Thermal Convection of Nanofluids: A Non-Fourier Perspective and Linear Stability Analysis

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***Abstract*—A linear stability analysis is implemented to examine the onset of thermal convection of nanofluids. The fluid is assumed to have a non-zero relaxation time, related to nanoparticle concentration. The fluid thus obeys the Cattaneo-Vernotte constitutive equation instead of Fourier's law of heat. It is found, in agreement with experiment, that both steady and oscillatory convection can set in upon loss of conduction for relatively low and high nanoparticle concentrations, respectively.**

***Keywords*—Nanofluids, non-Fourier, natural convection, linear stability.**

I. Introduction

THE composition of a nanofluid consists of a base fluid (e. g., water, oil or organic based liquid) containing a small volume fraction (1-5%) of nanoparticles (1-100nm in diameter). The exciting feature about nanofluids is that they allow substantial enhancement in heat transfer (as much as 40 percent [1]), despite the low volume fraction of the nanoparticles. This makes nanofluids extremely valuable, especially in processes where cooling is of primary concern, and thus the focus has now turned to convecting properties of nanofluids. One advantage that a fluid containing nanoparticles has over its milliparticle and microparticle counterparts is the small size of the nanoparticles, which is on the same order of magnitude as the molecules in the base fluid. This allows the solution to exist in a very stable manner without the occurrence of gravitational settling or particle agglomeration [1,2]. The most commonly used base fluids are water and organic fluids such as ethanol and ethylene glycol. The materials that have been utilized as nanoparticles include oxides of aluminum and silicon, as well as metals such as copper and gold [1]. Diamonds and nanotubes have also been widely experimented with [1]. If the fluid in a cooling process has improved thermal properties, then the workload of other components in the system (e. g., a pump) can be reduced. Better thermal conductivity and heat transfer coefficients would allow systems involving microelectronics to run with increased power while still maintaining appropriate operating temperatures, furthering the processing capabilities. The potential positive impact of nanofluids in many applications is very promising.

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Given the very small size of nanoparticles, it is not unconceivable that the nanofluid mixture may still be considered as a one-phase isotropic mixture, as opposed to viewing it as a two-phase liquid. In the latter case, a myriad of additional effects resulting from the interaction between the base fluid and nanoparticles must be incorporated [1], making cumbersome the problem formulation [3, 4]. These include the effects of particle inertia, drag, Brownian motion and thermophoresis, to cite a few. Alternatively, and similarly to non-Newtonian fluids [5], the nanofluid may be viewed as a non-Fourier fluid. Wei and Wang [6] established the equivalence between the two-phase and non-Fourier approaches for heat conduction. Essentially, a non-Fourier fluid follows the Cattaneo-Vernotte heat equation, where a time derivative of the heat flux is added, multiplied by a relaxation time.

The addition of the partial time derivative does not completely solve the problem of instantaneous thermal relaxation [7-9]. The Cattaneo-Vernotte equation is not a frame-invariant constitutive relation and, as such, is restricted to non-deformable media. Several objective derivatives have been applied to remedy this situation. However, they each have had their own shortcomings. The most promising modification comes from Christov [10] as well as Khayat [11], whose use of the Oldroyd's upper-convected derivative, which leads to the frame indifferent Cattaneo equation. It can also yield a single equation for the temperature field, an advantage other invariant formulations do not possess [10]. This equation replaces Fourier's law whenever the relaxation time is relevant, and quickly collapses back onto Fourier's law whenever it is not.

