Numerical Determinations with Finite Differences Method of Prismatic Beams Subjected to Torsion

Amado Stefan*, Marin Lupoae*, Daniel Constantin*, Catalin Baciu *

Abstract—**The paper analyzes the determination of tangential stress for two sections with is subject linear elastic torsion, with finite difference method. It discusses the constant and variable distance between the grid points.**

Index Terms **– tangential stress, Finite Differences Method, grid point.**

I. INTRODUCTION

Torsion of cylindrical shafts has been a topic in the classical theory of elasticity for a long time (Timoshenko and Goodier, 1970). The stiffness of a cylindrical shaft under torsional loading is often of interest in the study of torsion problems. Therefore, the uniform (St. Venant) and non-uniform torsion problem of structural components has long been the subject of theoretical and practical study in the field of solid mechanics (Chen et al., 2001).

Structural elements with very different cross sectional shapes are widely used in various engineering structures. The exact solutions for torsion have been found for some simple cross-sectional shapes such as circles, ellipses, and triangles. However, in the theory of elasticity, it is difficult to obtain analytical solutions for complicated cross-sections.

To solve general cross-sectional problems, numerical methods are usually necessary. For more complicated shapes, numerical methods are usually employed. Examples include the finite difference method (Ely and Zienkiewicz, 1960), the finite element method (Herrmann, 1965; Karayannis, 1995; Li at al., 2000), and the boundary element method (Jawson and Ponter, 1963; Friedman and Kosmatka, 2000; Sapountzakis, 2001; Sapountzakis and Mokos, 2001, 2003).The first two of these methods require the whole cross-section to be discretised into elements or grids.

For a complicated section, the finite difference method and the finite element method methods require a large number of elements or grids. The boundary element method requires discretisation at the boundary only, but in this method one has a singular integral on the boundary. Recently, mesh-free methods have developed fast as alternative solution methods (Di Paola et al., 2008).

Membrane, electrical, and fluid flow analogies have also been used in explaining torsion problems (Zhen-Min and Kexue, 1986).

They investigated the dependence of torsional properties for wall thickness. Using an approximated model and starting from Bredt's formulas (valid only for thin-walled closed sections), Serra (1996) obtained a formulation for the calculation of the torsional problem of solid cross-sections. Wang (1998) introduced the method of eigen function expansion and matching to solve the torsion problem of arbitrary shaped tubes described by curved and straight pieces. Najera and Herrera (2005) presented a method to approximate the torsional rigidity of non-circular solid cross sections encountered in mechanisms and machines.

Hematiyan and Doostfatemeh (2007) proposed a simple formulation for torsion analysis of moderately thick hollow tubes with polygonal shapes.

II. ANALYTICAL APPROACH

Consider a prismatic elastic bar with a constant cross section, and loaded with forces only in sections of the ends equivalent control torque. On side surface conditions are:

$$\begin{aligned} p_{nx} &= \tau_{xy} m + \tau_{xz} n = 0 \\ p_{ny} &= \sigma_y m + \tau_{yz} n = 0 \\ p_{nz} &= \tau_{yz} m + \sigma_z n = 0 \end{aligned} \qquad (1)$$

and the end of the beam sections:

$$\begin{aligned} T_y &= \iint_A \tau_{xy} \cdot dA = 0; \\ T_z &= \iint_A \tau_{xz} \cdot dA = 0; \\ M_x &= \iint_A \left(\tau_{xz} \cdot y - \tau_{xy} \cdot z\right) \cdot dA = M_t \end{aligned} \qquad (2)$$

Assumptions considered by Saint-Venant [1,3] are the following:

- cross-sections rotate with $\varphi(x)$ angle proportional to its distance from the origin, the contour section is unaltered,
- the warping of the cross section is the same for all sections.

If point A of a cross section is located at a distance x from the origin, with coordinates x, y, z, the position on the

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deformed shape is $x+u, y+v, z+w$ (Figure 1). The torsion angle of the section, $\varphi(x)$, is proportional to the position of point A:

$$\varphi(x)=\theta\cdot x \tag{3}$$

where θ is the specific twist angle.

