Periodic Solution of a Non-autonomous Neutral Delay Two-species Competitive System

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***Abstract*—The purpose of this paper is to investigate the existence of periodic solution of a general neutral delay two-species competitive non-autonomous system. With the help of the continuation theorem for composite coincidence degree and some techniques, a set of sufficient conditions are derived for the existence of at least one strictly positive periodic solution. Furthermore, some numerical simulations demonstrate our results.**

***Index Terms*—Periodic solution, Neutral delay, Continuation theorem, Composite coincidence degree.**

I. Introduction

TIME delay arises naturally in connection with system process and information flow for different part of dynamic systems. Practical systems with time delays now occupy a place of central importance in all areas of science, which have been received great interest and attention by many scholars, e.g. [1]–[3].

A neutral time-delay system contains time delays both in its state, and in its derivatives of state. Such system can be applied to many fields, such as population ecology [1], distributed networks containing lossless transmission lines [2], heat exchangers [3], robots in contact with rigid environments [4], and so on. Due to its wider application, neutral systems with constant or varying time delay have been of considerable interest by many authors for decades [5]–[11].

The well-known periodic single-species population growth models with periodic delay can be written as: $y'(t) = y(t)[r(t) - a(t)y(t) - b(t)y(t-\tau(t))]$, which was first proposed by Freedman and Wu in [12]. Furthermore, Liu established two corresponding periodic Lotka-Volterra competitive systems involving multiple delays in [13]:

$$\begin{cases} y_1'(t) = y_1(t)\left[r_1(t) - a_1(t)y_1(t) \right.\\ \qquad \left. - \sum\limits_{i=1}^{n} b_{1i}(t)y_1(t-\tau_i(t)) - \sum\limits_{j=1}^{m} c_{1j}(t)y_2(t-\rho_j(t))\right], \\ y_2'(t) = y_2(t)\left[r_2(t) - a_2(t)y_2(t) \right.\\ \qquad \left. - \sum\limits_{j=1}^{m} b_{2j}(t)y_2(t-\eta_j(t)) - \sum\limits_{i=1}^{n} c_{2i}(t)y_1(t-\sigma_i(t))\right], \end{cases} \tag{1}$$

where a_1, a_2, b_{1i}, b_{2j}, c_{1j}, $c_{2i} \in C(\mathbb{R}, [0, +\infty))$, τ_i, ρ_j, η_j, $\sigma_i \in C^1(\mathbb{R}, [0, +\infty))$ are ω-periodic functions. Here, the intrinsic growth rates $r_k(t) \in C(\mathbb{R}, \mathbb{R})$ are ω-periodic functions with $\int_0^\omega r_k(t)dt > 0$, $k = 1, 2$.

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In this paper, we consider periodic solution of the following two-species competition system with general periodic neutral delay:

$$\begin{cases} y_1'(t) = y_1(t)\left[r_1(t) - a_1(t)y_1(t) \right.\\ \qquad - \sum\limits_{i=1}^{n} b_{1i}(t)y_1(t-\tau_i(t)) - \sum\limits_{j=1}^{m} c_{1j}(t)y_2(t-\rho_j(t)) \\ \qquad \left. - e_1(t)y_1'(t-\delta_1(t))\right], \\ y_2'(t) = y_2(t)\left[r_2(t) - a_2(t)y_2(t) \right.\\ \qquad - \sum\limits_{j=1}^{m} b_{2j}(t)y_2(t-\eta_j(t)) - \sum\limits_{i=1}^{n} c_{2i}(t)y_1(t-\sigma_i(t)) \\ \qquad \left. - e_2(t)y_2'(t-\delta_2(t))\right]. \end{cases} \tag{2}$$

where $e_k(t) \in C^1(\mathbb{R}, [0, +\infty))$, $\delta_k(t) \in C^2(\mathbb{R}, [0, +\infty))$ $(k = 1, 2)$ are ω-periodic functions, other parameters are the ω-periodic functions as in (1).

The present paper is organized as follows: In the next section we introduce some notations and an important existence theorem developed in [9], [14]. By applying this theorem and some other techniques, we study the existence of positive periodic solutions of system (2) in Section 3. In Section 4, an illustrative example is given to demonstrate the effectiveness of the main result.

II. An existence lemma and notations

In this section, we shall summarize a few concepts and results from [9] and state an existence theorem.

For a fixed $\tau \geq 0$, let $\mathcal{C} := C([-\tau, 0]; \mathbb{R}^n)$. If $x \in C([\sigma - \tau, \sigma + \delta]; \mathbb{R}^n)$, for some $\delta > 0$ and $\sigma \in \mathbb{R}$, then $x_t \in \mathcal{C}$ for $t \in [\sigma, \sigma + \delta]$ is defined by $x_t(\theta) = x(t + \theta)$ for $\theta \in [-\tau, 0]$. The supremum norm in $\mathcal{C}$ is denoted by $\|\cdot\|$, that is, $\|\varphi\| = \max_{\theta\in[-\tau,0]} |\varphi(\theta)|$ for $\varphi \in \mathcal{C}$, where $|\cdot|$ denotes the norm in $\mathbb{R}^n$, and $|u| = \sum_{i=1}^{n} |u_i|$ for $u = (u_1, \cdots, u_n) \in \mathbb{R}^n$.

Consider the following neutral functional differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t) - b(t, x_t)] = f(t, x_t), \tag{3}$$

where $f : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$ is completely continuous and $b : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$ is continuous. Moreover, we assume:

(H_1) There exists $\omega > 0$ such that for every $(t, \varphi) \in \mathbb{R} \times \mathcal{C}$, we have $b(t + \omega, \varphi) = b(t, \varphi)$ and $f(t + \omega, \varphi) = f(t, \varphi)$.

(H_2) There exists a constant $k < 1$ such that $|b(t, \varphi) - b(t, \psi)| \leq k\|\varphi - \psi\|$ for $t \in \mathbb{R}$ and $\varphi, \psi \in \mathcal{C}$.

