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Conformal Transformation Using Symbolic Language Facilities and Applications in Electrical Engineering

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Abstract—**The method offers solutions in closed form for many interesting cases, avoiding the utilization of any numerical method like the finite element method. In the present work, the usage of a symbolic language is presented and certain new results have been obtained. The method permits to apply the conformal transformation avoiding the hand calculation of integrals for cases in which this calculation is very difficult by traditional procedures, involving some changes of variable suitable for the considered case. For this purpose, the Maple software has been used. An application for electrical engineering which has been considered led the author to find a new solution superior to the known ones.**

Index Terms—**Conformal transformation, Mapping, Complex potential function, Symbolic language, Maple.**

I. Introduction

Conformal transformation is a very useful tool for the study of various potential fields. The procedure offers closed form solutions for many interesting cases, when it is possible, avoiding the utilization of any numerical method like the finite element method, finite difference method, etc. In the present work, the usage of a symbolic language, for this purpose, is presented. The procedure permits to apply the conformal transformation avoiding the calculation of integrals, for cases in which they are very difficult or complicated to be found, in closed form. It also permits to control the procedure, what may be difficult or even impossible by traditional calculation. For this purpose the Maple software has been used. An application for electrical engineering has been considered. Simultaneously, some new results which show important differences relatively to the traditional treatments have been found.

II. Recall of the Procedure of the Conformal Transformation

Because, in literature the subject is treated in various manners, we shall recall the principle and properties of conformal transformation in the manner we have used, for being able to pass consistently to the usage of symbolic programs in this field. Consider two complex planes, z and w respectively, and an analytic univalent function:

$$w = f(z). \tag{1}$$

Equation (1) makes to each value z_1 representing a point on plane z to correspond a certain value w_1 representing a point on plane w. Analogously, to a succession of points on plane z, there corresponds a succession of points on plane w, so that to any curve $z_1 z_2$ on the z - plane, there corresponds a curve $w_1 w_2$, which is called the transformed curve on the w - plane by equation (1).

Analogously, assuring the correspondence of the points on the two planes z and w, an analytical function like (1), transforms any domain (D_z) interior or exterior to a boundary on the z - plane, into a domain (D_w) on the w - plane.

As known, a transformation like that above is called conformal transformation because two curves which cross each other at a certain angle on plane z keep the value and the sense of the angle, after their transformation on plane w.

As also known, at the base of the conformal transformation lies the Riemann theorem: the interior of a simply connected surface, of any domain (D_z) can be represented in a bi-univocal and conformal manner on the interior of a circle (D_w), adopting three arbitrary constants.

In many applications it is necessary to make successively two conformal transformations, one from the z - plane into the over half of the w - plane, the next from the over half plane into a rectangle domain (D_ζ) on a ζ - plane.

III. The Complex Potential and the Field Strength in the z-Plane

For any complex function as in next expression:

$$W(z) = U(x,y) + iV(x,y), \tag{2}$$

assumed to be holomorphic in a certain domain, since the derivative with respect to the complex co-ordinate

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z does not depend on the differentiation direction, there follows:

$$\frac{dW}{\partial z} = \frac{d(U + iV)}{d(x)} = \frac{d(U + iV)}{d(i\,y)}, \tag{3}$$

and by equating the real and imaginary parts:

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}; \quad \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}. \tag{4}$$

The field strength, having on a plane ζ two components, may generally be expressed as:

$$\begin{aligned} \underline{H}_\zeta &= -\frac{\partial V}{\partial \xi} - i\frac{\partial V}{\partial \eta} \\ &= -\frac{\partial V}{\partial \xi} - i\frac{\partial U}{\partial \xi} = -i\left(\frac{\partial U}{\partial \xi} - i\frac{\partial V}{\partial \xi}\right), \end{aligned} \tag{5}$$

which with relations above, and for simplicity, denoting $W = W_\zeta$, there follows:

$$\underline{H}_\zeta = -i\left(\frac{dW}{dz}\right)^*. \tag{6}$$

For the z - plane, it results that:

$$\underline{H}_z = -i\left(\frac{dW_z}{dz}\right)^* = \left(i\frac{\partial W_\zeta}{\partial z}\right)^* = \left(i\frac{dW_\zeta}{d\zeta}\frac{d\zeta}{dw}\frac{dw}{dz}\right)^*. \tag{7}$$

