{7i}$, $\dot{\boldsymbol{\theta}}_{7i} = \dot{\theta}_{7i} \cdot \mathbf{e}_{7i}$ - vectors of velocity of the active joints; $\mathbf{r}_{7i} = \mathbf{a}_{71i} + \mathbf{s}_{1i} + \mathbf{a}_{12i}$; $\dot{s}_{1i} \cdot \mathbf{e}_{1i}$, $\dot{\boldsymbol{\theta}}_{1i} = \dot{\theta}_{1i} \cdot \mathbf{e}_{1i}$, $\dot{s}_{2i} \cdot \mathbf{e}_{2i}$ - vectors of velocity of the passive joints; $\omega_P = \left[\omega_{P_{U_0}}, \omega_{P_{V_0}}, \omega_{P_{W_0}}\right]^T$ - a vector of angular velocity of a mobile platform; $\mathbf{f}_i = \overline{PO_{2i}} = \mathbf{g}_i - \mathbf{s}_{2i}$.

Dot-multiplying both sides of (6) by $\mathbf{a}_{12i}$ leads to

$$\mathbf{a}_{12i}^T \cdot \dot{\mathbf{r}}_P = \dot{s}_{7i}(\mathbf{a}_{12i}^T \cdot \mathbf{e}_{7i}) + \\ + \dot{\theta}_{7i}\mathbf{e}_{7i}^T \cdot (\mathbf{r}_{7i} \times \mathbf{a}_{12i}) - \omega_P^T \cdot (\mathbf{f}_i \times \mathbf{a}_{12i}),\ i = 1,\ 2,\ 3 \quad (7)$$

Equation (7) can be presented in the matrix form

$$\mathbf{J}_\mathbf{x}\dot{\mathbf{x}} = \mathbf{J}_\mathbf{q}\dot{\mathbf{q}} \quad (8)$$

by using the notation

$$\dot{\mathbf{x}} = \left[\dot{\mathbf{r}}_P^T, \omega_P^T\right]^T, \quad \dot{\mathbf{q}} = \left[\dot{s}_{71}, \dot{\theta}_{71}, \dot{s}_{72}, \dot{\theta}_{72}, \dot{s}_{73}, \dot{\theta}_{73}\right]^T,$$

where the Jacobian matrices $\mathbf{J}_\mathbf{x}$ and $\mathbf{J}_\mathbf{q}$ are defined as follows

$$\mathbf{J}_\mathbf{x} = \begin{bmatrix} \mathbf{a}_{12,1}^T & (\mathbf{f}_1 \times \mathbf{a}_{12,1})^T \\ \mathbf{a}_{12,2}^T & (\mathbf{f}_2 \times \mathbf{a}_{12,2})^T \\ \mathbf{a}_{12,3}^T & (\mathbf{f}_3 \times \mathbf{a}_{12,3})^T \end{bmatrix},$$

$$\mathbf{J}_\mathbf{q} = \begin{bmatrix} \mathbf{a}_{12,1}^T \cdot \mathbf{e}_{71} & 0 & 0 \\ \mathbf{e}_{71}^T(\mathbf{r}_{71} \times \mathbf{a}_{12,1}) & 0 & 0 \\ 0 & \mathbf{a}_{12,2}^T \cdot \mathbf{e}_{72} & 0 \\ 0 & \mathbf{e}_{72}^T(\mathbf{r}_{72} \times \mathbf{a}_{12,2}) & 0 \\ 0 & 0 & \mathbf{a}_{12,3}^T \cdot \mathbf{e}_{73} \\ 0 & 0 & \mathbf{e}_{73}^T(\mathbf{r}_{73} \times \mathbf{a}_{12,3}) \end{bmatrix}^T$$

IV. Singularity Analysis of the PM With 6 DOF

For a PM 3CCC the first type of singularity corresponds to the case when the matrix $\mathbf{J}_\mathbf{q}$ has rank deficiency, i.e. when at least the elements one of the rows of this matrix are zero. Then shall be fulfilled the following conditions

$$\mathbf{a}_{12i}^T \cdot \mathbf{e}_{7i} = 0, \quad (9)$$

$$\mathbf{e}_{7i}^T(\mathbf{r}_{7i} \times \mathbf{a}_{12i}) = 0. \quad (10)$$

From the first condition (9) $\mathbf{a}_{12i} \cdot \perp \mathbf{e}_{7i}$ is followed i.e. the vector $\mathbf{a}_{12i}$ determining the shortest distance between the axes of passive cylindrical joints is perpendicular to the axis of the input cylindrical joint. It is easy to see that this is possible in cases

a) if $\alpha_{71i} = 0$ or π, in always, (11)

b) if $0 < \alpha_{71i} < \pi$, in only $\theta_{1i} = 0$ or π. (12)

The case (11) defines the third type of singularity configuration when a finite change of the coordinate s_{7i} has no effect on the movement of the platform (Fig. 4).