Lemma 1: ([14]). Suppose that there exists a constant $M > 0$ such that:

1) for any $\lambda \in (0, 1)$ and any ω-periodic solution x of the system

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t) - \lambda b(t, x_t)] = \lambda f(t, x_t), \tag{4}$$

we have $|x(t)| < M$ for $t \in \mathbb{R}$;

2) $g(u) := \int_0^\omega f(s,\widehat{u})\mathrm{d}s \neq 0$ for $u \in \partial B_M(\mathbb{R}^n)$, where $B_M(\mathbb{R}^n) = \{u \in \mathbb{R}^n : |u| < M\}$, and $\widehat{u}$ denotes the constant mapping from $[-\tau, 0]$ to $\mathbb{R}^n$ with the value $u \in \mathbb{R}^n$;
3) $\deg(g, B_M(\mathbb{R}^n)) \neq 0$.

Then there exists at least one ω-periodic solution of the system

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t) - b(t, x_t)] = f(t, x_t), \tag{5}$$

that satisfies $\sup_{t\in\mathbb{R}}|x(t)| < M$.

The following remark is introduced by Fang (see Remark 1 in [15]).

Remark 1: ([15]). Lemma 1 still remains valid if the assumption (H_2) is replaced by

(H_2') there exists a constant $k < 1$ such that $|b(t,\varphi) - b(t,\psi)| \leq k\|\varphi - \psi\|$ for $t \in \mathbb{R}$ and $\varphi, \psi \in \{\varphi \in \mathcal{C} : \|\varphi\| < M\}$ with M as given in condition (1) of Lemma 1.

We will also need the following results.

Lemma 2: ([16]). Suppose $\varrho \in \mathcal{C}_\omega^1 = \{h : h \in C^1(\mathbb{R},\mathbb{R}), h(t+\omega) \equiv h(t)\}$ and $\varrho'(t) < 1$, $\forall\, t \in [0,\omega]$. Then the function $t - \varrho(t)$ has a unique inverse $v(t)$ satisfying $v \in C(\mathbb{R},\mathbb{R})$ with $v(a+\omega) = v(a) + \omega, \forall a \in \mathbb{R}$.

Remark 2: ([16]). By using Lemma 2, we see that if $g \in \mathcal{C}_\omega^0 = \{h : h \in C(\mathbb{R},\mathbb{R}), h(t+\omega) \equiv h(t)\}$, $\varrho \in \mathcal{C}_\omega^1$ and $\varrho'(t) < 1, \forall t \in [0,\omega]$. Then $g(v(t+\omega)) = g(v(t)+\omega) = g(v(t)), \forall t \in [0,\omega]$, where $v(t)$ is the inverse function of $t - \varrho(t)$, which together with $v \in C(\mathbb{R},\mathbb{R})$, implies that $g(v(t)) \in \mathcal{C}_\omega^0$.
Lemma 2 and Remark 2 can also be found in Lemma 4 of [17].

In the following, we denote

$$\bar{h} = \frac{1}{\omega}\int_0^\omega h(t)\mathrm{d}t, \qquad h_m = \min_{t\in[0,\omega]} h(t),$$
$$|h|_0 = \max_{t\in[0,\omega]}|h(t)|,$$

for a given $h \in \mathcal{C}_\omega^0$.

III. The main result

Theorem 1: Assume that the following conditions are satisfied.

1) The system of algebraic equations

$$\begin{cases} \left(\bar{a}_1 + \sum_{i=1}^n \bar{b}_{1i}\right)\mu_1 + \sum_{j=1}^m \bar{c}_{1j}\mu_2 = \bar{r}_1, \\ \sum_{i=1}^n \bar{c}_{2i}\mu_1 + \left(\bar{a}_2 + \sum_{j=1}^m \bar{b}_{2j}\right)\mu_2 = \bar{r}_2, \end{cases}$$

has a unique positive solution $\mu^* = (\mu_1^*, \mu_2^*)$;

2) $\tau_i'(t) < 1, \quad \rho_j'(t) < 1, \quad \eta_j'(t) < 1,$
$\sigma_i'(t) < 1, \quad \delta_k'(t) < 1, \quad \Gamma_{kl} > 0,$
$\bar{a}_1 + \sum_{i=1}^n \bar{b}_{1i} > 0, \quad \bar{a}_2 + \sum_{j=1}^m \bar{b}_{2j} > 0,$
$\bar{r}_1 > \frac{\bar{r}_2\sum_{j=1}^m \bar{c}_{1j}}{\bar{a}_2+\sum_{j=1}^m \bar{b}_{2j}}, \quad \bar{r}_2 > \frac{\bar{r}_1\sum_{i=1}^n \bar{c}_{2i}}{\bar{a}_1+\sum_{i=1}^n \bar{b}_{1i}},$
$(i = 1,\cdots,n;\ j = 1,\cdots,m;\ k,l = 1,2)$;

3) $k_0 := c\mathrm{e}^{M_0} < 1$.

Then system (2) has at least one positive $\omega-$periodic solution. Here we have