At each point of the domain (D_z), there corresponds a certain value of the complex flux expression $W = U + iV$. By conformal transformation of a domain (D_z) into a domain (D_w), at each point $w = w(z)$, obtained in domain (D_w), there corresponds the same value of W like in the corresponding point of domain (D_z) above. Assume that the lines $V = \text{const}$ are equipotential lines and $U = \text{const}$ are lines of field. These lines, according to their expressions, are orthogonal lines, and have the same meaning and expression in the corresponding domains of the two domains. One can write the expression of the potential difference between two points, denoted 1 and 2 in domain (D_z):

$$\Delta V = V_1 - V_2. \tag{8}$$

The potential difference between the corresponding points in domain (D_w) is given by the same relation and should have the same value as above.

The flux through an arc of curve which ties two points denoted by 1 and 2 on domain (D_z) will be:

$$\Delta\Psi = U_2 - U_1. \tag{9}$$

The lines of flux crossing the arc of curve of above are delimited by the lines passing through points 1 and 2. For the sake of clearness, we can consider the tube of flux of a plane-parallel configuration crossing the mentioned arc of curve and having the thickness equal to unity. The described flux in domain (D_w) is given by the same relation and should have the same value.

The potential and flux being the same for the case of each of the two planes it is convenient to carry out the calculations in the case in which they are simpler. For instance, in many cases the most convenient case is that of a rectangular domain, say on a ζ - plane, the upper and lower sides of the rectangle and the parallels to these are each of them equipotential lines, whereas the vertical sides and the parallels to these are lines of field.

The passage from the upper half plane to a rectangle is related to a strip, and is accomplished by the known conformal transformation:

$$\zeta = \ln(w). \tag{10}$$

In several applications, in Maple, it may be necessary to replace symbol ζ by t, the first being, in the mentioned software, protected. For practical applications, it is worth noting that if the potentials and fluxes keep their values after a conformal transformation, the quantities: electric capacitance, electric resistance, magnetic reluctance, also keep their values.

For practically solving a problem by using a conformal transformation, the following stages have to be browsed:

1. To find the adequate analytic function for transforming the given domain configuration into another convenient for solving the problem, in most cases it is the case of the rectangle above explained. Sometimes, several transformations are necessary for this purpose.

2. If inside the domain (D_w) the complex potential is:

$$W = w(z), \tag{11}$$

the same expression represents, apart from a constant, the complex potential in the interior of domain (D_z).

3. To perform the calculation of potential and fluxes for the domain D_w or (D_ζ), where there are easier. Using the complex potential function, the direct calculation of fluxes, used in traditional treatments, is no more required.

4. In many cases, it suffices to obtain the function for conformal transformation of the domain (D_z) of the z - plane into the interior of a circle or of rectangle, in a ζ - plane, because it is easy to obtain their conformal transformation into the domain of the explained rectangle, in the case of certain adequate boundary conditions. No general method exists for the conformal transformation formula, of any domain, but there are relations for particular cases very useful for many applications.

A very interesting case is that of the interior domain of a polygonal contour into the z - plane, with n vertices, situated at finite distance. For any vertex with

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ordinal number k , the oriented angle denoted by β_k with which the side has to be rotated for obtaining the direction of the next side should be given. Instead of the oriented angle, the value of the internal angle α_k at this vertex can be given, sometimes being easier for calculations. The following usual relations are fulfilled:

$$-\pi < \beta_k < \pi; \quad 0 < \alpha_k < 2\pi. \qquad (12)$$

The transformation of this domain into that of the superior half-plane is given by the relation:

$$\frac{\mathrm{d}z}{\mathrm{d}w} = A \prod_{k=1}^{n} \left(w - u_k\right)^{\frac{-\beta_k}{\pi}} . \qquad (13)$$

There follows:

$$z = A \int \prod_{k=1}^{n} \left(w - u_k\right)^{\frac{-\beta_k}{\pi}} \mathrm{d}w + C. \qquad (14)$$

A transformation of the form (13) or (14) is called, as known, a Schwarz-Christoffel transformation. Many treatment of this formula have been given in literature, among which [1-6]. However certain cases, especially if each angle β_k is not strictly positive [3], still raise difficulties. Cases of this type may be found in [1].

