An Analytical Solution for the Modified Lorenz System

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Abstract—**The well known Taylor series method will be presented her to derive series approximation to the solution of the nonlinear dynamical system of ordinary differential equations. The method is applied to the extended Lorenz system and it is found that only few terms of the series approximation is enough to characterize the chaotic properties of the system. The series estimation is good only for a very short period of time. To overcome this problem, the method is extended to longer time by taking smaller time steps and changing the initial conditions at each time step**

Index Terms— Analytic solutions, Chaotic system, Modified Lorenz system, Taylor series method.

1. Introduction

Literature is rich with different methods used to construct an approximation to the analytic solutions for nonlinear ordinary or partial differential equations, such methods include, but not limited to, the Adomian decomposition method [1,2,3,5, 23-29], the Homotopy analysis method (HAM), [4(a),4(b),15,16,17,30,31] , the homotopy perturbation method (HPM) [10,11], the variational iteration method (VIM) [7,8,9,12.13,14] and the Taylor series method.

Perturbation techniques are too strongly dependent upon the so called "small parameters" [19]. Thus, it is worthwhile developing some new analytic techniques independent upon small parameters. Liao [15,16,17] proposed such a kind of analytic technique, namely the Homotopy Analysis Method (HAM) The validity of the Homotopy analysis method was tested by many authors [4,16,17,30,31]. Adomian decomposition method and the variational iteration method were proven to be a special case of the homotopy analysis method.

Taylor series method is also a very well known old method used to solve initial value problems arises in science and engineering. It is a very simple technique used to derive the series expansion of the solution of the initial value problem whether it is linear or nonlinear.

The Taylor series method yields, without linearization, perturbation, transformation or discretization, an analytical solution in terms of an infinite power series with easily computable terms. The radius of convergence of the series depends on the type of the differential equation and on the number of terms used.

Our goal in this report is to use the Taylor series method to derive an approximation for the analytical solution for the chaotic extended Lorenz system.

2. The Taylor series method

Taylor series method is a simple techniques used very often in the literature to derive solutions for ordinary differential equations. For example, when the method is applied to the first order ordinary differential equation:

$$\begin{aligned} y'(t) &= f(y,t) \\ y(t_0) &= \alpha \end{aligned} \qquad (2.1)$$

With the assumption that the solution of the above initial value problem has a unique solution and the solution can be represented by the Taylor series of the form:

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n (t-t_0)^n \qquad (2.2) \\ &= a_0 + a_1(t-t_0) + a_2(t-t_0)^2 + \ldots\ldots \end{aligned}$$

Then

$$\begin{aligned} y'(t) &= \sum_{n=1}^{\infty} n a_n (t-t_0)^{n-1} \qquad (2.3) \\ &= a_1 + 2a_2(t-t_0) + 3a_3(t-t_0)^2 + \ldots\ldots \end{aligned}$$
(

Then substituting the above expansions in the differential equations (2.1) yields the following relation for the coefficients a_n :

$$a_n = \frac{y^n(t_0)}{n!} \qquad (2.4)$$

Accordingly, the series solution will be of the form

$$y(t) = \sum_{n=0}^{\infty} \frac{y^n(t_0)}{n!}(t-t_0)^n \qquad (2.5)$$

Then differentiating the series (2.3) with respect to t

$$\sum_{n=1}^{\infty} a_n (t-t_0)^{n-1} = f\left(\sum_{n=0}^{\infty} \frac{y^n(t_0)}{n!}(t-t_0)^n, t\right) \qquad (2.6)$$

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Then equating the coefficients leads to the following recurrence relation for a_n

$$a_0 = \alpha$$
$$a_n = \frac{d^{n-1} f(y,t)}{(n-1)!}\Big|_{t=t_0} \tag{2.7}$$

Then using any available software, such as Mathematica, one can easily compute the different terms $a_n; \quad n = 1,2,3,...$

3. Application to Lorenz system

The analysis presented in this paper is based upon the extended Lorenz system which was derived in [6]

$$\frac{dx_1}{dt} = \sigma(y_1 - x_1) = f_1(x_1,x_2,y_1,y_2,z,t)$$
$$\frac{dx_2}{dt} = -\sigma(y_2 + x_2) = f_2(x_1,x_2,y_1,y_2,z,t) \tag{3.1}$$
$$\frac{dy_1}{dt} = Rx_1 - y_1 - x_1 z = f_3(x_1,x_2,y_1,y_2,z,t)$$
$$\frac{dy_2}{dt} = -Rx_2 - y_2 + x_2 z = f_4(x_1,x_2,y_1,y_2,z,t)$$
$$\frac{dz}{dt} = x_1 y_1 - x_2 y_2 + bz = f_5(x_1,x_2,y_1,y_2,z,t)$$

Subject to the initial conditions

$$x_1(t_0) = x_{10},\quad x_2(t_0) = x_{20},\quad y_1(t_0) = y_{10}, \tag{3.2}$$
$$y_2(t_0) = y_{20},\quad z(t_0) = z_0$$