II. Problem formulation

A. Governing equations and boundary conditions

Consider a thin layer of a Newtonian non-Fourier liquid confined between the (X, Y) planes at $Z=0$ and $Z=D$, maintained at fixed temperatures $T_0+\delta T$ (hotter) and T_0 (cooler), respectively. The fluid layer is assumed to be of infinite horizontal extent. Convection sets in when buoyancy effect exceeds a critical threshold. The gravity acceleration vector is given by $\mathbf{g} = -g\mathbf{e}_z$, where $\mathbf{e}_z$ is the unit vector in the Z direction. The fluid density, ρ, is assumed to depend on the temperature T, following

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$$\rho = \rho_0 \left[1 - \alpha_T \left(T - T_0 \right) \right], \quad (1)$$

where α_T is the coefficient of volume expansion and ρ_0 is the mass density of the fluid at T_0. The fluid is assumed to be incompressible, of specific heat at constant pressure C_p, thermal conductivity k and viscosity μ. In this case, the general governing equations for a non-Fourier fluid comprise the conservation of mass, linear momentum and energy, as well as the constitutive equations. In this case, the conservation equations are given by

$$\nabla \cdot \mathbf{V} = 0, \quad (2$$

$$\rho_0 \left(\mathbf{V}_t + \mathbf{V} \cdot \nabla \mathbf{V} \right) = -\nabla P - \rho g \mathbf{e_z} + \mu \Delta \mathbf{V}, \quad (3)$$

$$\rho_0 c_p \left(T_t + \mathbf{V} \cdot \nabla T \right) = -\nabla \cdot \mathbf{Q}, \quad (4)$$

where ∇ and Δ are the gradient and Laplacian operators, respectively, and a subscript denotes partial differentiation. Here **V** = (U, W) is the velocity vector, P is the pressure, T is the temperature and **Q** is the heat flux vector. Note that the Boussinesq's approximation, which states that the effect of compressibility is negligible everywhere in the conservation equations except in the buoyancy term, is assumed to hold. In this work, the heat flux is assumed to be governed by [10, 11]

$$\tau \left(\mathbf{Q}_t + \mathbf{V} \cdot \nabla \mathbf{Q} - \mathbf{Q} \cdot \nabla \mathbf{V} \right) = -\mathbf{Q} - k \nabla T. \quad (5)$$

where τ is the relaxation time. The boundary conditions at the lower and upper surfaces are taken to correspond to free-free conditions. In this case

$$\mathbf{V}\left(X, Z = 0, t\right) \cdot \mathbf{e}_Z = \mathbf{V}\left(X, Z = D, t\right) \cdot \mathbf{e}_Z = 0,$$

$$\mathbf{V}_{ZZ}\left(X, Z = 0, t\right) \cdot \mathbf{e}_Z = \mathbf{V}_{ZZ}\left(X, Z = D, t\right) \cdot \mathbf{e}_Z = 0, \quad (6)$$

$$T\left(X, Z = 0, t\right) = T_0 + \delta T, \quad T\left(X, Z = D, t\right) = T_0.$$

Other boundary conditions could have been adopted, such as the rigid-rigid or rigid-free conditions. However, the free-free conditions are convenient and the most commonly used in the literature. Moreover, no qualitative change in behaviour is expected if one set of boundary conditions is used or another [5]. In fact, Khayat carried out a comparative study in the case of the similar problem of rotating flow [5].

The base state corresponds to the stationary heat conduction, which remains the same as for a Fourier fluid. In this case, the temperature, pressure gradient and heat flux are given by

$$T_B = -\left(Z/D \right) \delta T + T_0 + \delta T,$$

$$dP_B / dZ = -\rho_0 \left[1 - \alpha_T \delta T \left(1 - Z/D \right) \right] g, \quad (7)$$

$$\mathbf{Q}_B = \left(0, k \frac{\delta T}{D} \right).$$

B Dimensionless problem

The problem is conveniently cast in dimensionless form by taking the length, time and velocity scales as D, $\frac{D^2}{\kappa}$ and $\frac{\kappa}{D}$, respectively, where $\kappa = \frac{k}{\rho c_p}$ is the thermal diffusivity. Let $p = \frac{D^2}{\kappa \mu} \left(P - P_B \right)$ and $\theta = \frac{T - T_B}{\delta T}$ be the dimensionless pressure and temperature deviations from the base conduction state. In this case, one obtains the problem in dimensionless form:

$$\nabla \cdot \mathbf{v} = 0, \quad (8)$$

$$Pr^{-1} \left(\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + Ra \theta \mathbf{e}_Z + \Delta \mathbf{v}, \quad (9)$$

$$\theta_t + \mathbf{v} \cdot \nabla \theta = -\nabla \cdot \mathbf{q} + w, \quad (10)$$

$$C \left(\mathbf{q}_t - \mathbf{v}_Z + \mathbf{v} \cdot \nabla \mathbf{q} - \mathbf{q} \cdot \nabla \mathbf{v} \right) = -\mathbf{q} - \nabla \theta \quad (11)$$

where **v**(u, w) and **q** are the dimensionless velocity and heat flux vectors, respectively. The following non-dimensional groups have been introduced, namely the Prandtl number, Pr, the Rayleigh number, Ra, and the Cattaneo number, C, given by

$$Pr = \frac{\nu}{\kappa}, \quad C = \frac{\tau \kappa}{D^2}, \quad Ra = \frac{\delta T \alpha_T g D^3}{\nu \kappa} \quad (12)$$

Clearly, the Fourier limit is recovered by taking the limit $C \to 0$.

The problem can be simplified somewhat by casting it in terms of the scalar variable $j \equiv \nabla \cdot \mathbf{q}$. Thus, upon taking the divergence of equation (11), and noting the identity $\nabla \cdot \left(\mathbf{a} \cdot \nabla \mathbf{b} \right) = \nabla \mathbf{a} : \nabla \mathbf{b} + \mathbf{a} \cdot \nabla \left(\nabla \cdot \mathbf{b} \right)$, **a** and **b** being two general vectors, one obtains the following constitutive equation for j:

$$C \left(j_t + \mathbf{v} \cdot \nabla j \right) + j = -\Delta \theta, \quad (13)$$

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These equations must be solved subject to the following homogeneous boundary conditions:

$$\mathbf{v}(x,z=0,t)=\mathbf{v}(x,z=1,t)=0,$$

$$\theta(x,z=0,t)=\theta(x,z=1,t)=0.$$

where the continuity equation (8) is used. It is interesting to note the presence of two linear terms of non-Fourier origin, on the left-hand side of equation (11), namely the transient term and the velocity gradient in the z direction. This contrasts with the convection of viscoelastic fluids where the transient term is the only linear term that survives [5].

III. LINEAR STABILITY ANALYSI

Similarly to a Fourier fluid, the conduction of a non-Fourier fluid is lost to convection once a critical value of the Rayleigh number, Ra_c, is exceeded. However, in contrast to a Fourier fluid, and similarly to a viscoelastic fluid [5], non-Fourier conduction can be lost to steady or oscillatory convection, depending on the flow parameters. Let $\mathbf{\Phi}=(u,w,p,\theta,j)^T$ designate the vector set of unknown variables. Note that $\mathbf{\Phi}=\mathbf{0}$ corresponds to the conduction state.

The linear stability analysis of the conduction state is similar to the case of a viscoelastic fluid, except that in the current problem, the fluid at rest does recognize further the non-Fourier character given by the additional linear velocity gradient mentioned above, in addition to the transient term. Thus, the stability of the conduction state is examined to a small (infinitesimal) perturbation of the form

$$\mathbf{\Phi}(x,z,t)=e^{st+ikx}\hat{\mathbf{\Phi}}(z), \tag{14}$$

where k is the wavenumber of the perturbation in the x direction, and s dictates the time evolution of the disturbance. Thus, the conduction or base state is stable (unstable) if the real part of s is negative (positive). Following the standard procedure in linear stability analysis, the resulting z-dependent eigenfunctions are obtained upon substituting (14) into (8)-(10) and (13) and linearizing to obtain:

$$\begin{aligned}
&ik\hat{u}+D\hat{w}=0,\\
&Pr^{-1}s\hat{u}=-ik\hat{p}+\left(D^2-k^2\right)\hat{u},\\
&Pr^{-1}s\hat{w}=-D\hat{p}+\left(D^2-k^2\right)\hat{w}+Ra\hat{\theta},\\
&s\hat{\theta}=-\hat{j}+\hat{w},\\
&(Cs+1)\hat{j}=-\left(D^2-k^2\right)\hat{\theta}.
\end{aligned} \tag{14}$$

