Torsional Effect on the Nonlinear Electroelastic Response of a Circular Cylinder

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Abstract— **Polymeric materials (microscopic view) are formed by long chains having in many cases permanent dipoles in their repeating units or structural units. These materials form a class of smart materials as they suffer significant deformations under electric force fields and for this reason these materials are referred as electromechanically active elastomers (EMAE). Elastomers are lightweight materials that convert electrical energy into mechanical energy and vice-versa. Therefore these materials have potential for providing relatively cheap and light replacements for mechanical devices such as actuators, energy harvesters, robots, and biomedical prostheses. This paper is concerned with investigation of the effect of torsion on the mechanical response of a solid cylinder in an radial electric field on the basis of recently developed general theory of non linear electroelasticity.**

Index Terms— **Electro-sensitive, electro active polymer, orsion, Incompressible material.**

I. Introduction

Engineering applications of smart materials or electro sensitive (ES) elastomers are quite recent; the theoretical foundation of electromechanical interactions in solids framework date back to the 1950s and 1960s. In a recent series of papers [1-4], Dorfmann and Ogden have developed a theory of non-linear magnetoelasticity and applied it to a number of simple boundary value problems. Other recent developments in this area are described by Dorfmann et al. [5], Dorfmann and Brigadnov [6-7], Bustamante et al.[8], Ogden [9], Dorfmann et al. [10] and the relevant background to the equations governing magnetelasticity and electromagnetic-mechanical interactions by Maugin [11], Eringen and Maugin [12], Brown [13], Kovetz [14], Hutter [15-16] and Hutter et al. [17].

Electoactive elastomers exhibit a change in size or shape when stimulated by an electric field. The most common applications of this type of materials are in actuators where large deformations are required. Electromagnetic, piezoelectric or shape memory alloy actuators are either too heavy, too complex or too slow for such applications; electroactive polymers (EAP) however are relatively lightweight, rather simple and fast enough. As the most prospective practical research direction, EAP (Electro active polymers)have been utilizing in artificial muscles and refreshable Braille displays has emerged to aid visually impaired in fast reading and computer assisted communication.

The need to study the behaviour of new classes of polymers or elastomers which are rubber-like materials containing distributions of particles that react to electric/magnetic fields, in such polymers elastic modulus is low enough to allow large deformation.On the basis of theory of non-linear magnetoelasticity [1-4], Dorfmann and Ogden [18-19] have developed several alternative formulation of the equations of nonlinear electroelasticity and provide a theoretical framework for the analysis of boundary value problems that underpin the applications of the associated electromechanical interactions [18]. On the basis of this theory Dorfmann and Ogden [19] have solved the problems of azimuthal shear response of a thick-walled circular cylindrical tube, the extension and inflation characteristics of the same tube under a radial or an axial electric field (or both fields combined), and the effect of a radial field on the deformation of an internally pressurized spherical shell. Following the phenomenological approach proposed by Dorfmann and Ogden [18-19], Calleja et al. [20] have solved the problems of bifurcation namely biaxial and uniaxial stretching of a slab under electric field and effect of electric field on the inflation of spherical shell. Bustamante et al. [8] focused on the 'Maxwell stress' and 'Total stress' in the quasi-static context, based on the force, couple and energy balance equations, with particular reference to boundary conditions. Kumar and Kumar [21] have studied inhomogeneous deformations in electrosenstive materials.

The practical problems, including torsion given to cylinder is a well-known problem in the classical theory of finite strain. We do not attempt to provide a complete account of the relevant background, but instead refer the interested reader to the articles by Singh [22], Demiray and Suhubi [23], Kumar and Kumar [24] and books by Green and Zerna [25], Ogden [26], Eringen [27]. In this paper,on the basis of theory developed in [18-19], we have illustrate the effect of torsion on the mechanical response of solid circular cylinder in an radial electric field.

In Section 2, following Dorfmann and Ogden [18-19], we summarize briefly the basic electrical and mechanical balance laws for time-dependent electric fields. The general constitutive law for an isotropic electroelastic material is then discussed in Section 3.

In Section 4, we have examine the effect of torsion on the mechanical response of solid circular cylinder in an radial

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electric field and illustrated the results in Section 5. In Section 6, we have compared the results with limiting chain extensibility model (Gent's model) and exponential strain energy model.

II. Basic Equations

In this section, we recapitulate the basic equations formulated by Dorfmann and Ogden [18-19] in nonlinear electroelasticity.

(2.1). Kinematics

Consider a reference configuration, denoted β_0, of the material in which a material particle is lebelled by its position vector $\mathbf{X}$. This configuration may or may not be stress free. Let β denote the corresponding deformed configuration in which the particle $\mathbf{X}$ has position vector $\mathbf{x}$ and the deformation is defined by the mapping $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$ for $\mathbf{X} \in \beta_0$. The deformation gradient tensor, denoted $\mathbf{F}$,is

$$\mathbf{F} = Grad\,\chi, \tag{1}$$

where Grad is the gradient operator in β_0. We shall also use the notation $J = det\mathbf{F}$. By convention we take $J > 0$.