$c = \max\{|d_1|_0 + |d_2|_0, |c_1|_0 + |c_2|_0\}$,
$M_0 = \max\{|\ln\mu_1^*| + |\ln\mu_1^*|, K, \omega H_* + H_1 + H_2\}$,
$K = \max\{K_1, K_2\}$, $\bar{R}_k = \frac{1}{\omega}\int_0^\omega |r_k(t)|\mathrm{d}t$,
$K_k = \ln\frac{\bar{r}_k}{\vartheta_k} + \frac{\bar{r}_k}{\vartheta_k} + (\bar{R}_k + \Gamma_k\bar{r}_k)\omega$
$\Gamma_{11}(s) = a_1(s) - \frac{d_1'(\gamma_1(s))}{1-\delta_1'(\gamma_1(s))} + \sum\limits_{i=1}^n \frac{b_{1i}(u_{1i}(s))}{1-\tau_i'(u_{1i}(s))}$,
$\Gamma_{12}(s) = \sum\limits_{j=1}^m \frac{c_{1j}(v_{1j}(s))}{1-\rho_j'(v_{1j}(s))}$, $\Gamma_{21}(s) = \sum\limits_{i=1}^n \frac{c_{2i}(v_{2i}(s))}{1-\sigma_i'(v_{2i}(s))}$,
$\Gamma_{22}(s) = a_2(s) - \frac{d_2'(\gamma_2(s))}{1-\delta_2'(\gamma_2(s))} + \sum\limits_{j=1}^m \frac{b_{2j}(u_{2j}(s))}{1-\eta_j'(u_{2j}(s))}$,
$d_k(t) = \frac{e_k(t)}{1-\delta_k'(t)}$, $\Gamma_{12}^1 = \Gamma_{12}$, $\Gamma_{21}^1 = \Gamma_{21}$,
$\Gamma_{11}^1 = a_1(s) + \sum\limits_{i=1}^n \frac{b_{1i}(u_{1i}(s))}{1-\tau_i'(u_{1i}(s))} + \frac{|d_1'(\gamma_1(s))|}{1-\delta_1'(\gamma_1(s))}$,
$\Gamma_{22}^1 = a_2(s) + \sum\limits_{j=1}^m \frac{b_{2j}(u_{2j}(s))}{1-\eta_j'(u_{2j}(s))} + \frac{|d_2'(\gamma_2(s))|}{1-\delta_2'(\gamma_2(s))}$,
$P \triangleq \sum\limits_{i=1}^n (|b_{1i}|_0 + |c_{2i}|_0)\mathrm{e}^{K_1}$,
$Q \triangleq \sum\limits_{j=1}^m (|c_{1j}|_0 + |b_{2j}|_0)\mathrm{e}^{K_2}$, $\vartheta_k = \frac{(\Gamma_{kk})_m(1-\delta_k')_m}{(1-\delta_k')_m + |d_k|_0}$,
$H_* = \dfrac{P + Q + \sum_{k=1}^2 |r_k|_0 + \sum_{k=1}^2 |a_k|_0\mathrm{e}^{K_k}}{1 - \sum_{k=1}^2 |e_k|_0\mathrm{e}^{K_k}}$,
$\Gamma_k = \max\left\{\left(\frac{\Gamma_{kl}^1(s)}{\Gamma_{kl}(s)}\right)_0, l = 1,2\right\}$,
$H_1 = \max\left\{\left|\ln\frac{\bar{r}_1}{\bar{a}_1+\sum_{i=1}^n \bar{b}_{1i}}\right|, \left|\ln\frac{\bar{r}_1 - \frac{\bar{r}_2\sum_{j=1}^m \bar{c}_{1j}}{\bar{a}_2+\sum_{j=1}^m \bar{b}_{2j}}}{\bar{a}_1+\sum_{i=1}^n \bar{b}_{1i}}\right|\right\}$,
$H_2 = \max\left\{\left|\ln\frac{\bar{r}_2}{\bar{a}_2+\sum_{j=1}^m \bar{b}_{2j}}\right|, \left|\ln\frac{\bar{r}_2 - \frac{\bar{r}_1\sum_{i=1}^n \bar{c}_{2i}}{\bar{a}_1+\sum_{i=1}^n \bar{b}_{1i}}}{\bar{a}_2+\sum_{j=1}^m \bar{b}_{2j}}\right|\right\}$.

and $u_{1i}, v_{1j}, \gamma_1, u_{2j}, v_{2i}, \gamma_2$ represent the inverse functions of $t - \tau_i(t) = s$, $t - \rho_j(t) = s$, $t - \delta_1(t) = s$, $t - \eta_j(t) = s$, $t - \sigma_i(t) = s$ and $t - \delta_2(t) = s$, respectively.

To prove the above theorem, we make the change of variables

$$y_i(t) = \mathrm{e}^{x_i(t)}, \qquad i = 1,2. \tag{6}$$

Then the system (2) becomes

$$\begin{cases} x_1'(t) = r_1(t) - a_1(t)\mathrm{e}^{x_1(t)} \\ \quad -\sum\limits_{i=1}^n b_{1i}(t)\mathrm{e}^{x_1(t-\tau_i(t))} - \sum\limits_{j=1}^m c_{1j}(t)\mathrm{e}^{x_2(t-\rho_j(t))} \\ \quad -e_1(t)(1-\delta_1'(t))x_1'(t-\delta_1(t))\mathrm{e}^{x_1(t-\delta_1(t))}, \\ x_2'(t) = r_2(t) - a_2(t)\mathrm{e}^{x_2(t)} \\ \quad -\sum\limits_{j=1}^m b_{2j}(t)\mathrm{e}^{x_2(t-\eta_j(t))} - \sum\limits_{i=1}^n c_{2i}(t)\mathrm{e}^{x_1(t-\sigma_i(t))} \\ \quad -e_2(t)(1-\delta_2'(t))x_2'(t-\delta_2(t))\mathrm{e}^{x_2(t-\delta_2(t))}. \end{cases}$$

In fact, in this case, (2) should be reduced to

$$\begin{cases} x_1'(t) = r_1(t) - a_1(t)\mathrm{e}^{x_1(t)} \\ \quad -\sum\limits_{i=1}^n b_{1i}(t)\mathrm{e}^{x_1(t-\tau_i(t))} - \sum\limits_{j=1}^m c_{1j}(t)\mathrm{e}^{x_2(t-\rho_j(t))} \\ \quad -e_1(t)x_1'(t-\delta_1(t))\mathrm{e}^{x_1(t-\delta_1(t))}, \\ x_2'(t) = r_2(t) - a_2(t)\mathrm{e}^{x_2(t)} \\ \quad -\sum\limits_{j=1}^m b_{2j}(t)\mathrm{e}^{x_2(t-\eta_j(t))} - \sum\limits_{i=1}^n c_{2i}(t)\mathrm{e}^{x_1(t-\sigma_i(t))} \\ \quad -e_2(t)x_2'(t-\delta_2(t))\mathrm{e}^{x_2(t-\delta_2(t))}. \end{cases} \tag{7}$$

Let Θ denote the linear space of real value continuous $\omega-$periodic functions on $\mathbb{R}$. The linear space Θ is a Banach space with the usual norm $\|x\|_0 = \max_{t\in\mathbb{R}}\sum_{i=1}^2 |x_i(t)|$ for a given $x = (x_1, x_2) \in \Theta$.