The variables x_1, y_1 and z are respectively proportional to the convective velocity, the temperature difference between descending and ascending flows, and the mean convective heat flow used to appear in the standard Lorenz system, and σ, b and the so-called bifurcation parameter R are real constants. Throughout this paper, we set $\sigma = 10$, $b = -8/3$ and vary the parameter R. It is well-known that chaos sets in around the critical parameter value $R = 24.75$, [6] and [18]. Thus for the purpose of comparison, we shall consider two cases: $R = 20.5$ where the system is non-chaotic (in fact, it is in the state of transitional chaos) and $R = 23.5$ where the system exhibits chaotic behavior.
To apply the Taylor series method to solve the above system, we write the solution in the form of Taylor series as follows:

$$x_1(t) = \sum_{n=0}^{\infty} \frac{x_1^{\,n}(t_0)}{n!}(t-t_0)^n = a_0 + a_1(t-t_0) + a_2(t-t_0)^2 + ...$$
$$x_2(t) = \sum_{n=0}^{\infty} \frac{x_2^{\,n}(t_0)}{n!}(t-t_0)^n = \bar{a}_0 + \bar{a}_1(t-t_0) + \bar{a}_2(t-t_0)^2 + ...$$
$$y_1(t) = \sum_{n=0}^{\infty} \frac{y_1^{\,n}(t_0)}{n!}(t-t_0)^n = b_0 + b_1(t-t_0) + b_2(t-t_0)^2 + ... \tag{3.3}$$
$$y_2(t) = \sum_{n=0}^{\infty} \frac{y_2^{\,n}(t_0)}{n!}(t-t_0)^n = \bar{b}_0 + \bar{b}_1(t-t_0) + \bar{b}_2(t-t_0)^2 + ...$$
$$z(t) = \sum_{n=0}^{\infty} \frac{z^n(t_0)}{n!}(t-t_0)^n = c_0 + c_1(t-t_0) + c_2(t-t_0)^2 + ...$$

Accordingly, the coefficients $a_n, b_n, \bar{a}_n, \bar{b}_n$ and c_n can be calculated as follows:

$$a_0 = x_{10}$$
$$a_{n+1} = \frac{1}{(n+1)!}\frac{d^n(f_1(x_1,x_2,y_1,y_2,z,t)}{dt^n}, \quad n = 0,1,2,3.....$$
$$\bar{a}_0 = x_{20}$$
$$\bar{a}_{n+1} = \frac{1}{(n+1)!}\frac{d^n(f_2(x_1,x_2,y_1,y_2,z,t,t)}{dt^n}, \quad n = 0,1,2,3$$

$$b_0 = y_{10}$$
$$b_{n+1} = \frac{1}{(n+1)!}\frac{d^n(f_3(x_1,x_2,y_1,y_2,z,t)}{dt^n}, \quad n = 0,1,2,3..... \tag{3.4}$$
$$\bar{b}_0 = y_{20}$$
$$\bar{b}_{n+1} = \frac{1}{(n+1)!}\frac{d^n(f_4(x_1,x_2,y_1,y_2,z,t,t)}{dt^n}, \quad n = 0,1,2,3$$
$$c_0 = z_0$$
$$c_{n+1} = \frac{1}{(n+1)!}\frac{d^n(f_2(x,y,z,t)}{dt^n}, \quad n = 0,1,2,3.....$$

One of the drawbacks of this method is that the solution thus obtained does not converge for large values of t. To overcome this problem we solve the system (3.5) over the subintervals $[t_0,t_1],[t_1,t_2],[t_2,t_3],....,[t_{n-1},t_n = t]$. Accordingly, the initial values x_0, y_0, z_0 will be changed for each subinterval.

4. Discussion of the results

The Taylor series algorithm is coded in the computer package Mathematica and The values of the parameters are taken to be $\sigma = 10$, $b = -8/3$ and take the initial conditions $x(0) = 0$, $y(0) = 1$ and $z(0) = 0$. The time range studied in this work is [0, 80]. In addition to the case $R = 23.5$ which corresponds to a chaotic Lorenz system, we also consider the case $R = 20.5$, corresponding to a non-chaotic system, in our attempt to demonstrate the accuracy of the method for the solutions of both non-chaotic and chaotic systems.

4.1. Non-chaotic solutions

First we consider the case $R = 20.5$ which corresponds to non-chaotic case. The accuracy of the Taylor series method was tested by comparing the results with the results of the Runge-Kutta method of order 4 using the time step $\Delta t = 0.0002$. We choose this time step since a smaller one is computationally costly, and increasing the number of terms in the series solutions improves the accuracy of the solutions, but at the expense of increased computational efforts. The 5-term Taylor series solutions on the slightly larger time step $\Delta t = 0.0002$ Match the Runge-Kutta solutions to at least 5 decimal places. Obviously, further improvement can be made on the accuracy of the 5-term series solutions by taking a smaller time step. Figure 1 represents the time series solution of x(t), y(t) and z(t) for $t \in [0,20]$

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for the series results The x–y, x–z, y–z and x–y–z phase portraits obtained on $\Delta t = 0.025$ are also shown in Figure 2.