(2.2). Electric Balance Equations

When the material is deformed, the electric field variables may be defined as Eulerian quantities in the current configuration or as Lagrangian fields in the reference configuration. In this paper we start with the current configuration β and define the relevant electric field variables as the electric field $\mathbf{E}$, the electric induction $\mathbf{D}$ and the polarization density $\mathbf{P}$. These vectors are related by the standard equation

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}, \tag{2}$$

where the constant ε_0 is the vacuum electric permittivity (see, for example, Kovetz [14]). In vacuum, $\mathbf{P} = 0$ and equation (2) simplifies to $\mathbf{D} = \varepsilon_0\mathbf{E}$. In a material $\mathbf{P}$ measures the difference $\mathbf{D} - \varepsilon_0\mathbf{E}$ and is a material-dependent property that has to be given by a constitutive equation.

Here, initially, we take the basic variables to be the electric field $\mathbf{E}$ and the deformation gradient $\mathbf{F}$. Equation (2) then determines the electric induction $\mathbf{D}$ in terms of $\mathbf{F}$ and the field $\mathbf{E}$ when $\mathbf{P}$ is given by a constitutive equation.

For time-independent phenomena and in the absence of magnetic fields, free currents and free electric charges, the vectors $\mathbf{E}$ and $\mathbf{D}$ satisfy the equations

$$curl\,\mathbf{E} = \mathbf{0}, \qquad div\mathbf{D} = \mathbf{0}, \tag{3}$$

obtained by appropriate specialization of Maxwell's equations, where, respectively, curl and div are the curl and divergence operators in β.

The Lagrangian counterparts of the electric field and the electric induction, denoted by $\mathbf{E}_l$ and $\mathbf{D}_l$ respectively, are given by

$$\mathbf{E}_l = \mathbf{F}^T\mathbf{E}, \qquad \mathbf{D}_l = J\mathbf{F}^{-1}\mathbf{D}. \tag{4}$$

For details of the derivations of these connections we refer to, for example, Dorfmann and Ogden [18] and references therein. Standard identities ensure that equations (3) are equivalent to

$$Curl\mathbf{E}_l = \mathbf{0}, \qquad Div\mathbf{D}_l = \mathbf{0}, \tag{5}$$

provided $\boldsymbol{\chi}$ is suitably regular, where, respectively Curl and Div are the curl and divergence operators in β_0.

No corresponding pull-back operation for $\mathbf{P}$ arises naturally in a similar way. It is convenient, however, to define a Lagrangian form of $\mathbf{P}$, here denoted by $\mathbf{P}_l$, analogous to that for $\mathbf{D}$, by

$$\mathbf{P}_l = J\mathbf{F}^{-1}\mathbf{P}. \tag{6}$$

Using equations (4) and (6) in equation (2) we obtain

$$\mathbf{D}_l = \varepsilon_0 J\mathbf{c}^{-1}\mathbf{E}_l + \mathbf{P}_l, \tag{7}$$

where $\mathbf{c}^{-1}$ is the inverse of the right Cauchy-Green deformation tensor $\mathbf{c} = \mathbf{F}^T\mathbf{F}$.

(2.3). Mechanical Balance Laws

Let ρ_0 and ρ denote the mass densities in the reference and current configurations, respectively. Then in terms of the notation $J = det\mathbf{F}$, the conservation of mass equation has the form

$$J\rho = \rho_0. \tag{8}$$

If the electric body forces are included with the 'total' (Cauchy) stress tensor, denoted by σ, the equilibrium equation in the absence of mechanical body forces may be written in the simple form

$$div\sigma = \mathbf{0}, \tag{9}$$

balance of angular momentum ensuring symmetry of σ ,i.e., $\sigma^T = \sigma$.

The counterpart of the nominal stress tensor in elasticity theory, denoted here by $\mathbf{T}$, for the total stress is defined by

$$\mathbf{T} = J\mathbf{F}^{-1}\boldsymbol{\sigma}, \tag{10}$$

and the equilibrium equation (9) may then be written in the alternative form

$$Div\mathbf{T} = \mathbf{0}. \tag{11}$$

(2.4). Boundary Conditions

The electric field Let $\mathbf{E}$ and the electric induction vector $\mathbf{D}$ satisfy appropriate continuity conditions across any surface with in the material or the surface bounded the considered material. In the deformed configuration, in the absence of surface charge, the standard continuity conditions are

$$\mathbf{n}.[\mathbf{D}] = \mathbf{0}, \qquad \mathbf{n}\times[\mathbf{E}] = \mathbf{0}, \tag{12}$$

where a square bracket indicates a discontinuity across the surface and $\mathbf{n}$ is normal to the surface. By convention, on the material boundary $\mathbf{n}$ is taken to be the outward pointing normal. These equations may also be given in Lagrangian form (see, for example, [18]), but we omit the details here.

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For the mechanical quantities the function χ has to be continuous across any surface, as has the total traction vector $\boldsymbol{\sigma}\mathbf{n}$. The deformation $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$ may be prescribed on part of the bounded surface of the body, while the total traction vector on the remaining part of the surface must be continuous. The latter condition is given, in Eulerian form , by

$$[\boldsymbol{\sigma}]\mathbf{n} = \mathbf{0}, \tag{13}$$

where any applied mechanical traction contributes to the traction on the outside. The Maxwell stress there, denoted by $\boldsymbol{\sigma}_m$, must also be accounted for. If the exterior of the body is a vacuum, for example, then $\boldsymbol{\sigma}_m$ is given by

$$\boldsymbol{\sigma}_m = \varepsilon_0[\mathbf{E}\otimes\mathbf{E} - \frac{1}{2}(\mathbf{E}.\mathbf{E})\mathbf{I}], \tag{14}$$

where $\mathbf{I}$ is the identity tensor.