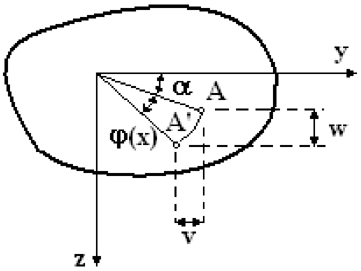

Figure 1 Displacement of point A

Assuming small deformations resulting displacement of point A as:

$$u=\theta\cdot\psi(y,z)$$
$$v=r\cdot\cos(\alpha+\varphi)-r\cdot\cos\alpha\cong-\varphi z \tag{4}$$
$$w=r\cdot\sin(\alpha+\varphi)-r\cdot\sin\alpha\cong\varphi y$$

where $\psi(y,z)$ is cross-sectional warping function.

Using the relationship between displacement and strain specific results:

$$\varepsilon_x=\varepsilon_y=\varepsilon_z=0 \tag{5}$$

$$\gamma_{xy}=\theta\left(\frac{\partial\psi}{\partial y}-z\right)$$
$$\gamma_{yz}=0 \tag{6}$$
$$\gamma_{xy}=\theta\left(\frac{\partial\psi}{\partial z}+y\right)$$

From the balance equations of elasticity theory in the absence of mass forces, while the $\sigma_x=\sigma_y=\sigma_z=\tau_{yz}=0$

$$\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}=0;$$
$$\frac{\partial\tau_{xy}}{\partial x}=0; \tag{7}$$
$$\frac{\partial\tau_{xz}}{\partial x}=0$$

hence shear stresses are of the form:

$$\tau_{xy}=G\theta\frac{\partial\Phi}{\partial z}$$
$$\tau_{xz}=-G\theta\frac{\partial\Phi}{\partial y}, \tag{8}$$

the constant product G·θ being introduced to obtain simple relations.

Function $\Phi=\Phi(y,z)$ is called stress function or function of Prandtl stress. It shows that:

$$\Delta\Phi=\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}=-2 \tag{9}$$

and

$$\Delta\psi=0. \tag{10}$$

where ψ is the warp function.

Provided that total of tension tangential to the contour tangent section, that function is constant tension on the contour:

$$\Phi(\bar{y},\bar{z})=0 \tag{11}$$

Using the conditions on sections of heads, resulting:

$$M_t=2\cdot G\cdot\theta\cdot\iint_A\Phi(y,z)dydz \tag{12}$$

Integral $I_t=2\cdot\iint_A\Phi(y,z)dydz$ is the conventional torsion moment of inertia of the bar.

III. Numerical Determinations with Finite Differences Method, Using Constant Distance Between Point

Applying the finite difference method for stress function equation (9), we obtain:

$$\frac{\Phi_{i-1,j}-2\cdot\Phi_{i,j}+\Phi_{i+1,j}}{h_y^2}+\frac{\Phi_{i,j-1}-2\cdot\Phi_{i,j}+\Phi_{i,j+1}}{h_z^2}= \tag{13}$$
$$=-2$$

this was considered a network of points on the section so that the index "i" varies as y coordinate and the index "j" varies as z coordinate. Considering the equal division points:

$$h_y=h_z=h \tag{14}$$

resulting:

$$\Phi_{i-1,j}+\Phi_{i+1,j}+\Phi_{i,j-1}+\Phi_{i,j+1}-4\Phi_{i,j}=-2\cdot h^2. \tag{15}$$

From equation (11), it follows that at a point determined by the indices "i" and "j" located on the perimeter section stress function is 0. For the "L" section the grid point how lies outside of the section domain, the stress function is considered also zero.

Using Gaussian reduction, the solution for the system for all the points of grade is:

$A\cdot\Phi=b$,

where A is the coefficient matrix and "b" is the free terms vector.

For the iterative method Gauss-Seidel the (15) is written as:

$$\Phi_{i,j}=\frac{1}{4}\left(\Phi_{i-1,j}+\Phi_{i+1,j}+\Phi_{i,j-1}+\Phi_{i,j+1}+2\cdot h^2\right) \tag{16}$$

and the algorithm is a two step process

$$\Phi_{i,j}^{(int)}=\frac{1}{4}\left(\Phi_{i-1,j}+\Phi_{i+1,j}+\Phi_{i,j-1}+\Phi_{i,j+1}+2\cdot h^2\right), \tag{17}$$

