Elaboration of a MATLB Program to Model Axisymmetric Shells

HAMADI Djamal[1], LABIODH Bachir[2], and C. A. D'Mello[3]

Abstract— **The objective of this work is the setting in numeric work of two finite elements for the axisymmetric shells; the first element is based on the Reissner-Mindlin theory and second is relative to Love-Kirchhoff theory. The MATLAB language is used for the programming of these two elements. In order to test the elaborated programs, some applications are carried out.**

Index Terms—**Modeling, Axisymmetric behaviour, Finite element, MATLAB programming**

I. Introduction

With the evolution of the finite element method several researchers developed many finite elements relating to axi symmetric shells, according to the theory of shells, can distinguish two types of finite elements can distinguished:

The finite elements where the effect of transverse shearing is taken into account (Reissner-Mindlin theory).

The finite elements where the effect of transverse shearing is not taken into account (theory of Love-Kirchhoff).

Several elements were developed since 1960, the first finite element formulated (1963) in the field of this types of structures hull be a truncated element for shells of revolution based on the theory of Love-Kirchhoff references ([1] [2] [3] [4]). Currently, the most used element of Kirchhoff is CAXI_K element [5], for this type the field of displacement U is linear and W is cubic. With regard to the elements based on the Reissner-Mindlin theory, CAXI_L element [5] was proposed and tested. A simple element and powerful based on the displacement model was formulated by Zienkiewicz and Taylor [6], the components of U and β are linear and W quadratic, the integration is done with 3 points of Gauss for the membrane, 2 points for the bending and 1 point for transverse shearing.

While taking as a starting point the element with mixed formulation in transverse shearing for the straight beams, an element with U and W linear and ß quadratic were formulated and tested by Despinoy [7] and Liu [8], by using a uniform integration with two Gauss points, this element implies a local elimination of two internal variables, but it presents a better performance than CAXI_L element.

The objective of this work is to setting numerically operational two finite elements for axisymmetric shells, for this, one carried out the development of two programs called Axisym CAXI_L and CAXI_K in MATLAB. The first one is dealing with the finite element CAXI_L which is based on the Reissner-Mindlin theory, and the second is related to CAXI_K element based on the Love-Kirchhoff theory.

II. Axisymmetric Shells Theory

A. Love Kirchhoff theory

Assumptions

The following assumptions have to be considered:
Geometrical assumptions of linearization: Displacements and strains remain small.
Assumption of material linearization: The material obeys the Hook's law.
The transverse normal stress is neglected $\sigma_z = 0$.
The cross-sections, normal in the medium plan not deformed, remain plane and perpendicular to the medium plan deformed $\gamma_{\alpha z} = 0$, $\gamma_{\beta z} = 0$ and $\varepsilon_z = 0$.

Displacement model

The relations efforts resulting-deformations are given by:

$$[N] = \lfloor H_m \rfloor\{e\} + \lfloor H_{mf} \rfloor\{\chi\} \quad (1a)$$

$$[M] = \lfloor H_{mf} \rfloor\{e\} + \lfloor H_f \rfloor\{\chi\} \quad (1b)$$

N: efforts resulting from membrane.
M: efforts resulting from bending (moments).
With $\langle e \rangle = \langle \; e_s \quad e_\theta \; \rangle \quad \langle \chi \rangle = \langle \; \chi_s \quad \chi_\theta \; \rangle$
e_s, e_θ: Deformation of membrane according to S (meridian) and θ (circumferentially).
χ_s, χ_θ : Curvatures according to s and θ.
The displacement model corresponds to:

$$W = W_{\text{int}} - W_{ext} = 0 \quad \text{(Principle of virtual work)} \quad (2a)$$

$$W_{\text{int}} = 2\pi \int \left(\langle e^* \rangle \left(\left[H_m\right]\{e\} + \left[H_{mf}\right]\{\chi\} \right) + \langle \chi^* \rangle \left(\left[H_{mf}\right]\{e\} + \left[H_f\right]\{\chi\} \right) \right) r ds \quad (2b)$$

e^*, χ^* : Deformations of membrane and virtual curvatures respectively.