Here D = d/dz. The dispersion relation in this case reads:

$$s^3+s^2\left(Pr\beta_n+\frac{1}{C}\right)+s\left(\beta_n\frac{Pr+1}{C}-\frac{k^2 Ra\,Pr}{\beta_n}\right)+\frac{Pr}{C\beta_n}\left(\beta_n^3-k^2Ra\right)=0 \tag{15}$$

where $\beta_n=k^2+n^2\pi^2$, and n is the mode number. In contrast to a Fourier fluid, the presence of the cubic term in (15) hints to the possibility of steady or oscillatory convection. Thus, the stability picture depends on C, Pr and Ra. This is a similar situation to viscoelastic fluids where the elasticity number appears instead of C [5]. The corresponding eigenvectors are given by

$$\begin{aligned}
&\hat{u}=\frac{in\pi}{k}\cos(n\pi z)\\
&\hat{w}=\sin(n\pi z)\\
&\hat{\theta}=\left(\frac{Cs+1}{Cs^2+s+\beta_n}\right)\sin(n\pi z)\\
&\hat{p}=-n\pi\frac{\left(Pr^{-1}s+\beta_n\right)}{k^2}\cos(n\pi z)\\
&\hat{j}=\left(\frac{\beta_n}{Cs^2+s+\beta_n}\right)\sin(n\pi z)
\end{aligned} \tag{16}$$

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In addition, the heat flux components deduce to

$$\hat{q}_x = -i\left(\frac{C}{sC+1}\frac{n^2\pi^2}{k} + \frac{k}{Cs^2+s+\beta_n}\right)\sin(n\pi z)$$

(17)

$$\hat{q}_z = n\pi\left[\frac{C}{Cs+1} - \frac{1}{Cs^2+s+\beta_n}\right]\cos(n\pi z)$$

For weakly non-Fourier fluids, steady convection is predicted upon loss of conduction, and one recovers the same critical Rayleigh number, Ra_c, as for a Fourier fluid, namely

$$Ra_c = \frac{\beta_n^3}{a^2}. \tag{18}$$

In this case, the n > 1 neutral curves are all above the n = 1 curve, with Ra_c displaying a minimum, $Ra_m = \frac{27\pi^4}{4}$ at $k_m = \frac{\pi}{\sqrt{2}}$. For strongly non-Fourier fluids, oscillatory convection sets in, and the corresponding neutral curves are obtained upon setting $s = i\omega$ in (15), ω being the frequency. Separating real and imaginary part gives

$$\omega = \frac{1}{C}\sqrt{C\beta - Pr^{-1} - 1}, \tag{19}$$

$$Ra_c = \frac{\beta\left(\beta C\, Pr^2 + Pr + 1\right)}{k^2 C^2 Pr^2}. \tag{20}$$

Clearly, oscillatory convection is possible only if

$$C > \frac{1+Pr}{Pr\,\beta}. \tag{21}$$

IV. RESULTS AND DISCUSSION

Typical marginal stability curves for several values of C are illustrated in figure 1 for Pr = 10. The curve C = 0 is the marginal stability curve for a Fourier fluid, and is independent of Pr. In this case, there is an exchange of stability between the pure conduction state and stationary convection. For C > 0, curves correspond to marginal stability for oscillatory convection (overstability) branch out from the Fourier curve. Two regimes are clearly distinguishable from figure 1. The weakly non-Fourier regime which is taken to correspond to $C < C_T = 0.0655$, where C_T is the level of non-Fourier character of the fluid above which transient convection is predicted to be first observed. Thus, oscillatory behaviour sets in for any wavenumber.

