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On the Chaoticity of a Class of Tent-like Interval Maps

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Abstract—**We show that a class of tent-like maps on the interval are topologically conjugate to the tent map and so are as chaotic as the tent map.**

Keywords: invariant, scrambled sets, transitive maps, transitive points

1 Introduction

Let I be a compact interval in the real line and let $f : I \longrightarrow I$ be a continuous map. It is well known [**1, 4**] that if f has a periodic point of least period not a power of 2, then there exist a number $\delta > 0$ and an uncountable subset S of I (called a δ-scrambled set of f) such that, for any $x \ne y$ in S, we have

$$\limsup_{n\to\infty} |f^n(x) - f^n(y)| \ge \delta \ \text{ and } \ \liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0.$$

In the theory of chaotic dynamical systems, the tent map $T(x) = 1 - |2x - 1|$, $0 \le x \le 1$ is often taken as an example to demonstrate its chaotic dynamics. In [**2**], we mention without proof that the tent map T has the property that, for any given countably infinite subset X of $[0, 1]$, T has a dense uncountable invariant 1-scrambled set Y of transitive points (a transitive point is a point with dense orbit) in $[0, 1]$ such that, for any points x in X and y in Y, we have

$$\limsup_{n\to\infty} |T^n(x) - T^n(y)| \ge \frac{1}{2} \ \text{ and } \ \liminf_{n\to\infty} |T^n(x) - T^n(y)| = 0.$$

In this note, we present a proof of this fact and extend the result to a class of tent-like maps.

2 The chaoticity of the tent map $T(x)$

Let $T(x) = 1 - |2x - 1|$ be the tent map on $I = [0, 1]$. Let $I(0) = [0, 1/2]$ and $I(1) = [1/2, 1]$. For $\alpha_i = 0$ or 1, let $I(\alpha_0\alpha_1 \cdots \alpha_n)$ denote a closed subinterval of $I(\alpha_0\alpha_1 \cdots \alpha_{n-1})$ of minimum length ([**1, 3**]) such that $T(I(\alpha_0\alpha_1 \cdots \alpha_n)) = I(\alpha_1\alpha_2 \cdots \alpha_n)$. Then, T maps the endpoints of $I(\alpha_0\alpha_1 \cdots \alpha_n)$ onto those of $I(\alpha_1\alpha_2 \cdots \alpha_n)$ and maps the interior of $I(\alpha_0\alpha_1 \cdots \alpha_n)$ onto the interior of $I(\alpha_1\alpha_2 \cdots \alpha_n)$ and the length of each $I(\alpha_0\alpha_1 \cdots \alpha_{n-1})$ is $1/2^n$. Let $\Sigma_2 = \{\alpha : \alpha = \alpha_0\alpha_1\alpha_2 \cdots$, where $\alpha_i = 0$ or $1\}$ be the compact metric space with metric d defined by $d(\alpha_0\alpha_1 \cdots, \beta_0\beta_1 \cdots) = \sum_{i=0}^{\infty} |\alpha_i - \beta_i|/2^{i+1}$ and let σ be the shift map on Σ_2 defined by $\sigma(\alpha_0\alpha_1\alpha_2 \cdots) = \alpha_1\alpha_2 \cdots$. For any $\alpha = \alpha_0\alpha_1\alpha_2 \cdots$ in Σ_2, let

$$I(\alpha) = \bigcap_{n=0}^{\infty} I(\alpha_0\alpha_1 \cdots \alpha_n).$$

Then it is easy to see that each $I(\alpha)\,(\subset I(\alpha_0))$ consists of one point, say $I(\alpha) = \{x_\alpha\}$, and

$$T(I(\alpha)) = I(\sigma\alpha).$$

Furthermore, it is also easy to see that if $< \alpha(n) >$ is a sequence of points in Σ_2 which converges to α, then $< T(x_{\alpha(n)}) >$ converges to $T(x_\alpha)$ in I. Now let $\bar{0} \in \Sigma_2$ denote the sequence consisting of all 0's. Then it is clear that $I(\bar{0}) = \{0\}$. Since $T(I(1\bar{0})) = I(\bar{0}) = \{\,0\,\}$ and since the point 1 is the only point in $I(1) = [1/2, 1]$ mapping to 0, we obtain that $I(1\bar{0}) = \{\,1\,\}$ (see [**3**, Proposition 20] for a more general case). These facts will be needed in the proof of Theorem 1 below.