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Djamal HAMADI, Civil Engineering and Hydraulics Department Faculty of Sciences and Technology, Biskra University B.P. 07000 – Algeria, fax: 00 21333741038; (e-mail: dhamadiuk@yahoo.com).
Bachir LABIODH Civil Engineering Department, Faculty of Sciences and Engineering Sciences, Biskra University, B.P. 07000, Algeria fax: 0021333741038; (e-mail: baclb2007@yahoo.fr).
Cedric D'Mello, School of Engineering and Mathematical Sciences, City University, Northampton Square London EC1V OHB, U.K. Fax: +44 (0) 20 7040 8570 (email: C.A. D'Mello@city.ac.uk)

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B. Reissner Mindlin theory

Assumptions

Geometrical assumptions of linearization: Displacements and strains remain small.

Assumption of material linearization: The material obeys the Hooke law.

The normal transverse stress is negligible: $\sigma_z = 0$.

Mixed models in transverse shearing

$$W = W_{\text{int}} - W_{ext} = 0 \tag{3a}$$

$$W_{\text{int}} = 2\pi \int \begin{pmatrix} \langle e^* \rangle \left([H_m]\{e\} + [H_{mf}]\{\chi\} \right) + \\ \langle \chi^* \rangle \left([H_{mf}][e] + [H_f]\{\chi\} \right) + \\ \gamma^* T_S + T_S^* \left(\gamma - H_c^{-1} T_S \right) \end{pmatrix} rds \tag{3b}$$

T_s, T^*_s: real and virtual shearing action following s.
H_c: shearing stiffness.

III. Elements Formulation

A. CAXI_L Element

The finite element CAXI_L [5] is a truncated element with two nodes, its formulation is based on the theory of Reissner-Mindlin theory. The model used for this element is the mixed model in transverse shearing. We suppose that the shell is modelled by a succession of truncated cones defined by the end nodes on the meridian curve.

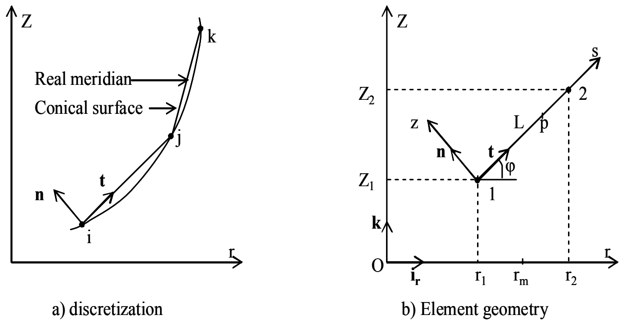

Fig.1. Truncated element CAXI_L (geometry)

The approximations of the displacement field of U, W and of β are linear in s and the shearing action Ts is constant.

$$U = N_1 U_1 + N_2 U_2 \qquad W = N_1 W_1 + N_2 W_2$$

$$\beta = N_1 \beta_1 + N_2 \beta_2 \tag{4}$$

with: $N_1 = 1 - s/L$; $N_2 = s/L$

The strains are:
Membrane strains are e_s, e_θ.
The curvatures are χ_s, χ_θ.
The transverse shearing is γ.

The element stiffness matrix can be evaluated numerically with the reduced integration method of internal work W^e_{int}:

$$W^e_{\text{int}} = \langle u^*_n \rangle [K] \{u_n\} \tag{5a}$$

With: $[K] = \lfloor K_{mf} \rfloor + \lfloor K_c \rfloor$
And:

$$[k_{mf}] = 2\pi \left(\lfloor B_m \rfloor^T \left(\lfloor H_m \rfloor \lfloor B_m \rfloor + \lfloor H_{mf} \rfloor \lfloor B_f \rfloor \right) + \lfloor B_f \rfloor^T \left(\lfloor H_{mf} \rfloor \lfloor B_m \rfloor + \lfloor H_f \rfloor \lfloor B_f \rfloor \right) r_m L \right)$$

$$[k_c] = \{k_{m/T}\} \frac{1}{K_T} \langle k_{m/T} \rangle = 2\pi \{B_c\} H_c r_m L \langle B_c \rangle \tag{5b}$$