III. Constitutive equations

To complete the formulation of boundary-value problems we need, in addition to the governing equations and boundary conditions, appropriate constitutive laws for the total stress tensor $\boldsymbol{\sigma}$ and for the polarization vector $\mathbf{P}$. Following Dorfmann and Ogden [18], we base the construction of constitutive laws on the existence of a free energy function, which may be regarded as a function of the deformation gradient $\mathbf{F}$ and one of the electric field vectors. Here, we take the independent variables initially to be $\mathbf{F}$ and $\mathbf{E}_l$, and in, the notation of Dorfmann and Ogden [18], we write the free energy (per unit mass) as

$$\boldsymbol{\Phi} = \boldsymbol{\Phi}(\mathbf{F}, \mathbf{E}_l). \tag{15}$$

It then follows (See, for example , Kovetz [14] and Dorfmann and Ogden [18] that the total Cauchy stress $\boldsymbol{\sigma}$ for a compressible material is given by

$$\boldsymbol{\sigma} = \rho\mathbf{F}\frac{\partial\boldsymbol{\Phi}}{\partial\mathbf{F}} + \boldsymbol{\sigma}_m, \tag{16}$$

where $\boldsymbol{\sigma}_m$ is given by equation (14). The standard requirements of objectivity show that $\boldsymbol{\Phi}$ depends on $\mathbf{F}$ only through $\mathbf{c} = \mathbf{F}^{\mathbf{T}}\mathbf{F}$, as in elasticity theory, and symmetry of the first term on the right-hand side of equation (16) then follows automatically and ensures symmetry of $\boldsymbol{\sigma}$. Note, however, that the $\boldsymbol{\sigma}_m$ inside and outside the material are in general different since $\mathbf{E}$ is different. In the absence of material $\boldsymbol{\Phi} = 0$ and $\boldsymbol{\sigma}$ reduces to the Maxwell stress (equation (14)).

The expression for the polarization vector in Eulerian form is given in terms of $\boldsymbol{\Phi}$ by

$$\mathbf{P} = -\rho\mathbf{F}\frac{\partial\Phi}{\partial\mathbf{E}_l}. \tag{17}$$

The corresponding Lagrangian forms of stress and polarization are obtained on use of equations (6) and (10), respectively. However, rather than giving these explicitly we now make use of a convenient alternative formulation of the constitutive law introduced by Dorfmann and Ogden [18]. This requires the definition of an amended (or 'total') free energy, denoted by $\Omega = \Omega(\mathbf{F}\mathbf{E}_l)$ and defined per unit reference volume (rather than per unit mass) within the material by

$$\boldsymbol{\Omega} = \rho_0\boldsymbol{\Phi} - \frac{1}{2}\varepsilon_0 J\mathbf{E}_l\cdot(\mathbf{c}^{-1}\mathbf{E}_l). \tag{18}$$

(Note that $\mathbf{F}$, and hence $\boldsymbol{\Omega}$, is not defined outside the material.). This allows us to write the total stress tensors $\boldsymbol{\sigma}$ and $\mathbf{T}$ in the compact forms

$$\boldsymbol{\sigma} = J^{-1}\mathbf{F}\frac{\partial\boldsymbol{\Omega}}{\partial\mathbf{F}}, \quad \mathbf{T} = \frac{\partial\Omega}{\partial\mathbf{F}}. \tag{19}$$

While it is the polarization that is given by equation (17), it is now the electric displacement that is given directly in terms of $\boldsymbol{\Omega}$. In Eulerian form the polarization and electric displacement are given by

$$\mathbf{P} = \mathbf{D} - \varepsilon_0\mathbf{E}, \quad \mathbf{D} = -J^{-1}\mathbf{F}\frac{\partial\boldsymbol{\Omega}}{\partial\mathbf{E}_l}, \tag{20}$$

and their Lagrangian counterpart by

$$\mathbf{P}_l = \mathbf{D}_l - \varepsilon_0 J\mathbf{c}^{-1}\mathbf{E}_l, \quad \mathbf{D}_l = \frac{\partial\boldsymbol{\Omega}}{\partial\mathbf{E}_l}. \tag{21}$$

The expressions listed above for the stresses require modification in the case of incompressible materials, which are subject to the constraint

$$J = det\mathbf{F} \equiv 1, \tag{22}$$

so that equation (18) becomes

$$\boldsymbol{\Omega} = \rho_0\boldsymbol{\Phi} - \frac{1}{2}\varepsilon_0\mathbf{E}_l\cdot(\mathbf{c}^{-1}\mathbf{E}_l). \tag{23}$$

The total stress tensors in equation (19) are replaced by

$$\boldsymbol{\sigma} = \mathbf{F}\frac{\partial\boldsymbol{\Omega}}{\partial\mathbf{F}} - p\mathbf{I}, \quad \mathbf{T} = \frac{\partial\Omega}{\partial\mathbf{F}} - p\mathbf{F}^{-1}. \tag{24}$$

where p is a Lagrarange multiplier associated with the constraint (equation (22)).The expressions for the electric induction and the polarization fields given in Eulerian and Lagrangian forms by equations (20) and (21), respectively, are unchanged but subject to equation (22).