IV. The Treatment of the Conformal Transformation Using the Maple Language

For the previously mentioned cases, the known calculations are difficult and their following is also difficult. In these cases, the symbolic languages for programming allow the possibility to avoid both difficulties. We shall explain the method on an example known in literature, in order to permit an easy comparison. We shall use the Maple language.

The advantages are the following:

a. No need to look for the expression of the integral, if it exists, the indefinite integral is automatically returned by the software, even if the expression is so long that the traditional solving could not be expected. Moreover, the display of the integral in closed form is not always, necessary because the evaluation of the stored result for the given data is also automatically performed by the program;

b. For the preparation of the program, it is convenient to keep unchanged the values of certain variables or names during the calculation, according to the required conditions. For this purpose, as we have experienced, the use of *procedures* is very useful;

c. An important remark, as principle, is that for conformal transformations we must work not with physical lengths that should be expressed in metres, but with the corresponding relative quantities. Therefore, the lengths which occur should be divided by another length, usually a segment which exists in the considered configuration.

As we have realized, the usage of a program prepared in Maple language, permits to avoid the calculation of fluxes, which can be replaced by using the complex potential function that remains unchanged by conformal transformations. Therefore it suffices to calculate the difference of two coordinates of two points for obtaining the flux passing through the respective tube of flux, in any transformed configuration.

At the same time, for obtaining the point of the minimum value of the component of the field strength it has been possible to calculate by Maple the derivative of the considered function, avoiding the classical treatments [1, 2].

As known, in classical treatments, the representation of the normal component of the field strength versus abscissa of the given configuration is very difficult. We succeeded to avoid this complication by using a parametric plot by a procedure found in Maple language.

It is useful to warn concerning calculations for the w - plane, because in the case of a parenthesis representing a square root, we can adopt one of the signs plus or minus, what cannot be automatically done, but only by tracking the continuity conditions. The program prepared by the author in Maple language has permitted to keep the necessary conditions.

V. The Considered Problem

The problem chosen for presenting the procedure is the plane-parallel configuration of Fig. 1. This configuration can also be considered as having a certain thickness, equal to unity.

For to fix the ideas, the contour in the figure represents the traces of the boundary of the interior domain. The configuration is considered to be the model of certain physical objects. The continuous lines constitute the walls delimiting the margins of ferromagnetic pieces. The space delimited by these walls is air or vacuum. The dashed lines represent the way along which we can travel the domain boundary by circumventing the singular points. The top continuous line is considered to be at any magnetic potential *V*, supposed to be positive. The down broken line, by a slot, is considered to be at magnetic potential zero.

The gap between the two armatures, upper and lower, called air-gap, has its minimum thickness denoted by δ and it is assumed constant, except the portion over the slot having the aperture breadth b_0 and assumed infinitely deep. Therefore, between the upper and lower armatures a potential magnetic field exists.

If the slot did not exist, the flux between the two armatures had any value Φ_0, but, due to the existence of the slot, it would be smaller, having a certain value $\Phi = \Phi_0 - \Phi_s$. One of the problems is to find the difference of the two fluxes, namely Φ_s .

This mathematical problem arose, the first time, for the calculation of the magnetic circuit of rotating electrical machines and was solved by Carter and minutely examined by Gibbs [1]. For this reason, a certain obtained coefficient is called Carter coefficient or factor. The solution has implied difficult calculations and demanded much preoccupation for determining the necessary transformations. Also it is worth noting that an analysis of the singular points has not been carried out.