We define the following maps:

$b : \mathbb{R}\times\mathcal{C} \to \mathbb{R}^2$,
$b(t,\varphi) = (b_1(t,\varphi), b_2(t,\varphi))$,
$b_1(t,\varphi) = -\frac{e_1(t)}{1-\delta_1'(t)}\mathrm{e}^{\varphi_1(-\delta_1(t))}$,
$b_2(t,\varphi) = -\frac{e_2(t)}{1-\delta_2'(t)}\mathrm{e}^{\varphi_2(-\delta_2(t))}$, $f : \mathbb{R}\times\mathcal{C} \to \mathbb{R}^2$,
$f(t,\varphi) = (f_1(t,\varphi), f_2(t,\varphi))$,

$$\begin{aligned}f_1(t,\varphi)&=r_1(t)-a_1(t)\mathrm{e}^{\varphi_1(0)}-\sum_{i=1}^{n}b_{1i}(t)\mathrm{e}^{\varphi_1(-\tau_i(t))}\\&\quad-\sum_{j=1}^{m}c_{1j}(t)\mathrm{e}^{\varphi_2(-\rho_j(t))}+(\tfrac{e_1(t)}{1-\delta_1'(t)})'\mathrm{e}^{\varphi_1(-\delta_1(t))},\\f_2(t,\varphi)&=r_2(t)-a_2(t)\mathrm{e}^{\varphi_2(0)}-\sum_{j=1}^{m}b_{2j}(t)\mathrm{e}^{\varphi_1(-\eta_j(t))}\\&\quad-\sum_{i=1}^{n}c_{2i}(t)\mathrm{e}^{\varphi_1(-\sigma_i(t))}+(\tfrac{e_2(t)}{1-\delta_2'(t)})'\mathrm{e}^{\varphi_2(-\delta_2(t))},\end{aligned}$$

where $\mathcal{C}:=C([-\tau,0];\mathbb{R}^2)$.

Clearly, $b:\mathbb{R}\times\mathcal{C}\to\mathbb{R}^2$ and $f:\mathbb{R}\times\mathcal{C}\to\mathbb{R}^2$ are complete continuation functions and system (7) takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t)-b(t,x_t)]=f(t,x_t). \tag{8}$$

In the proof of our main result below, we will use the following two lemmas.

Lemma 3: If the assumptions of Theorem 1 are satisfied and if $\Omega=\{\varphi\in\mathcal{C}:\|\varphi\|<M\}$, where $M>M_0$ is such that $k=ce^M<1$, then $|b(t,\varphi)-b(t,\psi)|\le k\|\varphi-\psi\|$ for $t\in\mathbb{R}$ and $\varphi,\psi\in\Omega$.

Proof. For $t\in\mathbb{R}$ and $\varphi,\psi\in\Omega$, we get

$$\begin{aligned}|b_i(t,\varphi)-b_i(t,\psi)|&\le d_i(t)|\mathrm{e}^{\varphi_i(-\delta_i(t))}-\mathrm{e}^{\psi_i(-\delta_i(t))}|\\&\le d_i(t)\mathrm{e}^{\theta_i\varphi_i(-\delta_i(t))+(1-\theta_i)\psi_i(-\delta_i(t))}\\&\quad\cdot|\varphi_i(-\delta_i(t))-\psi_i(-\delta_i(t))|,\end{aligned}$$

for some $\theta_i\in(0,1)$, $i=1,2$. Then we have

$$|b_i(t,\varphi)-b_i(t,\psi)|\le|d_i|_0\mathrm{e}^M\|\varphi-\psi\|,\ (i=1,2).$$

Hence,

$$\begin{aligned}|b(t,\varphi)-b(t,\psi)|&\le(|d_1|_0+|d_2|_0)\mathrm{e}^M\|\varphi-\psi\|\\&\le c\mathrm{e}^M\|\varphi-\psi\|=k\|\varphi-\psi\|.\end{aligned}$$

Thus, the proof is complete. □

Lemma 4: Assume that the assumption of theorem 1 are satisfied. Then every solution $x\in\Theta$ of the system

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t)-\lambda b(t,x_t)]=\lambda f(t,x_t),\qquad\lambda\in(0,1)$$

satisfies $\|x\|_0\le M_0$.

Proof. Let $\frac{\mathrm{d}}{\mathrm{d}t}[x(t)-\lambda b(t,x_t)]=\lambda f(t,x_t)$ for $x\in\Theta$, that is,

$$\begin{cases}X_1'(t)=\lambda\big[r_1(t)-a_1(t)\mathrm{e}^{x_1(t)}\\\quad-\sum\limits_{i=1}^{n}b_{1i}(t)\mathrm{e}^{x_1(t-\tau_i(t))}-\sum\limits_{j=1}^{m}c_{1j}(t)\mathrm{e}^{x_2(t-\rho_j(t))}\\\quad+\left(\frac{e_1(t)}{1-\delta_1'(t)}\right)'\mathrm{e}^{x_1(t-\delta_1(t))}\Big],\\X_2'(t)=\lambda\big[r_2(t)-a_2(t)\mathrm{e}^{x_2(t)}\\\quad-\sum\limits_{j=1}^{m}b_{2j}(t)\mathrm{e}^{x_2(t-\eta_j(t))}-\sum\limits_{i=1}^{n}c_{2i}(t)\mathrm{e}^{x_1(t-\sigma_i(t))}\\\quad+\left(\frac{e_2(t)}{1-\delta_2'(t)}\right)'\mathrm{e}^{x_2(t-\delta_2(t))}\Big],\end{cases} \tag{9}$$

where $X_i(t)=x_i(t)+\lambda\frac{e_i(t)}{1-\delta_i'(t)}\mathrm{e}^{x_i(t-\delta_i(t))}$ $(i=1,2)$.