Let $m \ge 5$ be a *fixed* integer and let $\alpha = \alpha_0\alpha_1\alpha_2 \cdots$ be a *fixed* transitive point in Σ_2. Then it is clear that the unique point x_α in $I(\alpha)$ is a transitive point in I. Let

$$X = \{x_1, x_2, \cdots, x_n, \cdots\}$$

be any given countably infinite subset of I. For each integer $n \ge 1$, there is a (not necessarily unique) element $\beta_{n,0}\beta_{n,1}\beta_{n.2} \cdots$ in Σ_2 such that $\{x_n\} = I(x_{\beta_{n,0}\beta_{n,1}\beta_{n.2}\cdots})$.

For simplicity, let $0^1 = 0, 0^2 = 00, 0^3 = 000, (01)^2 = 0101, (0011)^3 = 0011\,0011\,0011$, and so on.

For any integers $0 \le i < j$ and $n \ge 1$, let

$$C(x_n, i:j) \ = \beta_{n,i}\beta_{n,i+1} \cdots \beta_{n,j-1}\, 0$$

and

$$C^*(x_n, i:j) = \begin{cases} 10^{j-i}, & \text{if } \beta_{n,i} = 0, \\ 0^{j-i+1}, & \text{if } \beta_{n,i} = 1. \end{cases}$$

For any element $\gamma = \gamma_0\gamma_1\gamma_2 \cdots$ in Σ_2, we define a new element in Σ_2 by putting $\tau_\gamma = (\tau_\gamma)_0(\tau_\gamma)_1(\tau_\gamma)_2 \cdots =$ $\alpha_0\alpha_1 \cdots \alpha_{m!-2}0\ A_\gamma((m+1)!)\ A_\gamma((m+2)!)\ A_\gamma((m+3)!) \cdots$, where $A_\gamma(k!) = (\tau_\gamma)_{k!}(\tau_\gamma)_{k!+1}(\tau_\gamma)_{k!+2} \cdots (\tau_\gamma)_{(k+1)!-1} =$

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$\alpha_0\alpha_1\alpha_2\cdots\alpha_{k!-2}\,0$

$\gamma_0(0)^{(k-1)!-1}\,\gamma_1(0)^{(k-1)!-1}\,\cdots\,\gamma_{k-1}(0)^{(k-1)!-1}$

$1(0)^{k!-1}$

$B(x_1, 4k!) \quad B(x_2, 5k!) \quad \cdots \quad B(x_{k-3}, k\cdot k!),$

where $B(x_i,(3+i)k!)$ is a finite sequence of 0's and 1's of length $k!$ such that $B(x_i,(3+i)k!) =$

$C(x_i,(3+i)k! : (3+i)k! + [\frac{1}{2}(k-1)!-1])$

$\cdots$

$C(x_i,(3+i)k! + (j-1)[\frac{1}{2}(k-1)!-1] : (3+i)k! + j[\frac{1}{2}(k-1)!-1])$

$\cdots$

$C(x_i,(3+i)k! + (k-1)[\frac{1}{2}(k-1)!-1] : (3+i)k! + k[\frac{1}{2}(k-1)!-1])$

$C^*(x_i,(3+i)k! + \frac{1}{2}k! : (3+i)k! + \frac{1}{2}k! + [\frac{1}{2}(k-1)!-1])$

$\cdots$

$C^*(x_i,(3+i)k! + \frac{1}{2}k! + (j-1)[\frac{1}{2}(k-1)!-1] : (3+i)k! + \frac{1}{2}k! + j[\frac{1}{2}(k-1)!-1])$

$\cdots$

$C^*(x_i,(3+i)k! + \frac{1}{2}k! + (k-1)[\frac{1}{2}(k-1)!-1] : (3+i)k! + \frac{1}{2}k! + k[\frac{1}{2}(k-1)!-1]).$

Let $Y = \{T^n(x_{\tau_\gamma}) : \{x_{\tau_\gamma}\} = I(\tau_\gamma), \gamma\in\Sigma_2, n = 0,\,1,\,2,\,\cdots\}$. Then it is easy to check that the following result holds.

Theorem 1. *Let $T(x) = 1-|2x-1|$ be the tent map defined on $[0,1]$. Then for any given countably infinite subset X of $[0,1]$, there exists a dense invariant uncountable 1-scrambled set Y of transitive points in $[0,1]$ such that, for any $x\in X$ and any $y\in Y$, we have*

$$\limsup_{n\to\infty}|T^n(x)-T^n(y)| \ge \frac{1}{2} \ \text{ and } \ \liminf_{n\to\infty}|T^n(x)-T^n(y)| = 0.$$

3 The topologically conjugate class of tent-like maps

Now assume that, for some point $0<a<1$, $f\,(=f_a)$ is a continuous map from $[0,1]$ onto itself such that (i) $f(0)=0=f(1)$ and $f(a)=1$ and (ii) f is strictly increasing on $[0,a]$ and strictly decreasing on $[a,1]$. Note that the tent map T defined in section 2 is just a special case of f. We first show that if f satisfies (a) f has a dense orbit; or (b) f has dense periodic points; or (c) f has sensitive dependence on initial conditions (i.e., there exists a positive number δ such that for any point x in $[0,1]$ and any open neighborhood V of x there exist a point y in V and a positive integer n such that $|f^n(x)-f^n(y)|\ge\delta$), then $x<f(x)<1$ for all $0<x<a$.