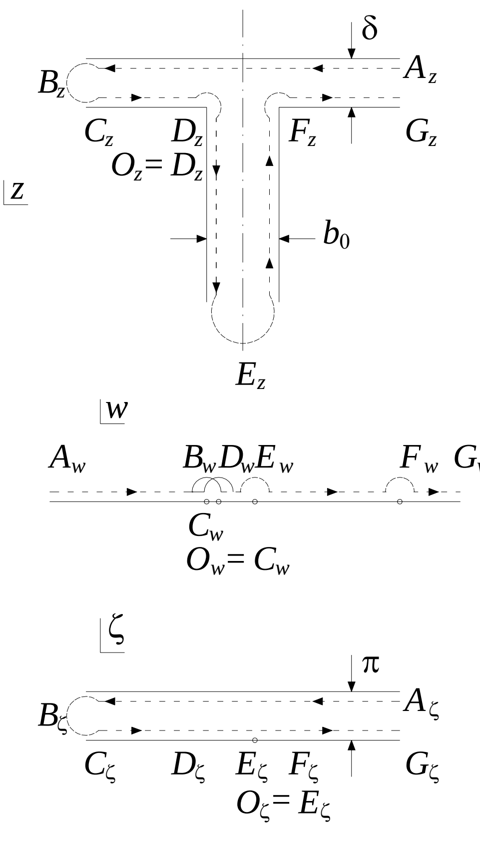

Fig. 1. Configuration of a magnetic system.

By using, as already mentioned a symbolic language, we obtained much easier the solution, and did a deeper analysis. As a result, it may be mentioned that the values of the field strength does not differ from those of [1], but for the fluxes the difference is sensible. The problem is not only mathematically interesting, but a correct solution can be used for solving similar technical problems.

VI. The Used Formulae and the Maple Built Programs

Before presenting the accurate solution, we have obtained, it may be added that, Maple allows also for a simple calculation of any flux, without implying the calculation in closed form of the occurring integral. It suffices to express the formula of the conformal transformation, and the relation for determining the fluxes. The Maple program returns the result very fast. However we realized that this procedure is not too precise. The reason is that the function which intervenes in the studied problem, when calculating the limit, shows relatively important oscillations in the vicinity of the singular point that reduce the precision of the result. However, the procedure may be useful in many problems as a fast orientation.

The steps of the program will now be presented as follows.

The Schwarz-Christoffel transformation (13), for passing from the z - plane to the w - plane of Fig. 1, is represented by the formula given by the literal expression:

$$\frac{dz}{dw} = A\frac{\sqrt{w-a}\sqrt{w-b}}{w(w-1)}. \tag{15}$$

In this case, the constants may not be determined by the general methods [3-6], because the angles β_k are not all strictly positive. The constants have been determined by calculating certain distances of the figure by circumventing, along portions of circles the singular points. However, we realized that these constants may be directly determined replacing the influence of any singular point (pole) by circumventing integrals, with the calculation of the residue for the same point, for which the Maple soft gives a simple statement *residue*, and keeping the half. We shall give only the results:

$$\frac{dz}{dw} = A\frac{\sqrt{w-a}\sqrt{w-b}}{w(w-1)}; \quad A = \frac{\delta}{\pi}; \quad a = \frac{1}{b}, \tag{16}$$

and having in view that according to Fig. 1, $b > 0$, we have:

$$\frac{dz}{dw} = A\frac{\sqrt{w^2 - \left(b_0^2 + 2\right)w + 1}}{w(w-1)}; \quad b_0 = \frac{b-1}{\sqrt{b}}. \tag{17}$$

In applying formula above, when not necessary, we shall disregard the constant A, for simplicity.

In the programs of [7], the obtaining of the solution of the Schwarz-Christoffel transformation is presented. As shown, the form is charged with many terms, but no reason for to find other simpler form. It is worth noting that the solution of the transformation is given not directly, but using a procedure. The reason is for to be able to change, at any step, the value of the variable.

VII. The Complex Potential and the Field Strength Components in the z-Plane

Formula (7) and (11) will be considered. The complex potential will be denoted $W = W_\zeta = C\,\zeta$, C being a constant. Then, for the z - plane, it results that:

$$\underline{H}_z = -i\left(\frac{dW_z}{dz}\right)^* = \left(i\frac{\partial W_\zeta}{\partial z}\right)^* = \left(i\frac{dW_\zeta}{d\zeta}\frac{d\zeta}{dw}\frac{dw}{dz}\right)^*. \tag{18}$$

There follows:

$$H_x = \mathrm{Re}\left(\underline{H}_z\right); \quad H_y = \mathrm{Im}\left(\underline{H}_z\right), \tag{19}$$

and:

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$$H_y = \mathrm{Im}\left(i \frac{\partial W_\zeta}{\partial \zeta} \frac{d\zeta}{dw} \frac{dw}{dz} \right)^* = -\left(\frac{V}{\delta}\right) \frac{w-1}{\sqrt{w^2 - \left(b_0^2 + 2\right) w + 1}} . \qquad (20)$$

The minimum of H_y, according to the program of [7], is reached for the value for which its derivative is zero. There follows that the value $w = -1$ rends minimum the considered function.