System (9) yields, after integrating from 0 to ω, that

$$\begin{cases}\int_0^\omega\Big[a_1(t)\mathrm{e}^{x_1(t)}+\sum\limits_{i=1}^{n}b_{1i}(t)\mathrm{e}^{x_1(t-\tau_i(t))}\\\quad+\sum\limits_{j=1}^{m}c_{1j}(t)\mathrm{e}^{x_2(t-\rho_j(t))}-d_1'(t)\mathrm{e}^{x_1(t-\delta_1(t))}\Big]\mathrm{d}t\\\quad=\int_0^\omega r_1(t)\mathrm{d}t=\bar{r}_1\omega,\\\int_0^\omega\Big[a_2(t)\mathrm{e}^{x_2(t)}+\sum\limits_{j=1}^{m}b_{2j}(t)\mathrm{e}^{x_2(t-\eta_j(t))}\\\quad+\sum\limits_{i=1}^{n}c_{2i}(t)\mathrm{e}^{x_1(t-\sigma_i(t))}-d_2'(t)\mathrm{e}^{x_2(t-\delta_2(t))}\Big]\mathrm{d}t\\\quad=\int_0^\omega r_2(t)\mathrm{d}t=\bar{r}_2\omega,\end{cases} \tag{10}$$

where $d_k(t)=\frac{e_k(t)}{1-\delta_k'(t)}$, $k=1,2$. Since $\tau_i'(t)<1$, the inverse function $t=u_{1i}(s)$ of $t-\tau_i(t)=s$, $t\in[0,\omega]$, exists. Then we have

$$\int_0^\omega b_{1i}(t)\mathrm{e}^{x_1(t-\tau_i(t))}\mathrm{d}t=\int_{-\tau_i(0)}^{\omega-\tau_i(\omega)}\frac{b_{1i}(u_{1i}(s))}{1-\tau_i'(u_{1i}(s))}\mathrm{e}^{x_1(s)}\mathrm{d}s. \tag{11}$$

According to Remark 2, we have

$$\int_0^\omega b_{1i}(t)\mathrm{e}^{x_1(t-\tau_i(t))}\mathrm{d}t=\int_0^\omega\frac{b_{1i}(u_{1i}(s))}{1-\tau_i'(u_{1i}(s))}\mathrm{e}^{x_1(s)}\mathrm{d}s. \tag{12}$$

Similarly,

$$\begin{cases}\int_0^\omega c_{1j}(t)\mathrm{e}^{x_2(t-\rho_j(t))}\mathrm{d}t=\int_0^\omega\frac{c_{1j}(v_{1j}(s))}{1-\rho_j'(v_{1j}(s))}\mathrm{e}^{x_2(s)}\mathrm{d}s,\\\int_0^\omega d_1'(t)\mathrm{e}^{x_1(t-\delta_1(t))}\mathrm{d}t=\int_0^\omega\frac{d_1'(\gamma_1(s))}{1-\delta_1'(\gamma_1(s))}\mathrm{e}^{x_1(s)}\mathrm{d}s,\\\int_0^\omega b_{2j}(t)\mathrm{e}^{x_2(t-\eta_j(t))}\mathrm{d}t=\int_0^\omega\frac{b_{2j}(u_{2j}(s))}{1-\eta_j'(u_{2j}(s))}\mathrm{e}^{x_2(s)}\mathrm{d}s,\\\int_0^\omega c_{2i}(t)\mathrm{e}^{x_1(t-\sigma_i(t))}\mathrm{d}t=\int_0^\omega\frac{c_{2i}(v_{2i}(s))}{1-\sigma_i'(v_{2i}(s))}\mathrm{e}^{x_1(s)}\mathrm{d}s,\\\int_0^\omega d_2'(t)\mathrm{e}^{x_2(t-\delta_2(t))}\mathrm{d}t=\int_0^\omega\frac{d_2'(\gamma_2(s))}{1-\delta_2'(\gamma_2(s))}\mathrm{e}^{x_2(s)}\mathrm{d}s,\end{cases} \tag{13}$$

where v_{1j}, γ_1, u_{2j}, v_{2i}, γ_2 are the corresponding inverse functions.

So from (10), (12) and (13), we can get

$$\int_0^\omega\sum_{k=1}^{2}\Gamma_{ik}(s)\mathrm{e}^{x_k(s)}\mathrm{d}s=\bar{r}_i\omega,\quad i=1,2 \tag{14}$$

From (9) we have

$$\begin{aligned}&\int_0^\omega\left|\left[x_1(t)+\lambda d_1(t)\mathrm{e}^{x_1(t-\delta_1(t))}\right]'\right|\mathrm{d}t\\&\quad=\lambda\int_0^\omega\Big|r_1(t)-a_1(t)\mathrm{e}^{x_1(t)}\\&\qquad-\sum_{i=1}^{n}b_{1i}(t)\mathrm{e}^{x_1(t-\tau_i(t))}-\sum_{j=1}^{m}c_{1j}(t)\mathrm{e}^{x_2(t-\rho_j(t))}\\&\qquad+d_1'(t)\mathrm{e}^{x_1(t-\delta_1(t))}\Big|\mathrm{d}t\end{aligned}$$

$$\leq \lambda \int_0^\omega |r_1(t)|\mathrm{d}t + \lambda \int_0^\omega \Big[a_1(t)\mathrm{e}^{x_1(t)} + \sum_{i=1}^n b_{1i}(t)\mathrm{e}^{x_1(t-\tau_i(t))} + \sum_{j=1}^m c_{1j}(t)\mathrm{e}^{x_2(t-\rho_j(t))} + |d_1'(t)|\mathrm{e}^{x_1(t-\delta_1(t))}\Big]\,\mathrm{d}t. \quad (15)$$