Suppose there is a fixed point $0<v<a$. If $f(x)>x$ for some $0<x<v$, let u be the smallest fixed point of f in $[x,v]$. Since f is strictly increasing on $[x,u]$, every point in $[x,u]$ is attracted to the fixed point u. So, f cannot satisfy any one of (a), (b) and (c). If $f(x)<x$ for some $0<x<v$, let w be the largest fixed point of f in $[0,x]$. Since f is strictly increasing on $[w,x]$, every point in $[w,x]$ is attracted to the fixed point w. So, f cannot satisfy any one of (a), (b) and (c). Therefore, if f satisfies any one of (a), (b) and (c), then f is strictly increasing and $x<f(x)<1$ on $(0,a)$, and f is strictly decreasing and $0<f(x)<1$ on $(a,1)$.

Let $I(0)=[0,a]$ and $I(1)=[a,1]$. For $\alpha_i=0$ or 1, let $I(\alpha_0\alpha_1\cdots\alpha_n)$ be any closed subinterval of $I(\alpha_0\alpha_1\cdots\alpha_{n-1})$ of minimum length **[1, 3]** such that $f(I(\alpha_0\alpha_1\cdots\alpha_n)) = I(\alpha_1\alpha_2\cdots\alpha_n)$. Hence, f maps the endpoints of $I(\alpha_0\alpha_1\cdots\alpha_n)$ onto those of $I(\alpha_1\alpha_2\cdots\alpha_n)$ and maps the interior of $I(\alpha_0\alpha_1\cdots\alpha_n)$ onto the interior of $I(\alpha_1\alpha_2\cdots\alpha_n)$. Consequently,

$$\begin{aligned} a &\notin \cup_{i=0}^n \operatorname{int}(f^i(I(\alpha_0\alpha_1\cdots\alpha_n))) \\ &= \cup_{i=0}^n \operatorname{int}(I(\sigma^i(\alpha_0\alpha_1\cdots\alpha_n))), \end{aligned}$$

where int(J) denotes the interior of the interval J. For any $\alpha=\alpha_0\alpha_1\cdots$ in Σ_2, let $I(\alpha)=\bigcap_{n=0}^\infty I(\alpha_0\alpha_1\cdots\alpha_n)$. Then $f(I(\alpha)) = I(\sigma\alpha)$ and each $I(\alpha)\,(\subset I(\alpha_0))$ is either a nondegenerate compact interval or consists of one point **[3]**. Furthermore, if $I(\alpha)$ is a nondegenerate compact interval then $a\notin\cup_{i\ge0}\operatorname{int}(f^i(I(\alpha)))$, f maps the endpoints of $I(\alpha)$ onto the endpoints of $f(I(\alpha))$ and the interior of $I(\alpha)$ onto the interior of $f(I(\alpha))$. Note that it is shown in [**3**, Propositions 20 & 21] that $I(\bar{0})=\{0\}$ and $I(1\bar{0})=\{1\}$ and, if $I(\alpha)\cap I(\beta)\ne\emptyset$ for some $\alpha\ne\beta$ in Σ_2, then for some point p in $[0,1]$ and some $k\ge0$ and $\gamma_i=0$ or 1, $0\le i\le k-1$, we have

$$\{\alpha,\beta\} = \{\gamma_0\gamma_1\cdots\gamma_{k-1}01\bar{0},\ \gamma_0\gamma_1\cdots\gamma_{k-1}11\bar{0}\},$$

$$I(\alpha)=I(\beta)=\{p\} \quad\text{and}\quad f^k(p)=a.$$

Conversely, if

$$\{\alpha,\beta\} = \{\gamma_0\gamma_1\cdots\gamma_{k-1}01\bar{0},\ \gamma_0\gamma_1\cdots\gamma_{k-1}11\bar{0}\},$$

then $I(\alpha)=I(\beta)=\{p\}$ for some point p and $f^k(p)=a$. These facts will be needed later.