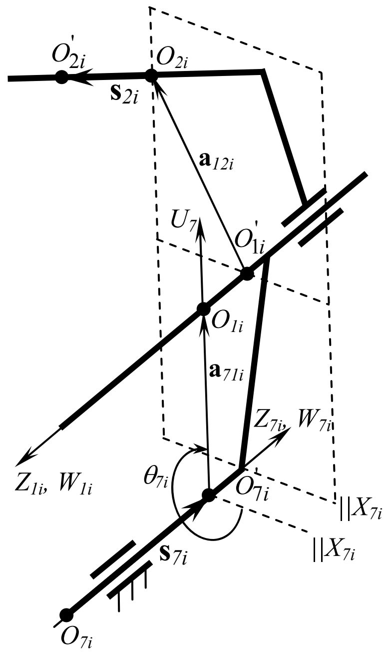

Fig. 4. The third type of singularity of the PM 3CCC

The case (12) implies that $\mathbf{a}_{12i} \parallel \mathbf{a}_{71i}$. At the same time the condition (10) implies that the three vectors $\mathbf{e}_{7i}$, $\mathbf{r}_{7i}$, $\mathbf{a}_{12i}$ must lie either in parallel planes or in the same plane. This arrangement of these vectors, subject to (12) holds only when the point O_{2i} defining the end of the vector $\mathbf{r}_{7i}$ lies in the plane $O'_{7i}U_{7i}W_{7i}$ (this plane is denoted by σ_i) (Fig. 5).

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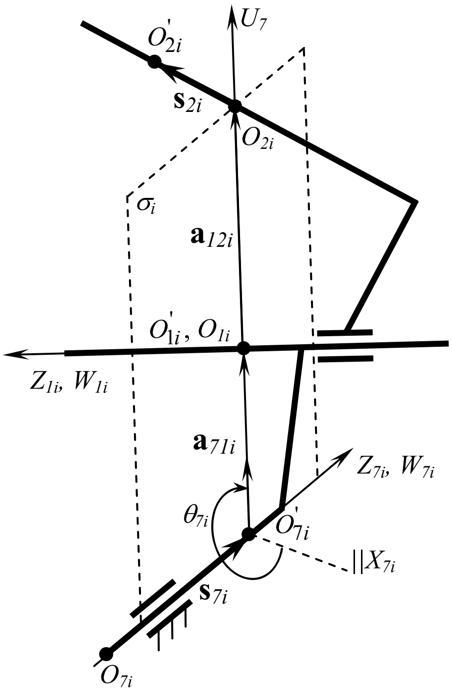

Fig. 5. The first type of singularity of the PM 3CCC

Since the vectors $\mathbf{a}_{12i}$, $\mathbf{a}_{71i}$ determine the shortest distance between the cylindrical joints, in this type of configuration the i-th leg is fully stretched or folded position. Therefore the set of Cartesian velocities of platform which corresponds to the velocities of point O_{2i} parallel to the vectors $\mathbf{a}_{12i}$, $\mathbf{a}_{71i}$ may not be reproduced. This set of the Cartesian velocities is given by the set of rotation of platform around an arbitrary line in the plane containing the point O_{2i} and orthogonal to the vectors $\mathbf{a}_{12i}$, $\mathbf{a}_{71i}$. In addition, any force applied to the platform and line of action which lies in the σ_i, and a pair of forces applied parallel to this plane do not affect the rotational component of the i-th cylindrical actuator. This is explained by the fact that the moments of force and a pair of forces with respect to the axis $O_{7i}Z_{7i}$, i.e. relative to the axis of rotation of the input joint are zero.