(3.1). Isotropy

Application of an electric field to an isotropic ES elastomer introduces, locally, a preferred direction analogous to that arising for transversely isotropic elastic solids. Following the analysis of such materials given in Spencer [28] and Ogden [29] , for example, we define an isotropic ES material as one for which $\boldsymbol{\Omega}$ is an isotropic function of $\mathbf{c}$ and $\mathbf{E}_l\otimes\mathbf{E}_l$.The form of $\boldsymbol{\Omega}$ is then reduced to dependence on the six independent invariants, denoted $I_1, I_2, \ldots. I_6$ of $\mathbf{c}$ and $\mathbf{E}_l\otimes\mathbf{E}_l$.For a compressible material, we choose the standard principal invariants of $\mathbf{c}$, namely

$$I_1 = tr\mathbf{c}, \quad I_2 = \frac{1}{2}[(tr\mathbf{c})^2 - tr(\mathbf{c}^2)], \quad I_3 = det\mathbf{c} = J^2, \tag{25}$$

while for the invariants depending on $\mathbf{E}_l$, we set

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$$I_4 = |\mathbf{E}_l|^2, \quad I_5 = \mathbf{E}_l \cdot (\mathbf{c}^{-1}\mathbf{E}_l), \quad I_6 = \mathbf{E}_l \cdot (\mathbf{c}^{-2}\mathbf{E}_l), \tag{26}$$

where tr is the trace of a second-order tensor. Note that I_5 and I_6 can also be written, respectively, as $\mathbf{E}\cdot\mathbf{E}$ and $\mathbf{E}\cdot(\mathbf{b}\_^{1}\mathbf{E})$, where $\mathbf{b} = \mathbf{F}\mathbf{F}^T$ is the left Cauchy-Green deformation tensor. The choice in the equation (26) is not, of course, unique and one could, for example , replace $\mathbf{c}\_^{1}$ by $\mathbf{c}$ in I_5 and I_6. For incompressible materials the invariant $I_3 \equiv 1$ is omitted.

For an incompressible isotropic material, therefore, $\mathbf{\Omega} = \mathbf{\Omega}(I_1, I_2, I_4, I_5, I_6)$, and the explicit form of the total stress tensor $\mathbf{\sigma}$ and the electric induction vector $\mathbf{D}$ are

$$\begin{aligned}\mathbf{\sigma} &= 2\mathbf{\Omega}_1\mathbf{b} + 2\mathbf{\Omega}_2(I_1\mathbf{b} - \mathbf{b}^2) - p\mathbf{I} - 2\mathbf{\Omega}_5\mathbf{E}\otimes\mathbf{E} \\ &- 2\mathbf{\Omega}_6(\mathbf{b}^{-1}\mathbf{E}\otimes\mathbf{E} + \mathbf{E}\otimes\mathbf{b}^{-1}\mathbf{E}),\end{aligned} \tag{27}$$

$$\mathbf{D} = -2(\mathbf{\Omega}_4\mathbf{b}\mathbf{E} + \mathbf{\Omega}_5\mathbf{E} + \mathbf{\Omega}_6\mathbf{b}^{-1}\mathbf{E}), \tag{28}$$

where the subscripts $1,2,4,5,6$ on $\mathbf{\Omega}$ signify partial differentiation with respect to I_1, I_2, I_4, I_5, I_6, respectively, and wherein the left Cauchy-Green deformation tensor $\mathbf{b}$ is used.

(3.2). Change of Independent Variables

In the solution of boundary-value problems involving non-uniform fields, it may in some circumstances be more convenient to select $\mathbf{D}_l$ as the independent electric variable instead of $\mathbf{E}_l$. This can be done, for example, by defining an energy function $\mathbf{\Omega}^* = \mathbf{\Omega}^*(\mathbf{F}, \mathbf{D}_l)$, complementary to $\mathbf{\Omega}$, via the partial Legendre-type transform

$$\mathbf{\Omega}^*(\mathbf{F}, \mathbf{D}_l) = \mathbf{\Omega}(\mathbf{F}, \mathbf{E}_l) + \mathbf{D}_l \cdot \mathbf{E}_l. \tag{29}$$

This requires that the relation (28), or more generally equation $(21)_2$, be invertible to give $\mathbf{E}_l$ in terms of $\mathbf{D}_l$ for each $\mathbf{F}$, a requirement that can be circumvented if one starts with a free energy that depends on $\mathbf{D}_l$ instead of $\mathbf{E}_l$.

The total stress tensor and the electric field in Lagrangian form, for compressible materials, are then simply

$$\mathbf{T} = \frac{\partial\mathbf{\Omega}^*}{\partial\mathbf{F}}, \quad \mathbf{E}_l = \frac{\partial\mathbf{\Omega}^*}{\partial\mathbf{D}_l}, \tag{30}$$

and the polarization is still given by equation $(21)_1$, but now with $\mathbf{D}_l$ as the independent variable and $\mathbf{E}_l$ given by equation $(30)_2$.