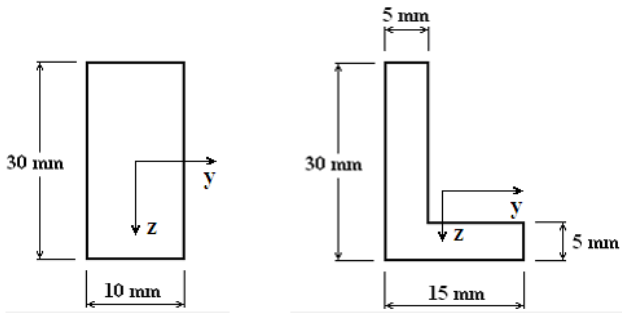

Figure 2 Dimensions of sections

$$\Phi_{i,j}^{(new)}=(1-\omega)\Phi_{i,j}^{(old)}+\omega\cdot\Phi_{i,j}^{(int)}. \tag{18}$$

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In the first step of (17), we calculate an intermediate update for $\Phi_{i,j}$. Then the true update of $\Phi_{i,j}$ is computed in (18). This true update is a weighted combination of the intermediate update and the old value of $\Phi_{i,j}$.

We considered two sections (Figure 2), Oy axis is horizontal and vertical Oz, bars and the steel $E = 2{,}1 \cdot 10^5\ \mathrm{N/mm^2}$ and $\mu = 0{,}3$. Both beams are subject to 10000 N·mm torque.

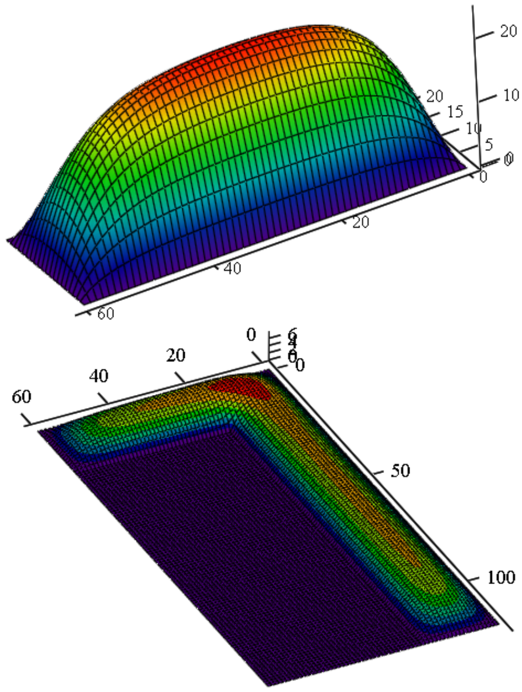

Figure 3 Stress function graph for the two sections

Solving the system leads to the following representation for the unknown stress function in two cases (Figure 3).

Using relation (12) is computed the value of the specific twist angle, using the Simpson formula

$$\theta = \frac{M_t}{2 \cdot G} \sum_{i=1}^{n-1} \sum_{j=1}^{m-1} \left(y_{i+1} - y_i\right)\left(z_{j+1} - z_j\right) \Phi\xi_{i,j} \,, \tag{19}$$

where “n” is the number of points in the Oy direction, “m” is the number of points in Oz direction and

$$\Phi\xi_{i,j} = \frac{\Phi_{i,j} + \Phi_{i+1,j} + \Phi_{i,j+1} + \Phi_{i+1,j+1}}{4} \,.$$

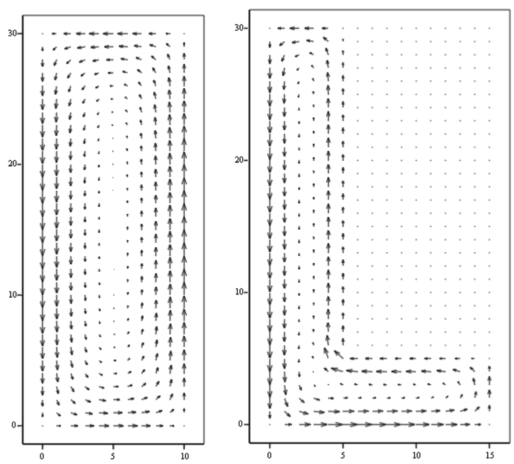

Figure 4 Tangential stresses in the vector representation

Stress function is known, using relations (8) is determined tangential stress, and a vectorial representation is shown in Figure 4. Figures 5 and 6 are presented and tangential stress variation for the two cases considered.