With:
$[k_c]$: transverse shearing stiffness matrix.
$[k_{mf}]$: membrane bending stiffness matrix for an isotropic material

$$H_c = k.G.h$$

$$G = E/2(1+v)$$

$$K = \frac{5}{6} \quad \text{(Transverse shearing correction factor)}$$

$$[H_m] = \frac{\text{Eh}}{(1-\nu^2)} \begin{vmatrix} 1 & \nu \\ \nu & 1 \end{vmatrix} \qquad [H_f] = \frac{\text{Eh}^3}{12(1-\nu^2)} \begin{vmatrix} 1 & \nu \\ \nu & 1 \end{vmatrix}$$

$$[H_{mf}] = \frac{\text{Eh}^3 S}{12(1-\nu^2)\text{r}_\text{m}} \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \tag{6}$$

Resulting efforts (normal effort and bending moment):

$$[N] = \lfloor H_m \rfloor \{e\} + \lfloor H_{mf} \rfloor \{\chi\}$$
$$[M] = \lfloor H_{mf} \rfloor \{e\} + \lfloor H_f \rfloor \{\chi\}$$

B. CAXI_K Element

This finite element is a truncated element. Its formulation based on the Kirchhoff theory [5] and the displacement model. The curvilinear components U (s) and W (s) are defined by linear approximations and cubic of hermitian type respectively. The numerical integration used is of Gauss type with two points for the evaluation of the stiffness matrix $[k^e]$.

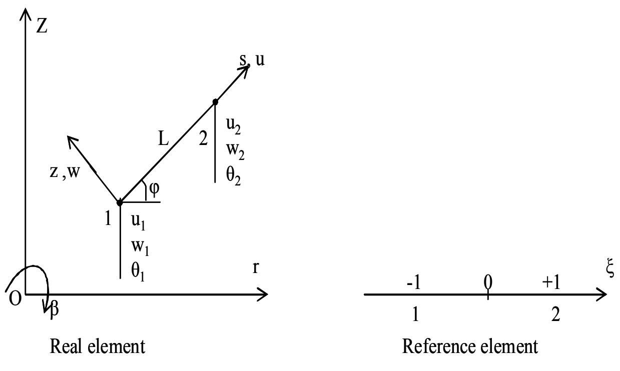

Fig.2. Truncated element CAXI_K

$$[k]_{loc} = 2\pi \int_0^L [k_\xi] ds = 2\pi \int_{-1}^{1} [k_\xi] \frac{L}{2} d\xi \tag{7}$$

With:

$$\lfloor k_\xi \rfloor = \left(\lfloor B_m \rfloor^T \left(\lfloor H_m \rfloor \lfloor B_m \rfloor + \lfloor H_{mf} \rfloor \lfloor B_f \rfloor \right) + \lfloor B_f \rfloor^T \left(\lfloor H_{mf} \rfloor \lfloor B_m \rfloor + \lfloor H_f \rfloor \lfloor B_f \rfloor \right) \right)$$

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The numerical integration according to the Gauss method is:

$$[k]_{loc} = 2\pi \sum_{i=1}^{2} \left[k_{\xi}\left(\xi = \xi_i\right)\right]\omega_i \frac{L}{2}$$

With $\xi_i = \pm 1/\sqrt{3}$ and $\omega_i = 1$ (8)