For an isotropic material, $\mathbf{\Omega}^*$ depends on the invariants I_1, I_2, I_3 defined in equation (25) and on the three invariants based on $\mathbf{D}_l$, for which we use the notation K_4, K_5, K_6. We choose to define these as

$$K_4 = \mathbf{D}_l \cdot \mathbf{D}_l, \quad K_5 = \mathbf{D}_l \cdot (\mathbf{c}\mathbf{D}_l), \quad K_6 = \mathbf{D}_l \cdot (\mathbf{c}^2\mathbf{D}_l). \tag{31}$$

For an incompressible material, the Eulerian form of the total stress $\mathbf{\sigma}$ and the electric field $\mathbf{E}$ based on $\mathbf{\Omega}^*$ have the explicit forms

$$\begin{aligned}\mathbf{\sigma} &= 2\mathbf{\Omega}_1^*\mathbf{b} + 2\mathbf{\Omega}_2^*(I_1\mathbf{b} - \mathbf{b}^2) - p\mathbf{I} + 2\mathbf{\Omega}_5^*\mathbf{D}\otimes\mathbf{D} \\ &+ 2\mathbf{\Omega}_6^*(\mathbf{D}\otimes\mathbf{b}\mathbf{D} + \mathbf{b}\mathbf{D}\otimes\mathbf{D}),\end{aligned} \tag{32}$$

$$\mathbf{E} = -2(\mathbf{\Omega}_4^*\mathbf{b}^{-1}\mathbf{D} + \mathbf{\Omega}_5^*\mathbf{D} + \mathbf{\Omega}_6^*\mathbf{b}\mathbf{D}). \tag{33}$$

The polarization is again given by equation $(20)_1$ with $\mathbf{D} = \mathbf{F}\mathbf{D}_l$. Here , we define $\mathbf{\Omega}_l^*$ to be $\partial\mathbf{\Omega}^*/\partial I_i$ for $i = 1,2$ and $\partial\mathbf{\Omega}^*/\partial K_i$ for $K = 4,5,6$.

(3.3). The Reference Configuration

If the material is not subject to any mechanical boundary tractions or mechanical body forces then, in general, application of an electric field will induce the material to deform, a phenomenon known as electrostriction. Let the resulting configuration to taken as the reference configuration, which we now denote by ß_r to distinguish it from ß_0. These two reference configurations can be taken to coincide if appropriate loads are applied to the body, which will result in a (residual) stress distribution throughout the material. In such a case we denote the values of $\mathbf{\sigma}, \mathbf{E}, \mathbf{D}$ and $\mathbf{P}$ in this configuration by $\mathbf{\sigma}_0, \mathbf{E}_0, \mathbf{D}_0$ and $\mathbf{P}_0$, respectively. Again we focus on incompressible materials. With $I_3 \equiv 1$ and $\mathbf{F} = \mathbf{I}$ the invariants (25), (26) and (31) reduce to

$$\begin{aligned}&I_1 = I_2 = 3, \quad I_4 = I_5 = I_6 = \mathbf{E}_0 \cdot \mathbf{E}_0, \\ &K_4 = K_5 = K_6 = \mathbf{D}_0 \cdot \mathbf{D}_0.\end{aligned} \tag{34}$$

Then, in terms of $\mathbf{\Omega}$ and $\mathbf{\Omega}^*$ the expression for the total stress tensor $\mathbf{\sigma}_0$ simplify to

$$\mathbf{\sigma}_0 = [2(\mathbf{\Omega}_1 + 2\mathbf{\Omega}_2) - p]\mathbf{I} - 2(\mathbf{\Omega}_5 + 2\mathbf{\Omega}_6)\mathbf{E}_0\otimes\mathbf{E}_0, \tag{35}$$

and

$$\mathbf{\sigma}_0 = [2(\mathbf{\Omega}_1^* + 2\mathbf{\Omega}_2^*) - p]\mathbf{I} - 2(\mathbf{\Omega}_5^* + 2\mathbf{\Omega}_6^*)\mathbf{D}_0\otimes\mathbf{D}_0, \tag{36}$$

respectively, with $\mathbf{\Omega}_i$ and $\mathbf{\Omega}_i^*$ evaluated for the appropriate subset of invariants (34).

The corresponding expressions for electric field vectors may be simplified to defining $\mathbf{\Omega}_0(I_4) \equiv \mathbf{\Omega}(3,3,I_4,I_4,I_4)$ and $\mathbf{\Omega}_0^*(K_4) \equiv \mathbf{\Omega}^*(3,3,K_4,K_4,K_4)$. Then, we obtain the specializations of $\mathbf{D}$ in equation (28) and $\mathbf{P}$ as

$$\mathbf{D}_0 = -2\mathbf{\Omega}_0'(I_4)\mathbf{E}_0, \quad \mathbf{P}_0 = \mathbf{D}_0 - \varepsilon_0\mathbf{E}_0, \tag{37}$$

where the prime signifies differentiation with respect to I_4 .