The second type of singularity occurs when matrix $\mathbf{J_x}$ lacks the range, that is when rows or columns of this matrix are linearly dependent. This type of singularities in contrast to the first is located inside the workspace. For the corresponding configuration of the PM3CCC the nonzero output Cartesian velocities $\dot{\mathbf{x}}$ of the platform are displayed in the zero vector by conversion $\mathbf{J_x}$. These speeds of platform are possible even in still actuators of input joints. Let's rewrite the matrix $\mathbf{J_x}$ in the form

$$\mathbf{J_x} = \begin{bmatrix} a_{12,1X} & a_{12,1Y} & a_{12,1Z} & n_{1X} & n_{1Y} & n_{1Z} \\ a_{12,2X} & a_{12,2Y} & a_{12,2Z} & n_{2X} & n_{2Y} & n_{2Z} \\ a_{12,3X} & a_{12,3Y} & a_{12,3Z} & n_{3X} & n_{3Y} & n_{3Z} \end{bmatrix}, \tag{13}$$

where the elements of the first three columns are the components of one group of vectors $\mathbf{a}_{12i}$, and elements of the last three columns - the second group of components of the vectors $\mathbf{n}_i = \mathbf{f}_i \times \mathbf{a}_{12i}$ relative to an inertial coordinate system, i.e.

$$\mathbf{a}_{12i} = [a_{12iX}, a_{12iY}, a_{12iZ}]^T,$$
$$\mathbf{n}_i = \mathbf{f}_i \times \mathbf{a}_{12i} = [n_{iX}, n_{iY}, n_{iZ}]^T, \; i = 1, 2, 3., \tag{14}$$

For definiteness assume that the vectors $\mathbf{g}_i$ lie in a plane perpendicular to the axes of the platform and kinematic cylindrical joints of platform.

Consider the case where the three columns of vectors are linearly dependent. This case means that all the vectors of this group are parallel to each other and parallel to a common plane.

Let's assume without proof the the following theorem.

Theorem. Two vectors laying in different and mutually intersecting planes are parallel if and only if each of these vectors are parallel to the line along which their planes are intersected.

The statement 1. The parallelism of all vectors $\mathbf{n}_i$, $i = 1,2,3$ is possible only if each of these vectors are parallel to the axis PZ_P of the coordinate system $PX_PY_PZ_P$ associated with the mobile platform.

This statement means that in this case the vectors $\mathbf{f}_i$ and $\mathbf{a}_{12i}$, $i = 1,2,3$ according to the definition of the vector product of two vectors must lie in the plane PX_PY_P. Then all cylindrical joints axis of the platform must also lie in the plane PX_PY_P (this plane is denoted by Ω) (Fig. 6). Thus we find that the last z-components of vectors $\mathbf{a}_{12i}$, $i = 1,2,3$ with respect to the local coordinate system $PX_PY_PZ_P$ should be zero, i.e. ${}^P a_{12iZ} = 0$, $i = 1,2,3$.

We can get the form (8) with respect to the coordinate system $PX_PY_PZ_P$ associated with the platform. For this case the third, fourth and fifth columns of the matrix $\mathbf{J_x}$ will be zero. Then the zero space of the matrix $\mathbf{J_x}$ is generated by the vector $[0, 0, 1, 1, 1, 0]^T$. This zero space corresponds to the set of local rotations of the platform relative to an arbitrary axis of plane PX_PY_P and local displacement along the axis Pz_P with fixed actuators. In addition, the force acting along the axis Pz_P and a pair of forces applied to the platform in a parallel axis Pz_P of plane do not affect to actuators, i.e. manipulator is not able to resist these loads.

The statement 2. All vectors $\mathbf{n}_i$, $i = 1,2,3$ can be simultaneously parallel to one arbitrary plane if all axis of cylindrical joints of platform are orthogonal to this plane.

Assume that all axis of cylindrical joints are orthogonal to the plane PX_PY_P of the platform (Fig.7). Then the vectors $\mathbf{a}_{12i}$ and $\mathbf{n}_i$ $i = 1,2,3$ lie in planes parallel to the plane PX_PY_P and therefore ${}^P a_{12iZ} = 0$ and ${}^P n_{iZ} = 0$, $i = 1,2,3$. That is the third and fourth columns of the matrix $\mathbf{J_x}$, compiled on a system of coordinates $PX_PY_PZ_P$, will be zero. In this case the zero space of matrix $\mathbf{J_x}$ is generated by vector $[0, 0, 1, 1, 0, 0]^T$. This zero space corresponds to a local rotation of the platform relative to the axis PZ_P and its finite movement along the axis PZ_P with fixed actuators. In addition, the