For $C < C_T$, however, oscillatory convection is still possible for a range of wave numbers corresponding to $k > k_i$ (the point of intersection between the C = 0 and C > 0 curves). When $C > C_T$, oscillatory convection is predicted to be observed first. In this case, the conductive state loses its stability to oscillatory convection. It is thus appropriate to define a weakly (strongly) non-Fourier fluid, as a fluid for which $C < C_T$ ($C > C_T$). Note that the critical wavenumber corresponding to the onset of oscillatory convection is always larger than k_i. Thus, the convective pattern becomes increasingly difficult to detect for the more non-Fourier fluids. Note that, in the strongly non-Fourier regime, stationary convection remains possible for a small range of k values (shown to the extreme left of the intersection point with the C = 0 curve). This range diminishes and eventually vanishes as C increases. Above a certain non-Fourier level, only oscillatory convection is predicted for any wave number.

The corresponding frequency, ω, curves are shown in figure 2. For small C, oscillatory behaviour is clearly possible when $k > k_i$, with k_i being the intersection point of the ω curve and the k axis (also point of intersection of the C = 0 and C > 0 curves). The frequency increases with wavenumber. Thus, oscillatory rolls tend to be smaller in size. As C increases, the non-zero frequency range increases (the zero frequency range decreases), with the point of intersection continuously moving to the left (see also figure 1). Interestingly, the frequency decreases with C for relatively large k. For C > 0.1115, oscillatory behaviour is predicted for any wavenumber, the point of intersection always being at k = 0 and infinite Rayleigh number.

V. CONCLUSION

In conclusion, the present study examines the linear stability of the conduction state of a NF once a threshold temperature difference is exceeded. The NF is modeled as a non-Fourier fluid obeying a generalized Cattaneo-Vernotte heat equation. It is found that steady (oscillatory) convection sets in for (weakly) non-Fourier fluids, corresponding to low (high) NP concentration.

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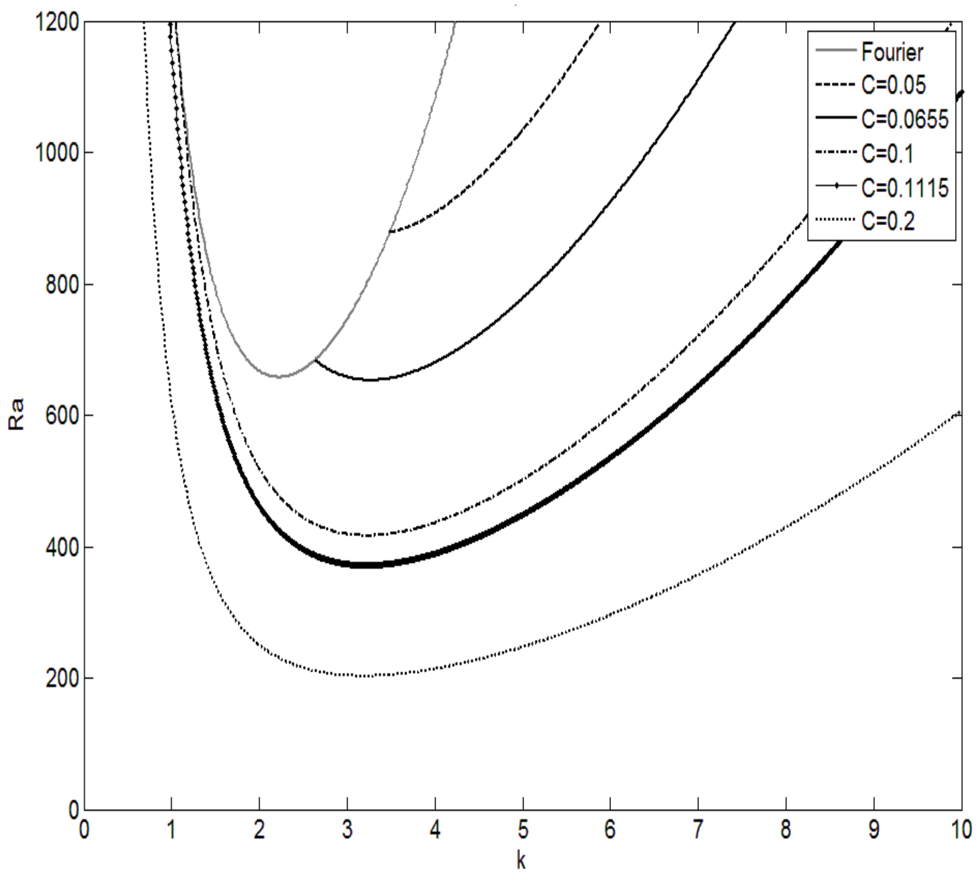