Similarly, for $\mathbf{E}$ in equation (33) and $\mathbf{P}$ the specializations are

$$\mathbf{E}_0 = 2\mathbf{\Omega}_0^*(K_4)\mathbf{D}_0, \quad \mathbf{P}_0 = \mathbf{D}_0 - \varepsilon_0\mathbf{E}_0, \tag{38}$$

where the prime signifies differentiation with respect to K_4

In this configuration $\mathbf{E}_0, \mathbf{D}_0$ and $\mathbf{\sigma}_0$ must satisfy the equations

$$Curl\mathbf{E}_0 = 0, \quad Div\mathbf{D}_0 = 0, \quad Div\boldsymbol{\sigma}_0 = 0. \quad (39)$$

(3.4). Non-homogeneous Deformations

For a general boundary-value problem involving non-homogeneous deformations we are required to solve the equations

$$Div\mathbf{D} = 0, \quad Curl\,\mathbf{E} = 0, \quad Div\boldsymbol{\sigma} = 0, \quad (40)$$

for the deformation field χ and $\mathbf{E} = -grad\,\phi$ (where ϕ is potential function) in respect of the formulation based on $\boldsymbol{\Omega}$. In this case, these equations are taken together with

$$\boldsymbol{\sigma} = 2\boldsymbol{\Omega}_1\mathbf{b} + 2\boldsymbol{\Omega}_2(I_1\mathbf{b} - \mathbf{b}^2) - p\mathbf{I} - 2\boldsymbol{\Omega}_5\mathbf{E}\otimes\mathbf{E} + 2\boldsymbol{\Omega}_6(\mathbf{b}^{-1}\mathbf{E}\otimes\mathbf{E} + \mathbf{E}\otimes\mathbf{b}^{-1}\mathbf{E}), \quad (41)$$

and

$$\mathbf{D} = -2(\boldsymbol{\Omega}_4\mathbf{bE} + \boldsymbol{\Omega}_5\mathbf{E} + \boldsymbol{\Omega}_6\mathbf{b}^{-1}\mathbf{E}). \quad (42)$$

On the other hand, for the formulation based on $\boldsymbol{\Omega}_1^*$, equations (40) are appended with

$$\boldsymbol{\sigma} = 2\boldsymbol{\Omega}_1^*\mathbf{b} + 2\boldsymbol{\Omega}_2^*(I_1\mathbf{b} - \mathbf{b}^2) - p\mathbf{I} + 2\boldsymbol{\Omega}_5^*\mathbf{D}\otimes\mathbf{D} + 2\boldsymbol{\Omega}_6^*(\mathbf{D}\otimes\mathbf{bD} + \mathbf{bD}\otimes\mathbf{D}), \quad (43)$$

and

$$\mathbf{E} = -2(\boldsymbol{\Omega}_4^*\mathbf{b}^{-1}\mathbf{D} + \boldsymbol{\Omega}_5^*\mathbf{D} + \boldsymbol{\Omega}_6^*\mathbf{bD}). \quad (44)$$

In each case appropriate boundary conditions need to be specified.

IV. Mathematical Formulation and Solution

Consider a long solid circular cylinder of radius A composed of an incompressible isotropic hyperelastic material subjected to the condition that the cylinder is not allowed to stretch axially or contract radially. On using cylindrical coordinates (R,Θ,Z) in the undeformed configuration and (R,θ,z) in the current configuration, we may thus write

$$r = R, \quad \theta = \Theta + \tau Z, \quad z = Z, \quad (45)$$

where τ denote the twist per unit length of the rod.

The component of matrix of the deformation gradient tensor $\mathbf{F}$, denoted by F is

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \tau R \\ 0 & 0 & 1 \end{bmatrix}, \quad (46)$$

The resulting matrices of the left and right Cauchy-Green deformation tensors $\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathbf{T}}$ and $\mathbf{c} = \mathbf{F}^{\mathbf{T}}\mathbf{F}$, written b and c, are

$$b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+\tau^2R^2 & \tau R \\ 0 & \tau R & 1 \end{bmatrix}, \qquad c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \tau R \\ 0 & \tau R & 1+\tau^2R^2 \end{bmatrix}, \quad (47)$$

and the associated principal invariants are, from equation (25).

$$I_1 = 3 + \tau^2R^2, \quad I_3 = 1. \quad (48)$$

We focus on the formulation based on $\boldsymbol{\Omega}$. Since the Lagrangian field $\mathbf{E}_l$ is the independent electric variable, we may choose it to be the field $\mathbf{E}_0$. We take $\mathbf{E_0}$ in the radial direction. Then the components of $\mathbf{E}$ in the deformed configuration follow from the component form of equation $\mathbf{E}_l = \mathbf{F}^T\mathbf{E}$ as

$$E_r = 0, \quad E_\theta = E_{0\Theta}, \quad E_z = 0. \quad (49)$$

From equation (26) we then calculate the invariants

$$I_4 = E_{0\Theta}^2, \quad I_5 = I_6 = (1+\tau^2R^2)I_4. \quad (50)$$

The resulting components of σ, obtained from equation (27), are

$$\sigma_{rr} = -p + 2\Omega_1 + 2\Omega_2(1+\tau^2R^2),$$
$$\sigma_{\theta\theta} = -p + 2\Omega_1(1+\tau^2R^2) + 2\Omega_2(2-\tau^4R^4 - 2\tau^2R^2) - 2(\Omega_5 + 2\Omega_6)I_4,$$
$$\sigma_{zz} = -p + 2\Omega_1 + 4\Omega_2, \quad (51)$$
$$\sigma_{z\theta} = 2\tau R(\Omega_1 + \Omega_2 + \Omega_6 I_4), \quad (52)$$