After the evaluation of $[k]_{loc,}$ and before the assembling of the matrices; it is necessary to transform the variables $\{u_n\}_{loc}$ defined in the local coordinate of the element according to the nodal variables of the cylindrical reference. The transformation matrix [T] is given by:

$$[T] = \begin{vmatrix} [t] & [0] \\ 0 & 1 \end{vmatrix} \quad [t] = \begin{vmatrix} [t]^T & 0 \\ [0] & [\mathrm{t}] \end{vmatrix}$$

$$[Q] = [\ t \quad n\ ] = \begin{vmatrix} C & S \\ S & C \end{vmatrix} \qquad (9)$$

Thus we can write: $\left[k^e\right] = [T]^T [k]_{loc} [T]$

IV. Numerical Implementation

As it is well known, all the programs based on the finite element method include a few characteristic subroutines:
Reading, checking and organisation of the data describing the meshes (nodes and elements), physical parameters (elasticity modulus … etc), applied loadings and boundary conditions
Construction of the elementary stiffness matrices and vectors, then the assembly of those, to form the global matrix and the total vector of the applied loadings
Resolution of the system of equations after taking into account the boundary conditions
Printing the results after calculation of the additional variables (stresses, reactions… etc)

A. General Algorithm

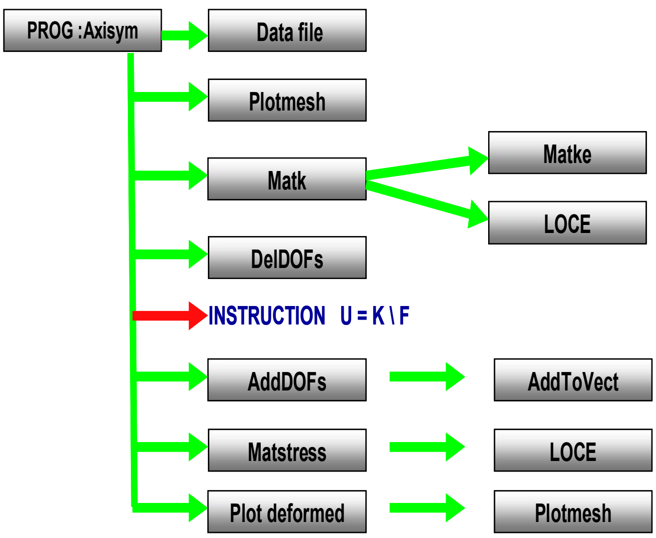

Fig.3. General Algorithm of the Axisym Program

The two elaborate programs (Axisym CAXI_L, and Axisym CAXI_K) relating to finite elements CAXI_L, CAXI_K, presented above are written with MATLAB language, each program consists of a principal function Axisym and subroutines.

B. Description of Axisym program

Principal function Axisym

It calls the various subroutines or functions for different calculations of an axisymmetric structure, using the formulations described above for each element.

Data file

It is a file function where all the relative data with the problem are introduced:
Table of the nodes coordinates.
Table of the elements connectivity
Elements thickness
Elasticity modulus
Poisson's ratio
Boundary conditions (numbers of the fixed degrees of freedom)
Applied loading vector
The MATLAB instruction: [data] = feval (str2func (Name of the data file); open the data file and allows the various subroutines or functions to read the relating data.

Plot mesh function

This function gives a graphic posting of the meshes, the coordinates and the numbering of the nodes, which allows, with the layout the seized structure, a visual checking of the data (coordinates table and connectivity).

Matk function

This function contains the process of assembling the element stiffness matrices Ke provides by the function matke. The do loop is done on the elements, by using the function LOCE; which provides the insertion of each element term in the global stiffness matrix.

LOCE Function

According the connectivity table; this function provides the localization table (numbers) of the degrees of freedom for each element.

Matke function

The element stiffness matrix can be evaluated by Matke function.

DelDOFs function

The boundary conditions are fixed displacements on the levels of the supports, a vector E in the data file is used to specify the degrees of freedom to be fixed. To apply this condition; we eliminate the corresponding lines and columns to the vector E from the matrices K and the applied load vector F (since the loading is evaluated for all the nodes including those of the fixed supports).