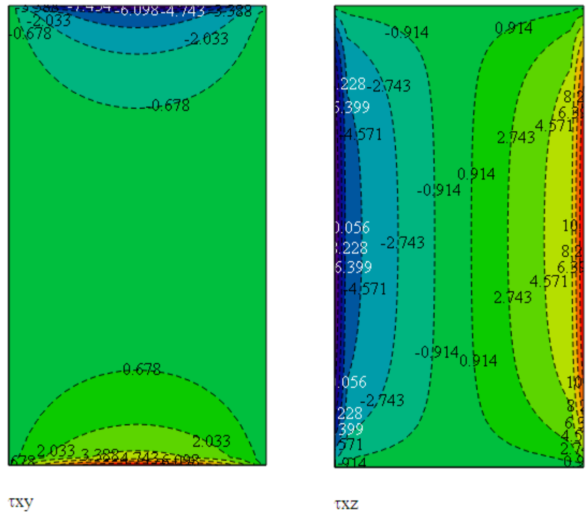

Figure 5 Variation of shear stress for rectangular section

For interior points, the tangential stress is computed using

$$\begin{aligned} \tau xz_{i,j} &= -G \cdot \theta \frac{\Phi_{i+1,j} - \Phi_{i-1,j}}{2 \cdot h}, \\ \tau xy_{i,j} &= G \cdot \theta \frac{\Phi_{i,j+1} - \Phi_{i,j-1}}{2 \cdot h}. \end{aligned} \tag{20}$$

For boundary points

$$\begin{aligned} \tau xz_{i,j} &= -G \cdot \theta \frac{\Phi_{i+1,j} - \Phi_{i,j}}{h}, \\ \tau xy_{i,j} &= G \cdot \theta \frac{\Phi_{i,j+1} - \Phi_{i,j}}{h}, \end{aligned} \tag{21}$$

If the point located by (i,j) laying on boundary and the section domain is in right (for τ_{xz}), or down (for τ_{xy}),

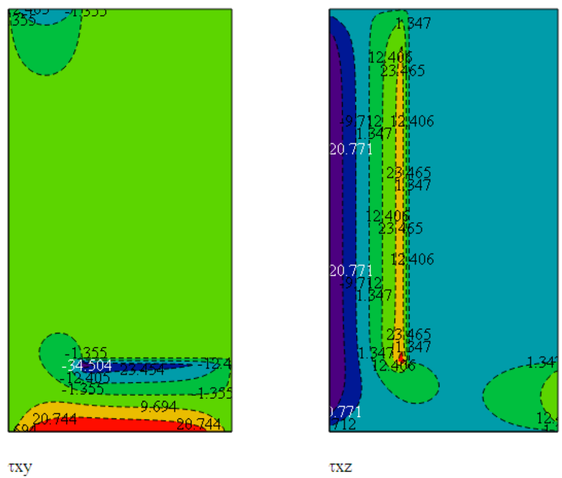

Figure 6 Variation of shear stress for “L” section

$$\begin{aligned} \tau xz_{i,j} &= -G \cdot \theta \frac{\Phi_{i,j} - \Phi_{i-1,j}}{h}, \\ \tau xy_{i,j} &= G \cdot \theta \frac{\Phi_{i,j} - \Phi_{i,j-1}}{h}, \end{aligned} \tag{22}$$

If the point is laying on boundary the section domain is in left (for τ_{xz}), or up (for τ_{xy})

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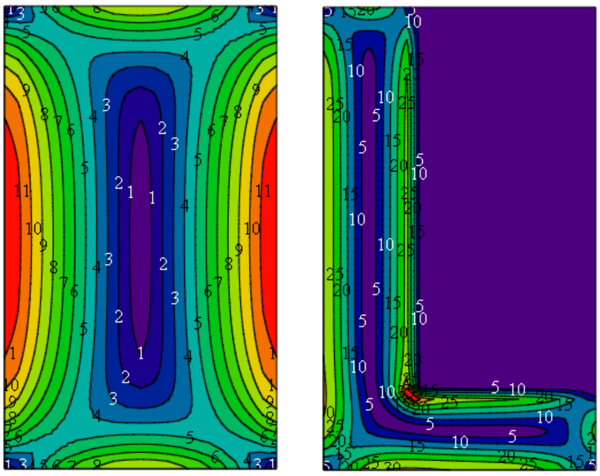

Figure 7 Variation of resultant shear stress for the two sections

Table I presents the tangential stress variations and specific twist angle depending on the grid step for rectangular section.