VIII. The Plotting of Curves

Although the computation returns the values of the components in terms of w, and not of x, we avoided any difficulty by using the parametric plotting allowed by the Maple software. We prepared and used the programs of [7]. Some results may be seen in Fig. 2 and Fig. 3, for two configurations. As expected, the minimum value of the normal component of the field strength is reached on the slot axis. Here is the case to mention that the called axis has the abscissa $w = -1$.

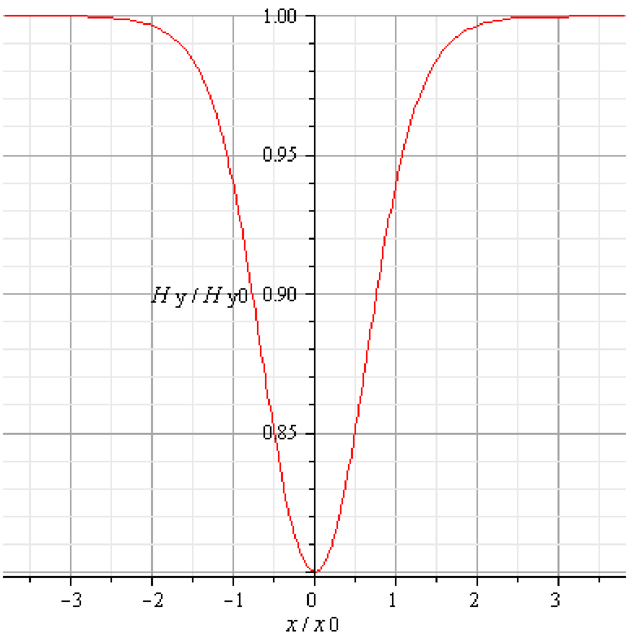

Fig. 2. Normal component of the field strength for the case $b_0 = 1.5\,\delta$.

IX. The Flux Difference Computing Programs

For the calculation of the flux reduction due to the existence of the slot, we have to calculate both fluxes, the flux in the case in which the normal component of the field strength is considered having a constant value corresponding to the minimum value of the air-gap thickness, say Φ_0, and that which actually exists for the real thickness, say Φ. Both fluxes will be calculated using the complex potential function, and a separate direct flux calculation is no more necessary, at the difference of known works.

Due to the performed transformations, the calculation should refer to the $\zeta = \xi + i\,\eta$ - plane domain, corresponding to the interval $w \in (-\infty, \ \ -1]$; that corresponds to the half of the length of the upper line (i.e., armature) with positive abscissa. The inverse values of w, e.g. -0.001 and $-1000.$, give symmetric values of ξ with respect to $\xi = 0$.

Remark. The last mention is of outstanding interest, not found in traditional treatments, because it means that for to obtain the total flux it suffices to know only the half of it, either for positive abscissa or for the negative abscissa.

In the case of infinite length as in [1], each of the two fluxes, Φ_0 and Φ is infinite but their difference is finite, say $\Phi_s = \sigma$. For this reason, we have calculated, the fluxes for the interval $w \in [c, \ \ -1]$, the quantity c being negative. Then we have subtracted the results using the formula in closed form. Several terms, containing the quantity c reduced each other. For obtaining the fluxes, we have used the expression of