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Instruction U = K\F

Once the boundary conditions are applied, it only remains to solve the discrete system. The solution in displacement (with the nodes) can be obtained with instruction MATLAB U = K \ F to solve the linear equations system (Cholesky method).

AddDOFs function

With this function, the degrees of freedom of the fixed nodes are added to the displacement vector; that is removed by the DelDOFs function. The do loop is made on the numbers of the degrees of freedom add by using the AddToVect function.

AddToVect function

This function provides the resulting displacement vector for each degree of freedom.

Matstress function

With this function, we can calculate the stresses, using the formulation presented for each element.

Plottedeformed function

Finally, with this function we can obtain the deformed shape of the structure.

V. Numerical Applications

The purpose of this section is the validation of the elaborated programs. Also the comparison of the results obtained with the presented elements and those given by ANSYS.

A. Circular Plate (without transverse shearing)

A circular plate is subjected to uniformly distributed load; its geometry and mechanical properties are presented on the fig.4.

Table I shows the results obtained with the Axisym programs for elements CAXI_L and CAXI_K and the analytical solution [10] without taking into account the assumption of transverse shearing(TS).

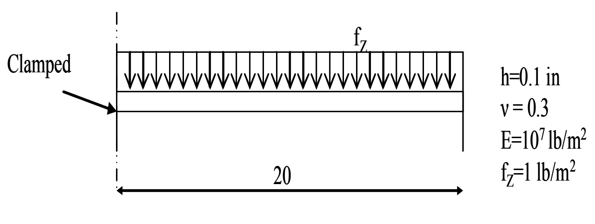

Fig.4. Circular plate subjected to uniformly distributed load

TABLE I: Value of Maximum Displacement (in)

Meshes	Analy Sol. [10]	Prog:CAXI_L	Error	Prog:CAXI_K	Error
2 elements	0,17	0,2113	24,29%	0,1705	0,29%
4 elements		0,1759	3,47%	0,1700	0,00%

Comment

Both elements CAXI_L and CAXI_K converge towards the analytical solution in a monotonous way.

B. Hemisphere

A hemisphere shell structure (Fig.5) is modelled by truncated element CAXI_K (24 elements). The results obtained will be compared with the analytical solution [10], and with those given by ANSYS (Fig.6). All results are presented in Table II.

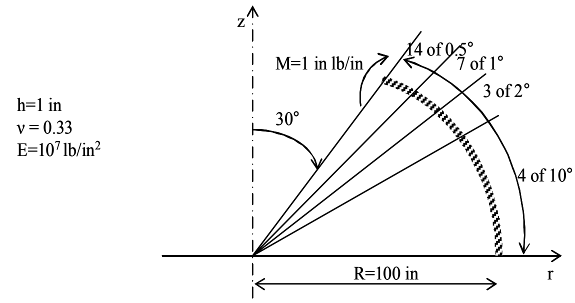

Fig.5: Hemisphere

TABLE II: Value of Maximum Displacement (in)

Mesh	Analyt. Sol. [10]	ANSYS	Prog:CAXI_K	Error
24 elements	1,60E-05	1,59E-05	1,587E-05	0,81%