Table I Tangential stress variations

h (mm)	$\tau_{xy_{max}}$	$\tau_{xz_{max}}$	θ
1	8,298	11,3729	$1,5911\cdot10^{-5}$
0,5	8,8087	11,8847	$1,5736\cdot10^{-5}$
0,333	8,9965	12,0719	$1,5705\cdot10^{-5}$
0,25	9,0942	12,1689	$1,5696\cdot10^{-5}$
0,2	9,1542	12,2282	$1,5693\cdot10^{-5}$
0,1667	9,1949	12,2682	$1,5692\cdot10^{-5}$
0,1429	9,2244	12,2969	$1,5692\cdot10^{-5}$
0,125	9,2468	12,3186	$1,5692\cdot10^{-5}$

Maximum stress values determined using trigonometric series for solution of stress function for rectangular section (analytical solution), lead to solutions, in the middle of large side:

$$\tau_{xz_{max}} = \frac{Mt}{\alpha h b^2}\cdot\frac{8}{\pi^2}\cdot\sum_{n=1,3,5,...}^{\infty}\frac{1}{n^2}\left(1-\frac{1}{\cosh\left(n\pi h/2b\right)}\right) = 12,4844\ N/mm^2$$

where

$$\alpha = \frac{1}{3}\left(1-\frac{192}{\pi^5}\frac{b}{h}\sum_{n=1,3,5,...}^{\infty}\frac{1}{n^5}\tanh\left(\frac{n\pi h}{2b}\right)\right).$$

In the middle of small side $\tau_{xy_{max}} = 9,4007\ N/mm^2$, and the specific twist angle $\theta = 1,5692\cdot10^{-5}\ \frac{rad}{mm}$.

Conventional torsion moment of inertia of the section is $I_t = 2\cdot\iint_A \Phi(y,z)\,dydz$, and calculated $I_t = 3944,02\ mm^4$ for rectangular section and $I_t = 786,5\ mm^4$ for “L” section.

Analyzing the variation of $\tau_{xz_{max}}$ from Figure 8 we see that the process converge, but it is a difference between the calculated value and the analytical solution.

The value of $\tau_{xz_{max}}$ depends by the distance between the

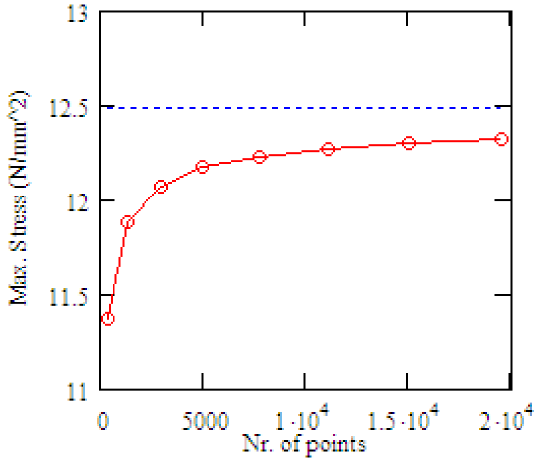

Figure 8 Variation of $\tau_{xz_{max}}$ with number of points

point laying on boundary and the neighbor point inside the section domain. Smaller value lead to better estimation of $\tau_{xz_{max}}$ and an increased size of unknowns system.

IV. Numerical Determinations with Finite Differences Method, Using Variable Distance Between Points

Utilizing variable distance between mesh points leads to a more accurate estimate of $\tau_{xz_{max}}$ using a small number of grid points - $11\times21=231$ points (figure 9).

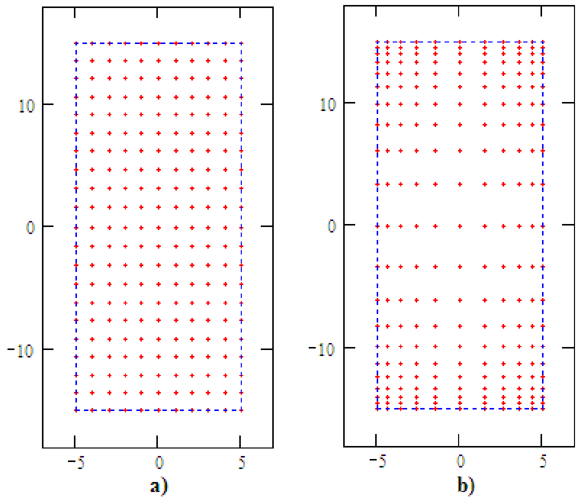

Figure 9 a) Equal distance grid points; b) Variable distance grid points with a same number of points to case a)