In view of (10)-(14) and by a similar analysis, we have

$$\begin{aligned}
&\int_0^\omega \Big[a_1(t)\mathrm{e}^{x_1(t)} + \sum_{i=1}^n b_{1i}(t)\mathrm{e}^{x_1(t-\tau_i(t))} + \sum_{j=1}^m c_{1j}(t)\mathrm{e}^{x_2(t-\rho_j(t))} + |d_1'(t)|\mathrm{e}^{x_1(t-\delta_1(t))}\Big]\,dt \\
&= \int_0^\omega \sum_{k=1}^2 \Gamma_{1k}^1(s)\mathrm{e}^{x_k(s)}\mathrm{d}s \\
&= \int_0^\omega \sum_{k=1}^2 \left(\frac{\Gamma_{1k}^1(s)}{\Gamma_{1k}(s)}\right) \Gamma_{1k}(s)\mathrm{e}^{x_k(s)}\mathrm{d}s \\
&\leq \int_0^\omega \sum_{k=1}^2 \left(\frac{\Gamma_{1k}^1(s)}{\Gamma_{1k}(s)}\right)_0 \Gamma_{1k}(s)\mathrm{e}^{x_k(s)}\mathrm{d}s \\
&\leq \Gamma_1 \int_0^\omega \sum_{k=1}^2 \Gamma_{1k}(s)\mathrm{e}^{x_k(s)}\mathrm{d}s, \qquad (16)
\end{aligned}$$

It follows from (14)-(16) that

$$\int_0^\omega \left|\Big[x_1(t) + \lambda d_1(t)\mathrm{e}^{x_1(t-\delta_1(t))}\Big]'\right| dt \leq (\bar{R}_1 + \Gamma_1\bar{r}_1)\omega. \qquad (17)$$

Similarly

$$\int_0^\omega \left|\Big[x_2(t) + \lambda d_2(t)\mathrm{e}^{x_2(t-\delta_2(t))}\Big]'\right| dt \leq (\bar{R}_2 + \Gamma_2\bar{r}_2)\omega, \qquad (18)$$

From(14), we have

$$\begin{aligned}
\bar{r}_1\omega &= \int_0^\omega \Big[\Gamma_{11}(t)\mathrm{e}^{x_1(t)} + \Gamma_{12}(t)\mathrm{e}^{x_2(t)}\Big]\,\mathrm{d}t \\
&= \int_0^\omega \Big[\vartheta_1\mathrm{e}^{x_1(t)} + \vartheta_1 d_1(t)\mathrm{e}^{x_1(t-\delta_1(t))}\Big]\,\mathrm{d}t \\
&\quad + \int_0^\omega \Big[\Gamma_{11}(t)\mathrm{e}^{x_1(t)} + \Gamma_{12}(t)\mathrm{e}^{x_2(t)} - \vartheta_1\mathrm{e}^{x_1(t)} - \vartheta_1 d_1(t)\mathrm{e}^{x_1(t-\delta_1(t))}\Big]\,\mathrm{d}t. \qquad (19)
\end{aligned}$$

Similarly to (10)-(14) we can get

$$\begin{aligned}
&\int_0^\omega \Big[\Gamma_{11}(t)\mathrm{e}^{x_1(t)} + \Gamma_{12}(t)\mathrm{e}^{x_2(t)} - \vartheta_1\mathrm{e}^{x_1(t)} - \vartheta_1 d_1(t)\mathrm{e}^{x_1(t-\delta_1(t))}\Big]\,\mathrm{d}t \\
&= \int_0^\omega \left[\left(\Gamma_{11}(t) - \vartheta_1 - \vartheta_1\frac{d_1(\gamma_1(t))}{1-\delta_1'(\gamma_1(t))}\right)\mathrm{e}^{x_1(t)} + \Gamma_{12}(t)\mathrm{e}^{x_2(t)}\right]\mathrm{d}t.
\end{aligned}$$

As $\vartheta_1 = \frac{(\Gamma_{11})_m(1-\delta_1')_m}{(1-\delta_1')_m + |d_1|_0}$, it follows

$$\Gamma_{11}(t) - \vartheta_1 - \vartheta_1\frac{d_1(\gamma_1(t))}{1-\delta_1'(\gamma_1(t))} \geq 0.$$

So we find from (19) that

$$\bar{r}_1\omega \geq \int_0^\omega \vartheta_1\mathrm{e}^{x_1(t)} + \vartheta_1 d_1(t)\mathrm{e}^{x_1(t-\delta_1(t))}\mathrm{d}t. \qquad (20)$$

By the mean value theorem, we see that there exist points ξ_1 such that

$$\bar{r}_1 \geq \vartheta_1\mathrm{e}^{x_1(\xi_1)} + \vartheta_1 d_1(\xi_1)\mathrm{e}^{x_1(\xi_1-\delta_1(\xi_1))}, \qquad (21)$$

which implies that

$$x_1(\xi_1) < \ln\frac{\bar{r}_1}{\vartheta_1}, \qquad d_1(\xi_1)\mathrm{e}^{x_1(\xi_1-\delta_1(\xi_1))} < \frac{\bar{r}_1}{\vartheta_1}. \qquad (22)$$

By (17) and (22), we can see

$$\begin{aligned}
&x_1(t) + \lambda d_1(t)\mathrm{e}^{x_1(t-\delta_1(t))} \\
&\leq x_1(\xi_1) + \lambda d_1(\xi_1)\mathrm{e}^{x_1(\xi_1-\delta_1(\xi_1))} + \int_0^\omega \left|\Big[x_1(t) + \lambda d_1(t)\mathrm{e}^{x_1(t-\delta_1(t))}\Big]'\right| \mathrm{d}t \\
&< \ln\frac{\bar{r}_1}{\vartheta_1} + \frac{\bar{r}_1}{\vartheta_1} + (\bar{R}_1 + \Gamma_1\bar{r}_1)\omega := K_1.
\end{aligned}$$

Similarly

$$x_2(t) + \lambda d_2(t)\mathrm{e}^{x_2(t-\delta_2(t))} < \ln\frac{\bar{r}_2}{\vartheta_2} + \frac{\bar{r}_2}{\vartheta_2} + (\bar{R}_2 + \Gamma_2\bar{r}_2)\omega := K_2.$$

where $\vartheta_2 = \frac{(\Gamma_{22})_m(1-\delta_2')_m}{(1-\delta_2')_m + |d_2|_0}$.