and $\sigma_{r\theta} = \sigma_{zr} = 0$. From equation (28) the components of the vector $\mathbf{D}$ are obtained as

$$D_r = 0,\ D_\theta = -2[\Omega_4(1+\tau^2R^2) + \Omega_5 + \Omega_6]E_{0\Theta},$$
$$D_z = -2[\Omega_4(\tau R) + \Omega_6(-\tau R)]E_{0\Theta}, \quad (53)$$

while equation $(20)_1$ gives the corresponding components of $\mathbf{P}$ as

$$P_r = 0,$$
$$P_\theta = -2[\Omega_4(1+\tau^2R^2) + \Omega_5 + \Omega_6]E_{0\Theta} - \varepsilon_0 E_{0\Theta},$$
$$P_z = -2[\Omega_4(\tau R) + \Omega_6(-\tau R)]E_{0\Theta}. \quad (54)$$

For the considered circular symmetry the equilibrium equation $(40)_1$ and by using equations (51), (52), the component of σ can be written

$$\sigma_{rr}(R) = -\int_R^A [2\Omega_1\tau^2R + 2\Omega_2(\tau^4R^3 + 3\tau^2R - R^{-1}) + 2(\Omega_5 + 2\Omega_6)R^{-1}\Omega_4]dR,$$
$$\sigma_{\theta\theta}(R) = \sigma_{rr}(R) + 2\Omega_1\tau^2R^2 + 2\Omega_2(-\tau^4R^4 - 3\tau^2R^2) - 2(\Omega_5 + 2\Omega_6)I_4,$$
$$\sigma_{zz}(R) = \sigma_{rr}(R) - 2\Omega_2\tau^2R^2, \quad (55)$$
$$\sigma_{z\theta} = 2\tau R(\Omega_1 + \Omega_2 + \Omega_6 I_4). \quad (56)$$

The resultant applied moment and axial force necessary to maintain deformation are given by

$$M = \int_0^{2\pi}\int_0^A \sigma_{z\theta}R^2 dR d\theta = 4\pi\tau\int_0^A R^3(\Omega_1 + \Omega_2 + \Omega_6 I_4)dR. \quad (57)$$

and

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$$N=\int_0^{2\pi}\int_0^A\sigma_{zz}RdRd\theta=2\pi\tau^2\int_0^A-\Omega_1R^3dR$$
$$+2\pi\int_0^A[2\Omega_2(R^5/4+\tau^2R^3/2-R^2)$$
$$+2(\Omega_5+\Omega_6)RI_4)]dR. \tag{58}$$

In view of equations (49) and (50) there remain two independent variables in Ω, namely R and I_4. It is convenient to define a reduced form of Ω as a function of these two variables only. Accordingly, we define the appropriate specialization , denoted ω, by

$$\omega(R,I_4)=\Omega(I_1,I_2,I_4,I_5,I_6), \tag{59}$$

with equations (49) and (50), it follows that

$$\omega_R=2\tau R(\Omega_1+\Omega_2+\Omega_6I_4),$$
$$\omega_4=\Omega_4(1+\tau^2R^2)+\Omega_5+\Omega_6, \tag{60}$$

where the subscript R and 4 on ω indicate partial differentiation with respect to R and 4 respectively.The expressions for $\sigma_{z\theta}$ and D_θ and P_θ then simplify to

$$\sigma_{z\theta}=\omega_R,\quad D_\theta=-2\omega_4E_{0\Theta},$$
$$P_\theta=-(2\omega_4+\varepsilon_0)E_{0\Theta}. \tag{61}$$

Equation $(61)_1$ is exactly the same as that arising in elasticity theory in the absence of the electric field, but here ω depends on the electric field through I_4.

V. ILLUSTRATION

A simple illustration of the above theory is provided by the model

$$\Omega=\frac{\mu(I_4)}{k}[\frac{(I_1-1)^k}{2^k}-1]+\nu(I_4), \tag{62}$$

wherein μ and ν are the function of I_4 and k is a constant such that $k\geq\frac{1}{2}$. In the absence of the electric field $I_4=0$ with $\nu(0)$ taken to be 0, (62) reduces to a special class of models considered by Jiang and Ogden [30], with $\mu(0)(>0)$ being the shear modulus of the material. In respect of (62), equation $(61)_1$ yields

$$\sigma_{z\theta}=2\tau R\mu(I_4)[\frac{2+\tau^2R^2}{2}]^{k-1}, \tag{63}$$

and we note , in particular that this does not involve the function ν. Equation (63) describes shear response of the considered class of materials, with the gradient of the $\sigma_{z\theta}$ vs. R curve dependent on the electric field strength through I_4. Thus, $\mu(I_4)$ characterizes the dependence of the shear modulus on the electric field.

The resulting components of σ, obtained from equations (55) and (56), are

$$\sigma_{rr}(R)=-\tau^2(A^2-R^2)\mu(I_4)[\frac{2+\tau^2R^2}{2}]^{k-1},$$
$$\sigma_{\theta\theta}(R)=-\tau^2(A^2-3R^2)\mu(I_4)[\frac{2+\tau^2R^2}{2}]^{k-1},$$
$$\sigma_{zz}(R)=-\tau^2(A^2-R^2)\mu(I_4)[\frac{2+\tau^2R^2}{2}]^{k-1}. \tag{64}$$

and $\sigma_{r\theta}=\sigma_{zr}=0$.