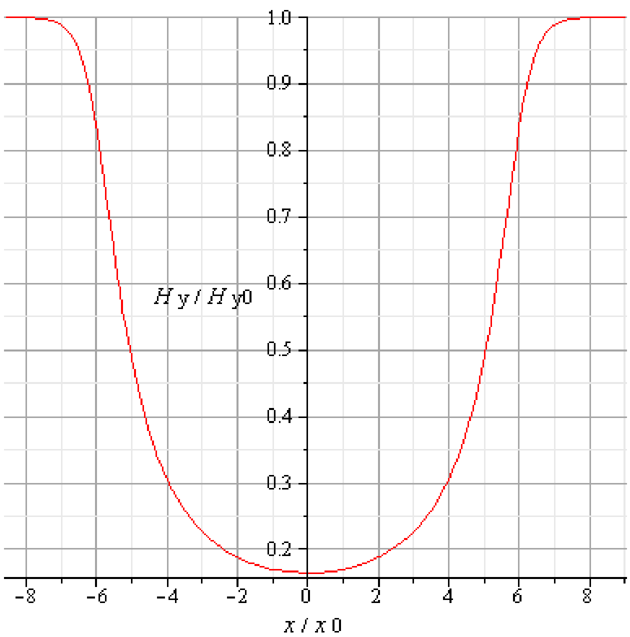

Fig. 3. Normal component of the field strength for the case $b_0 = 12\,\delta$.

the complex potential function. After having obtained the difference (i.e., a sum of two terms one of them negative), we have considered the quantity c approaching to zero, being easier than for infinity. Here, the Maple software has presented a difficulty; because it could not remove the indeterminate form presented by the sum as the quantity c is tending to zero. We have removed, the indeterminate form of the sum, by putting them in the form of a quotient and using the rule of de l'Hospital. Then, a relatively simple formula has been obtained. Both literal expression and numerical results have been obtained.

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For easier solving this and other cases, we have no more used the transformation formula based on relation (16), but (17). It has the advantage of returning the result expressed only in terms of the given data and leading to a simpler final literal expression.

The computing expressions we obtained may be found in [7], for both cases: $c \neq 0$ and $c = 0$.

In this stage, the obtained formula can be used for one of the two purposes: to calculate the difference for a finite length of the armature but keeping the symmetry of the configuration, what cannot be performed by the results of the known literature, and for the case of the infinite length, given in the known literature, but surely, due to the mentioned circumstances we obtained different results. Namely, for the relative value (i.e., length divided by δ and field strength divided by H_{y0}) we obtained $\sigma = -0.330$, whereas by the traditional procedure the value is $\sigma = -0.220$.

X. The Carter factor

In practice, one of the most important results is the Carter factor, which refers to an alternative succession of slots and teeth, the pitch, i.e., the distance between the axes of two successive slots having any value t. The traditional derivation passes from the calculation of the infinite length of the pitch, as explained above, to a finite length accepting very large approximations. With the formulae we established above, with much smaller approximation we can calculate it. It is the ratio between the flux in the case in which the thickness of the air-gap were constant and equal to its minimum value and the actual flux in the real case. Both results from the complex potential function and the relations are given in a program of [7]. For example, we shall consider the case $c = -0.001$ to which there corresponds $x = 2.363674365$ relative units with respect to the air-gap thickness.

According to the definition above, the Carter factor will be given by the formula:

$$k_C = \frac{z_d}{z_t}; \quad z_d = z_t + \sigma, \tag{21}$$

where, according to [7], z_d (in program, the symbols are *zd* and *zt*) represents the length on the over armature, on the z - plane, between the points corresponding to abscissae $w = -1$ and $w = c, \forall c < 0$ on the w - plane; and z_t represents the length on the over armature on the zeta – plane which is just the flux between the two points corresponding to those above on the w - plane. Both quantities are proportional, apart from a constant factor, to the complex potential. For the case of Fig. 2, we have obtained $k_C = 1.07496873$. As a verification, in the ideal case, for $c \to 0, \forall c < 0$, there follows $k_C \to 1$.

XI. Conclusion

A new method, efficient in many cases, for using the conformal transformation for the calculation of potential fields has been presented. The method is based on the utilization of symbolic languages and permits to directly obtain the integral of the Schwarz-Christoffel formula, regardless its complication, provided the solution in closed form exists, what is not possible by the traditional methods. Moreover, the proposed procedure has allowed the obtaining of new results of higher accuracy than previously. Also, the analysis of the singular points could be carried out and taken into consideration.

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