Figure 1. Marginal stability curves for Pr = 10. Critical Rayleigh number plotted against the wavenumber, k.

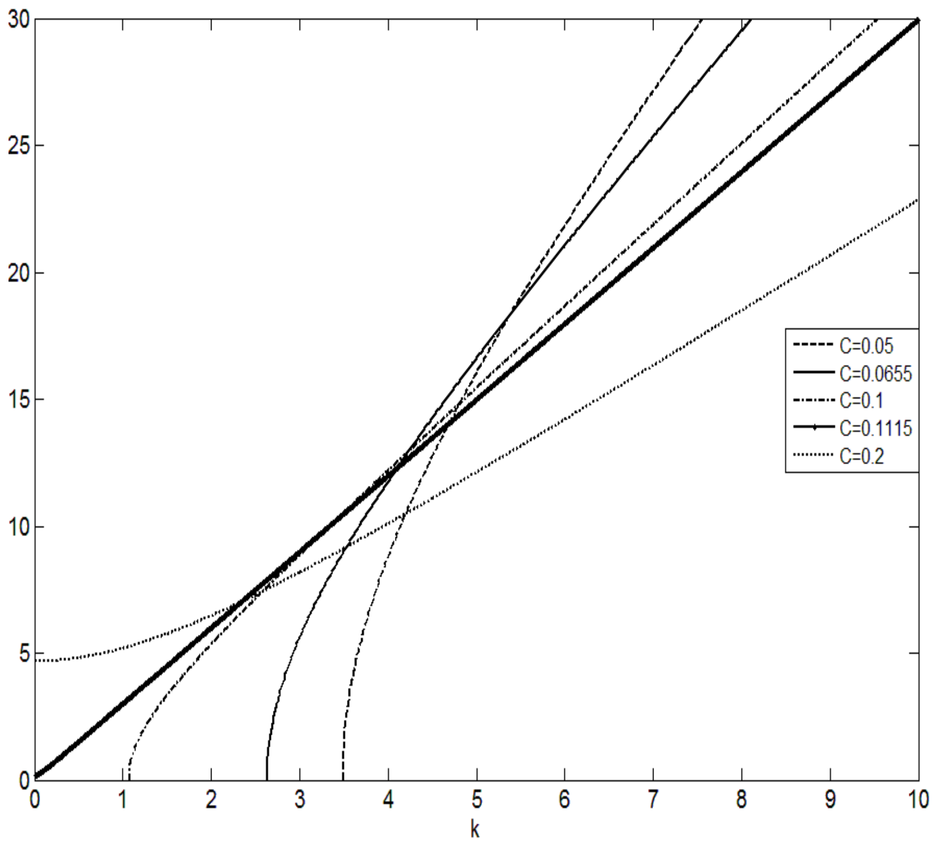

Figure 2. Critical frequency of oscillation plotted against wavenumber for Pr = 10/

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