Assume that $I(\alpha)=\bigcap_{n=0}^\infty I(\alpha_0\alpha_1\cdots\alpha_n)$ is a nondegenerate interval. Then so is $f^i(I(\alpha))$ for every $i\ge0$ since f is not constant on any interval. Since $a\notin\cup_{i\ge0}\operatorname{int}(f^i(I(\alpha)))$, f^i is strictly monotonic on $I(\alpha)$ for every $i\ge1$. Assume that f satisfies any one of (a), (b) and (c), and assume that, for some integer $m\ge1$, $f^m(I(\alpha))$ is a nondegenerate interval such that $\operatorname{int}(f^m(I(\alpha)))\cap$

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$\mathrm{int}(I(\alpha)) \neq \emptyset$ (this happens when f satisfies (a) or (b)). Then $f^m(I(\alpha)) = I(\alpha)$ and f^m maps the endpoints of $I(\alpha)$ onto itself and f^m is monotonic on $I(\alpha)$. By resorting to f^{2m} if necessary, we may assume that f^m is increasing on $I(\alpha)$ and fixes both endpoints of $I(\alpha)$. But then, one endpoint of $I(\alpha)$ which is a fixed point of f^m attracts all points of $\mathrm{int}(I(\alpha))$ (under f^m) which clearly contradicts the assumption that f satisfies any one of (a), (b) and (c). If f satisfies (c) and $f^i(I(\alpha))$ and $f^j(I(\alpha))$ have disjoint interiors whenever $i \neq j$, then since the interval $[0,1]$ has finite length, we must have $\lim_{n\to\infty}$ diameter$(f^n(I(\alpha))) = 0$ which contradicts the assumption that f has sensitivity. This shows that if f satisfies any one of (a), (b) and (c), then $I(\alpha)$ consists of exactly one point for every α in Σ_2 and every point of $[0,1]$ belongs to $I(\alpha)$ for some (not necessarily unique) α in Σ_2.

In the following, assume that f satisfies any one of (a), (b) and (c). For the sake of clarity, we write $I_f(\alpha)$ instead of $I(\alpha)$ to emphasize the role of f. For every x in $[0,1]$, there is an α in Σ_2 such that $I_f(\alpha) = \{x\}$. If there is another $\beta \neq \alpha$ in Σ_2 such that $I_f(\beta) = \{x\}$, then it follows from the above that, for some $k \geq 0$, $\{\alpha, \beta\} = \{\gamma_0\gamma_1\cdots\gamma_k 01\bar{0}, \gamma_0\gamma_1\cdots\gamma_k 11\bar{0}\}$. But then $I_T(\alpha) = I_T(\beta) = \{w\}$ for some w with $T^k(w) = 1/2$. So, the map $\psi : [0,1] \to [0,1]$ defined by letting $\psi(I_f(\alpha)) = I_T(\alpha)$ is well-defined (cf. [**3**, Theorem 22]). It is easy to see that ψ is a homeomorphism such that $\psi(0) = 0, \psi(a) = 1/2, \psi(1) = 1$ and $(\psi f)(I_f(\alpha)) = \psi(f(I_f(\alpha)))$

$$\psi(I_f(\sigma\alpha)) = I_T(\sigma\alpha) = TI_T(\alpha) = (T\psi)(I_f(\alpha)).$$

Therefore, f is topologically conjugate to T through ψ. This, together with Theorem 1 above, implies the following result.

Theorem 2. *Let $0 < a < 1$ and let f be a continuous map from $[0,1]$ onto itself such that (i) $f(0) = 0 = f(1)$ and $f(a) = 1$ and (ii) f is strictly increasing on $[0,a]$ and strictly decreasing on $[a,1]$. Then the following statements are equivalent:*

(a) *f has a dense orbit.*

(b) *f has dense periodic points.*

(c) *f has sensitive dependence on initial conditions.*

Furthermore, if f has a dense orbit, then f is topologically conjugate to the tent map $T(x) = 1 - |2x-1|$ on $[0,1]$ and, for any countably infinite subset X of $[0,1]$, f has a dense uncountable invariant 1-scrambled set Y of transitive points in $[0,1]$ such that, for any $x \in X$ and $y \in Y$, we have

$$\limsup_{n\to\infty} |f^n(x) - f^n(y)| \geq \min\{a, 1-a\}$$

and

$$\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0.$$

References

[1] Block, L.S., Coppel, W.A., *Dynamics in One Dimension,* Lecture Notes in Math., no. 1513, Springer-Verlag, 1992.

[2] Du, B.-S., “On the nature of chaos,” arXiv:math/0602585 (2006).

[3] Evens, M.J., Humke, P.D., Lee, C.-M., O’malley, R.J., “Characterizations of turbulent one-dimensional mappings via ω-limit sets,” *Trans. Amer. Math. Soc.,* V326, N1, 261-280, 7/91; Corrigendum: *Trans. Amer. Math. Soc.,* V333, N2, pp. 939-940, 10/92.

[4] Li, T.-Y., Yorke, J.A., “Period three implies chaos,” *Amer. Math. Monthly,* V82, N10, pp. 985-992, 12/75.

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