As $\lambda d_i(t)\mathrm{e}^{x_i(t-\delta_i(t))} > 0$, $i = 1, 2$, we can find that

$$x_i(t) < K_i, \qquad i = 1, 2. \qquad (23)$$

Besides, from (9) we get

$$\begin{cases}
x_1'(t) = \lambda \Big[r_1(t) - a_1(t)\mathrm{e}^{x_1(t)} - \sum_{i=1}^n b_{1i}(t)\mathrm{e}^{x_1(t-\tau_i(t))} - \sum_{j=1}^m c_{1j}(t)\mathrm{e}^{x_2(t-\rho_j(t))} \\
\qquad - e_1(t)x_1'(t-\delta_1(t))\mathrm{e}^{x_1(t-\delta_1(t))}\Big], \\
x_2'(t) = \lambda \Big[r_2(t) - a_2(t)\mathrm{e}^{x_2(t)} - \sum_{j=1}^m b_{2j}(t)\mathrm{e}^{x_2(t-\eta_j(t))} - \sum_{i=1}^n c_{2i}(t)\mathrm{e}^{x_1(t-\sigma_i(t))} \\
\qquad - e_2(t)x_2'(1-\delta_2(t))\mathrm{e}^{x_2(t-\delta_2(t))}\Big].
\end{cases}$$

Then by (23) we have

$$\begin{cases}
|x_1'(t)| \leq \lambda \Big[r_1(t) + a_1(t)\mathrm{e}^{x_1(t)} + \sum_{i=1}^n b_{1i}(t)\mathrm{e}^{x_1(t-\tau_i(t))} + \sum_{j=1}^m c_{1j}(t)\mathrm{e}^{x_2(t-\rho_j(t))} \\
\qquad + e_1(t)|x_1'(t-\delta_1(t))|\mathrm{e}^{x_1(t-\delta_1(t))}\Big], \\
\qquad \leq |r_1|_0 + |a_1|_0\mathrm{e}^{K_1} + \sum_{i=1}^n |b_{1i}|_0\mathrm{e}^{K_1} + \sum_{j=1}^m |c_{1j}|_0\mathrm{e}^{K_2} + |e_1|_0|x_1'|_0\mathrm{e}^{K_1}, \\
|x_2'(t)| \leq |r_2|_0 + |a_2|_0\mathrm{e}^{K_2} + \sum_{j=1}^m |b_{2j}|_0\mathrm{e}^{K_2} + \sum_{i=1}^n |c_{2i}|_0\mathrm{e}^{K_1} + |e_2|_0|x_2'|_0\mathrm{e}^{K_2}.
\end{cases}$$

Furthermore, we have

$$\begin{aligned}\|x'\|_0 &\le |x_1'(t)|_0 + |x_2'(t)|_0 \\ &\le \sum_{k=1}^{2}|r_k|_0 + \sum_{k=1}^{2}|a_k|_0 \mathrm{e}^{K_k} + \sum_{i=1}^{n}(|b_{1i}|_0 + |c_{2i}|_0)\mathrm{e}^{K_1} \\ &\quad + \sum_{j=1}^{m}(|c_{1j}|_0 + |b_{2j}|_0)\mathrm{e}^{K_2} + \sum_{k=1}^{2}|e_k|_0\|x'\|_0 \mathrm{e}^{K_k}.\end{aligned}$$

By the assumption (3) of Theorem 1, we see

$$\sum_{k=1}^{2}|e_k|_0 \mathrm{e}^{K_k} \le \sum_{k=1}^{2}|e_k|_0 \mathrm{e}^{K} \le \sum_{k=1}^{2}|e_k|_0 \mathrm{e}^{M_0} < 1.$$

Then

$$\|x'\|_0 < H_*, \tag{24}$$

Now, recalling (14) we can see that

$$\bar{r}_i\omega = \sum_{k=1}^{2}\int_0^{\omega}\Gamma_{ik}(s)\mathrm{e}^{x_k(s)}\mathrm{d}s, i = 1, 2,$$

and by using the extended integral mean value theorem, we can find points $\eta_k \in [0, \omega]$ $(k = 1, 2)$ such that

$$\begin{aligned}\bar{r}_i\omega &= \sum_{k=1}^{2}\int_0^{\omega}\Gamma_{ik}(s)\mathrm{e}^{x_k(s)}\mathrm{d}s \\ &= \sum_{k=1}^{2}\mathrm{e}^{x_k(\eta_k)}\int_0^{\omega}\Gamma_{ik}(s)\mathrm{d}s, \qquad i = 1, 2.\end{aligned} \tag{25}$$

Since $t = u_{1i}(s)$ is the inverse function of $t - \tau_i(t) = s$, $t \in [0, \omega]$, and in view of the Lemma 2, we can see $u_{1i}(\omega) = u_{1i}(0) + \omega$, so

$$\begin{aligned}\int_0^{\omega}\frac{b_{1i}(u_{1i}(s))}{1 - \tau_i'(u_{1i}(s))}\mathrm{d}s &= \int_{u_{1i}(0)}^{u_{1i}(\omega)}\frac{b_{1i}(t)(1 - \tau_i'(t))}{1 - \tau_i'(t)}\mathrm{d}t \\ &= \int_{u_{1i}(0)}^{u_{1i}(0)+\omega} b_{1i}(t)\mathrm{d}t = \bar{b}_{1i}\omega.\end{aligned}$$

Similarly,

$$\int_0^{\omega}\frac{d_1'(\gamma_1(s))}{1 - \delta_1'(\gamma_1(s))}\mathrm{d}s = \int_{\gamma_1(0)}^{\gamma_1(\omega)}\frac{d_1'(t)(1 - \delta_1'(t))}{1 - \delta_1'(t)}\mathrm{d}t = 0,$$

$$\int_0^{\omega}\Gamma_{12}(s)\mathrm{d}s = \int_0^{\omega}\sum_{j=1}^{m}\frac{c_{1j}(v_{1j}(s))}{1 - \rho_j'(v_{1j}(s))}\mathrm{d}s = \sum_{j=1}^{m}\bar{c}_{1j}\omega.$$