The points in positive Oyz space, figure 19, generate the distance between points function by a geometrical progression with ry ratio for Oy axis and rz ratio for Oz axis. Appling symmetry generates the rest of points:

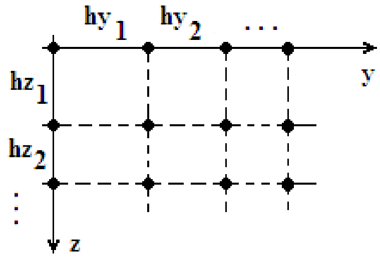

Figure 10 Grid of points with variable distance

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$$hy_i = h0y \cdot ry^{i-1}$$
$$hz_j = h0z \cdot rz^{j-1} \tag{23}$$

Selecting the number of interval for each semi axe (ny, nz) and the ratio (ry, rz), the length of first interval is

$$hoy = \frac{b}{2} \cdot \frac{1-ry}{1-ry^{ny}}, \quad hoz = \frac{h}{2} \cdot \frac{1-rz}{1-rz^{nz}}. \tag{24}$$

Using central finite difference scheme for obtaining the general relation, we can write:

$$\frac{\partial \Phi\left(y_{i+1/2}, z_j\right)}{\partial y} = \frac{\Phi_{i+1,j} - \Phi_{i,j}}{y_{i+1} - y_i}; \tag{25}$$

$$\frac{\partial \Phi\left(y_{i-1/2}, z_j\right)}{\partial y} = \frac{\Phi_{i,j} - \Phi_{i-1,j}}{y_i - y_{i-1}} \tag{26}$$

$$\frac{\partial^2 \Phi\left(y_i, z_j\right)}{\partial y^2} = \frac{\dfrac{\partial \Phi\left(y_{i+1/2}, z_j\right)}{\partial y} - \dfrac{\partial \Phi\left(y_{i-1/2}, z_j\right)}{\partial y}}{y_{i+1/2} - y_{i-1/2}} = = \frac{\dfrac{\Phi_{i+1,j} - \Phi_{i,j}}{y_{i+1} - y_i} - \dfrac{\Phi_{i,j} - \Phi_{i-1,j}}{y_i - y_{i-1}}}{0.5 \cdot \left(y_{i+1} - y_{i-1}\right)}. \tag{27}$$

Similarly,

$$\frac{\partial^2 \Phi\left(y_i, z_j\right)}{\partial z^2} == \frac{\dfrac{\Phi_{i,j+1} - \Phi_{i,j}}{z_{i+1} - z_i} - \dfrac{\Phi_{i,j} - \Phi_{i,j-1}}{z_i - z_{i-1}}}{0.5 \cdot \left(z_{i+1} - z_{i-1}\right)}. \tag{28}$$

Solving the problem for various value of ratio, it observed that for some value of ratio the value of maximal tangential stress match the analytical solution, but the process is not convergent, figure 11. In fact, for lower value of ratio the error is increasing because the numerical determination of the integral of conventional torsion moment of inertia.

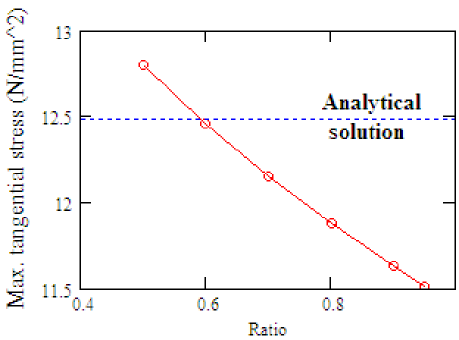

Figure 11 Variation of $\tau_{xz_{max}}$ with ratio number

V. Conclusions

Accelerating convergence results can be made using a grid with variable distance between grid points, with higher density points to the frontier section and a low density at the interior. Thus a smaller number of points (less time of calculation) can achieve good results. The finite difference method allows the study of stress distribution for sections in which an analytical approach is difficult. However, there are sections for which that method poses difficulties, such as curved boundary domains.

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