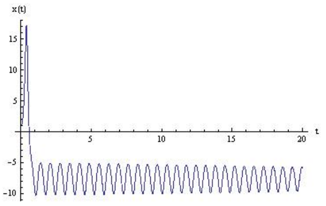

Figure (1a)

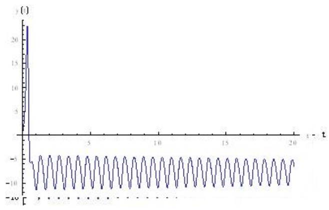

Figure (1b)

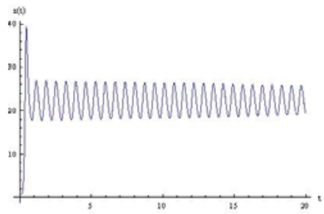

Figure 1(a).

Figure 1 Time series of the solutions using 5-term Taylor series (a) x(t), (b) y(t) and (c) z(t) for R=20.5

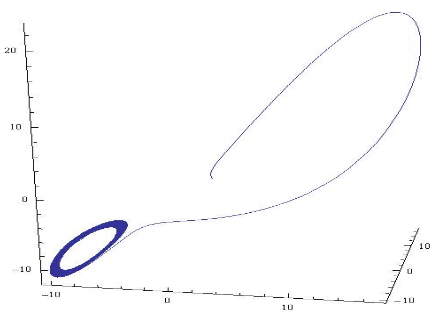

Figure 2. Phase portraits using 5-term Taylor series for $\Delta t = 0.0001$ and $R = 20.5$.

4.2. Chaotic solutions

The system of equations Eq. 3.4 with $R = 23.5$ and the other parameters as given above exhibits chaotic solutions, and so we should expect solutions which are highly sensitive to time step. As expected, the solutions of the chaotic system become less accurate as time progresses. So based on these observations we choose the RK4 solutions on the time step $\Delta t = 0.025$ as the benchmark for our comparison purposes.

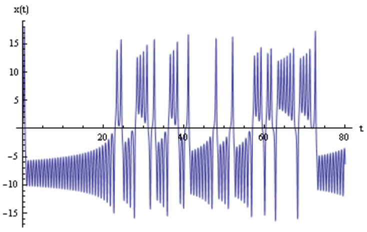

Figure (3a)

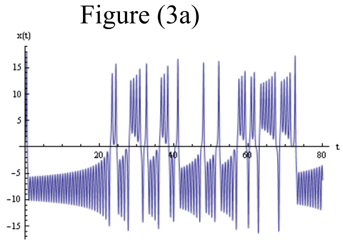

Figure (3b)

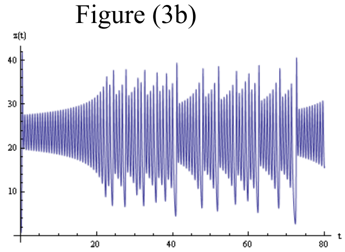

Figure (3.c)

Figure3 Time series of the solution using 5-term series solutions for $\Delta t = 0.0002$ and 400000 points are used. $R = 23.5$. (a) x(t), (b) y(t), and (c) z(t)

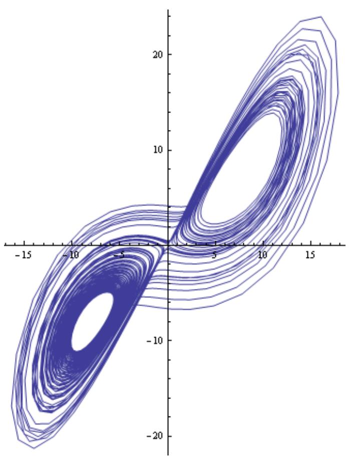

Figure 4. Phase portraits for x(t) and y(t) using 5-term series solution for $\Delta t = 0.0002$ and $R = 23.5$.

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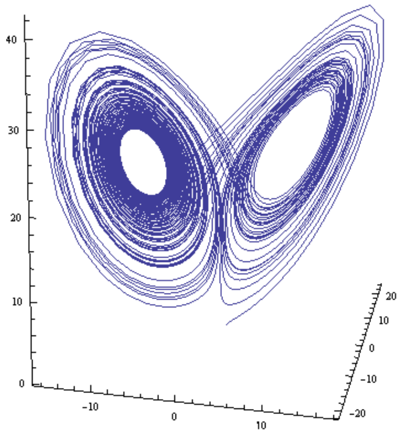

Figure 5: Phase plot of x(t), y(t0 and z(t) for the chaotic case.

In Figure 3 we plot the 5-term series solutions on $\Delta t = 0.0002$. In Figure 5 we reproduce the well-known x–y, x–z, y–z and x–y–z phase portraits of the chaotic Lorenz system using the 5-term series solutions and $\Delta t = 0.0002$. The results presented here indicate that the Taylor series method is very efficient in deriving an approximation to the analytic solution of the Lorenz system for the two cases considered.

5. Conclusion

In this paper, the Taylor series method was employed to solve the Lorenz system. The method was tested for 8 and on the two cases considered. The first case considered was the case when $R = 20.5$ which corresponds to non-chaotic case, and the chaotic case corresponds to $R = 23.5$ was also considered.

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