The resultant applied moment necessary to maintain deformation are given by using equations (57) and (62)

$$M=4\pi\tau\int_0^AR^3[\mu(I_4)(\frac{2+\tau^2R^2}{2})^{k-1}]dR. \tag{65}$$

Case 1. when $k<1$ (for example $k=3/4$). Then solution of equation (65) is

$$M=\frac{32\pi\mu(I_4)}{\tau^3}[\frac{(1+\frac{\tau^2A^2}{2})^{\frac{7}{4}}}{7}-\frac{(1+\frac{\tau^2A^2}{2})^{\frac{3}{4}}}{3}+\frac{4}{21}]. \tag{66}$$

Case 2. when $k=1$, Then solution of equation (65) is

$$M=\pi\tau A^4\mu(I_4). \tag{67}$$

The result is consistent with the result obtained by Kanner and Horgan (31) for neo-Hookean materials.

Case 3. when $k>1$ (for example $k=2$), Then solution of equation (65) is

$$M=4\pi\tau[\frac{A^4}{4}+\frac{\tau^2A^6}{12}]\mu(I_4). \tag{68}$$

Similarly the resultant axial force necessary to maintain deformation are given by

$$N=-2\pi\tau^2\int_0^AR^3[\mu(I_4)(\frac{2+\tau^2R^2}{2})^{k-1}]dR. \tag{69}$$

Case 1. when $k<1$ (for example $k=3/4$), Then solution of equation (65) is

$$N=-\frac{16\pi\mu(I_4)}{\tau^2}[\frac{(1+\frac{\tau^2A^2}{2})^{\frac{7}{4}}}{7}-\frac{(1+\frac{\tau^2A^2}{2})^{\frac{3}{4}}}{3}+\frac{4}{21}]. \tag{70}$$

Case 2. when $k=1$, Then solution of equation (65) is

$$N=-\pi\tau^2A^4\mu(I_4)/2. \tag{71}$$

The result is consistent with the result obtained by Kammer and Horgan (31) for neo-Hookean materials.

Case 3. when $k>1$ (for example $k=2$), Then we get the equation

$$N=-2\pi\tau^2[\frac{A^4}{4}+\frac{\tau^2A^6}{12}]\mu(I_4). \tag{72}$$

Also the function ν , on the other hand, enters the expression $(61)_2$ for D_θ and the corresponding expression for P_θ form equation $(61)_3$.In particular, in the reference configuration, we have

$$P_{0\theta}=-[2\nu'(I_4)+\varepsilon_0]E_{0\Theta},\quad I_4=E_{0\Theta}^2 \tag{73}$$

This shows that $\nu'(I_4)$ characterizes the polarization in the reference configuration or, equivalently, the relation between the electric field and the electric displacement.

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VI. DISCUSSION

In this paper, we have obtained the exact solution for resultant applied moment and resultant axial force for different values of k (equations 66-68 and 70-72).

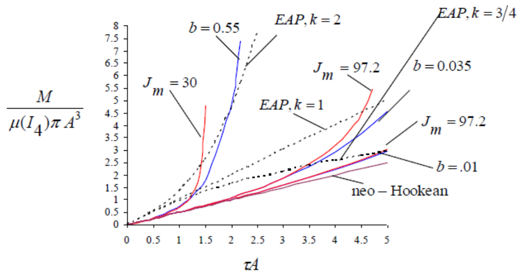

Fig.1 Resultant applied moment for simple torsion for the Gent's model (red lines), exponential model (blue lines), neo-Hookean model (grey line) and electroactive polymers (EAP)(dotted lines).

In Fig. 1, we plot M for the different models mainly Gent's model, HS model, exponential model and compare them with the model discussed in this paper. As in all models with the increase in τA the curves diverges for all values of k. However, due to effect of radial electric field the curves for the present model predicts slightly more rapid approach towards divergence as compared to small values of J_m, J and soft biological tissues. The results are consistent with neo-Hookean model for $k=1$.

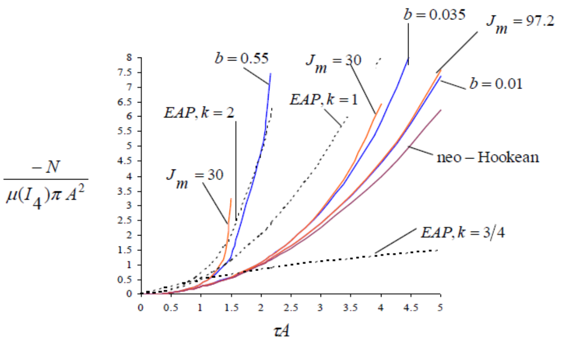

Fig.2 Resultant compressive axial force for simple torsion for the Gent's model (red lines),exponential model (blue lines), neo-Hookean model (grey line) and electroactive polymers (EAP) (dotted lines).

The resultant axial force are plotted in Fig. 2 and it is verified that the resultant axial force required to maintain pure torsion is compressive for all the models. In the absence of such a force, the bar would be elongate on twisting reflecting the celebrated Poynting effect. In Fig. 2 the compressive axial force for different models are compared. The curves for different values of k shows the similar trends to those discussed in Fig. 1 and results are consistent with neo-Hookean model for $k=1$.

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