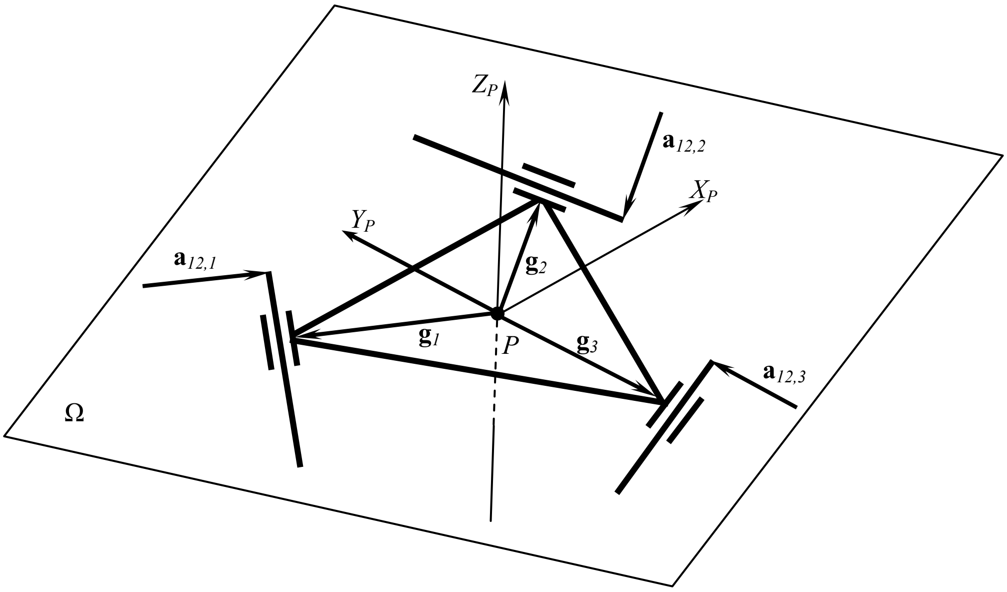

Fig. 6. The second type of singularity of the PM 3CCC

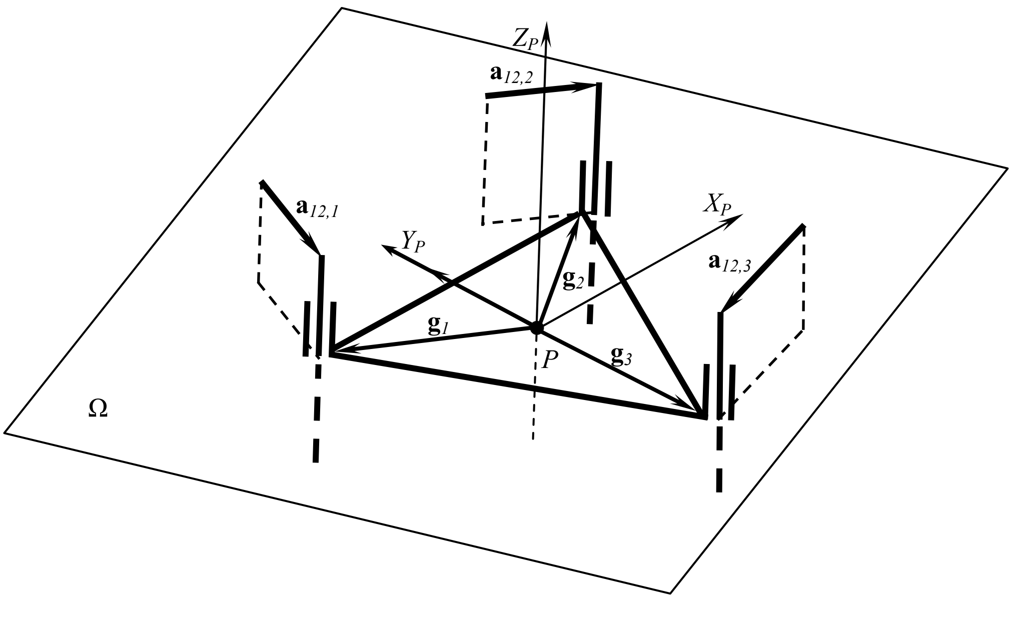

Fig 7. The third type of singularity of the PM 3CCC

force acting along the axis PZ_P and a pair of forces applied to the platform in a parallel axis PZ_P of a plane do not affect to the actuators, i.e. parallel manipulator is not able to resist these loads.

Thus, we found another case that led to the singularity of the third kind when the axis of the cylindrical joints of the platform are perpendicular to its plane, which leads to the finite no controlled movements of platform in the orthogonal direction to the plane of platform in the fixed actuators.

V. Conclusion

On the basis of the equations of closed kinematic chains of the legs of a PM3CCC the velocity vectors are formed.

On the basis of the Jacobi matrix the conditions for the existence of singular configurations are defined and their geometrical and physical interpretations are given. These results can be used in designing of construction of such manipulators. Besides the obtained conditions of configuration singularities can be expressed through constants and variable parameters of the parallel manipulator that is important for the control of such configurations.

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