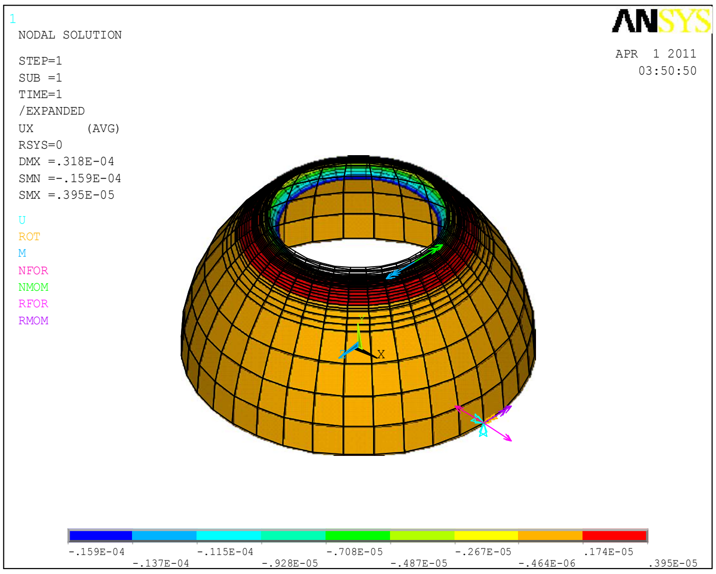

Fig. 6 Deformed structure

Comment

The results obtained by the element CAXI_K are acceptable

C. Circular cone

A circular cone (Fig.7) is modelled by ANSYS and Axisym program with CAXI_L element. Figure (8) shows the variation of the results difference between the Axisym program and ANSYS.

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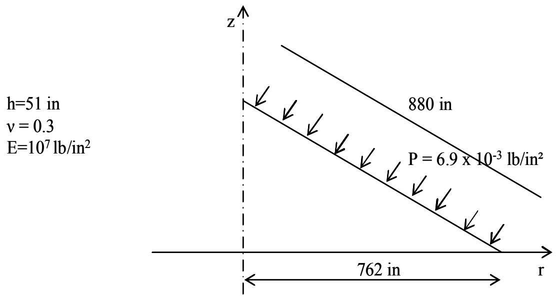

Fig. 7 Circular cone

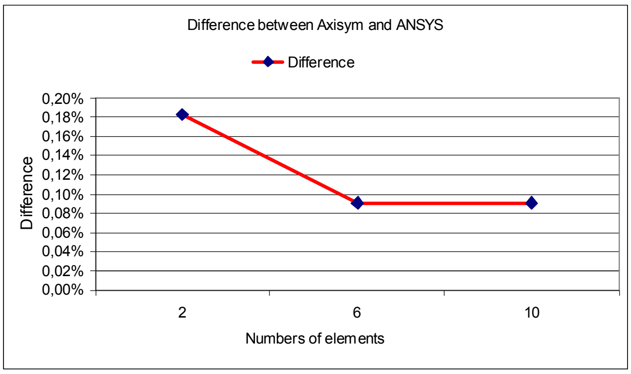

Fig. 8 Variation of the difference between Axisym and ANSYS

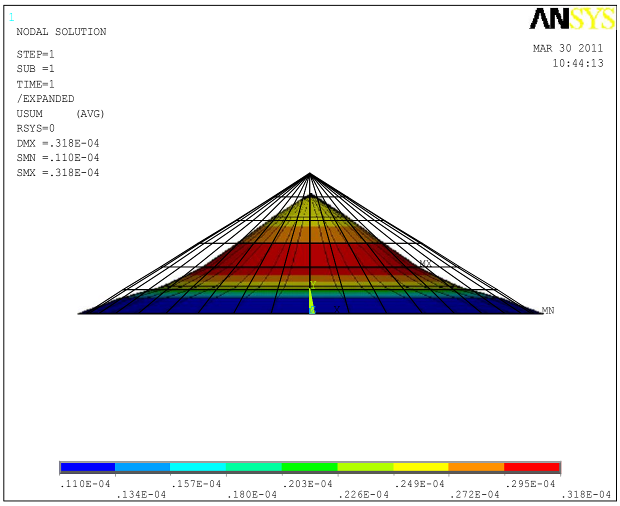

Fig. 9 Deformed structure

Comment

It is noticed that the results obtained by Axisym program using CAXI_L element and those given by ANSYS converge in a similar way.

VI. Conclusion

From the results obtained above, the following conclusions can be drawn:

It can be observed that, the programs Axisym CAXI_L and CAXI_K give good results, which confirm the validation of the elaborated programs.

The presented CAXI_L and CAXI_K prove their efficiency in analysing axisymmetric shells structures.

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