Thus

$$\begin{aligned}\int_0^{\omega}\Gamma_{11}(s)\mathrm{d}s &= (\bar{a}_1 + \sum_{i=1}^{n}\bar{b}_{1i})\omega, \\ \int_0^{\omega}\Gamma_{12}(s)\mathrm{d}s &= \sum_{j=1}^{m}\bar{c}_{1j}\omega.\end{aligned} \tag{26}$$

Similarly

$$\begin{aligned}\int_0^{\omega}\Gamma_{21}(s)\mathrm{d}s &= \sum_{i=1}^{n}\bar{c}_{2i}\omega, \\ \int_0^{\omega}\Gamma_{22}(s)\mathrm{d}s &= (\bar{a}_2 + \sum_{j=1}^{m}\bar{b}_{2j})\omega.\end{aligned} \tag{27}$$

It follows from (25), (26) and (27) that

$$\begin{aligned}\bar{r}_1 &= \mathrm{e}^{x_1(\eta_1)}(\bar{a}_1 + \sum_{i=1}^{n}\bar{b}_{1i}) + \mathrm{e}^{x_2(\eta_2)}\sum_{j=1}^{m}\bar{c}_{1j}, \\ \bar{r}_2 &= \mathrm{e}^{x_1(\eta_1)}\sum_{i=1}^{n}\bar{c}_{2i} + \mathrm{e}^{x_2(\eta_2)}(\bar{a}_2 + \sum_{j=1}^{m}\bar{b}_{2j}).\end{aligned} \tag{28}$$

From (28), we have

$$\begin{aligned}x_1(\eta_1) &\le \ln\frac{\bar{r}_1}{\bar{a}_1 + \sum_{i=1}^{n}\bar{b}_{1i}}, \\ x_2(\eta_2) &\le \ln\frac{\bar{r}_2}{\bar{a}_2 + \sum_{j=1}^{m}\bar{b}_{2j}}.\end{aligned} \tag{29}$$

On the other hand, from (28) and (29) we get

$$\begin{aligned}\bar{r}_1 &= \mathrm{e}^{x_1(\eta_1)}(\bar{a}_1 + \sum_{i=1}^{n}\bar{b}_{1i}) + \mathrm{e}^{x_2(\eta_2)}\sum_{j=1}^{m}\bar{c}_{1j} \\ &\le \mathrm{e}^{x_1(\eta_1)}(\bar{a}_1 + \sum_{i=1}^{n}\bar{b}_{1i}) + \frac{\bar{r}_2\sum_{j=1}^{m}\bar{c}_{1j}}{\bar{a}_2 + \sum_{j=1}^{m}\bar{b}_{2j}}, \\ \bar{r}_2 &= \mathrm{e}^{x_1(\eta_1)}\sum_{i=1}^{n}\bar{c}_{2i} + \mathrm{e}^{x_2(\eta_2)}(\bar{a}_2 + \sum_{j=1}^{m}\bar{b}_{2j}) \\ &\le \frac{\bar{r}_1\sum_{i=1}^{n}\bar{c}_{2i}}{\bar{a}_1 + \sum_{i=1}^{n}\bar{b}_{1i}} + \mathrm{e}^{x_2(\eta_2)}(\bar{a}_2 + \sum_{j=1}^{m}\bar{b}_{2j}).\end{aligned}$$

Therefore, by the assumption (2) of Theorem 1 we obtain

$$\begin{aligned}x_1(\eta_1) &\ge \ln\frac{\bar{r}_1 - \frac{\bar{r}_2\sum_{j=1}^{m}\bar{c}_{1j}}{\bar{a}_2 + \sum_{j=1}^{m}\bar{b}_{2j}}}{\bar{a}_1 + \sum_{i=1}^{n}\bar{b}_{1i}}, \\ x_2(\eta_2) &\ge \ln\frac{\bar{r}_2 - \frac{\bar{r}_1\sum_{i=1}^{n}\bar{c}_{2i}}{\bar{a}_1 + \sum_{i=1}^{n}\bar{b}_{1i}}}{\bar{a}_2 + \sum_{j=1}^{m}\bar{b}_{2j}}.\end{aligned} \tag{30}$$

(29) and (30) imply

$$|x_i(\eta_i)| \le H_i, \quad i = 1, 2 \tag{31}$$

From (24) and (31), we have

$$|x_i| \le |x_i(\eta_i)| + \int_0^{\omega}|x_i'|\mathrm{d}t \le H_i + \int_0^{\omega}|x_i'|\mathrm{d}t,\ i = 1, 2.$$

Hence,

$$\|x\|_0 \le \sum_{i=1}^{2}H_i + \int_0^{\omega}\|x'\|_0\mathrm{d}t \le \sum_{i=1}^{2}H_i + H_*\omega \le M_0. \tag{32}$$

Obviously, M_0 is independent of λ, the proof is complete. □

Based on the above results, we can now apply Lemma 1 and Remark 1 to (7) and obtain a proof of Theorem 1.

Proof of Theorem 1. Obviously, for M as give in Lemma 3, condition (1) in Lemma 1 is satisfied. Let $g(\mu) = (g_1(\mu), g_2(\mu))$.
Since

$$g_1(\mu) = \omega\left[\bar{r}_1 - \left((\bar{a}_1 + \sum_{i=1}^{n}\bar{b}_{1i})\mathrm{e}^{\mu_1} + \sum_{j=1}^{m}\bar{c}_{1j}\mathrm{e}^{\mu_2}\right)\right],$$

$$g_2(\mu) = \omega\left[\bar{r}_2 - \left(\sum_{i=1}^{n}\bar{c}_{2i}\mathrm{e}^{\mu_1} + (\bar{a}_2 + \sum_{j=1}^{m}\bar{b}_{2j})\mathrm{e}^{\mu_2}\right)\right].$$

and $M > |\ln\mu_1^*| + |\ln\mu_2^*|$, we have $g(\mu) \ne 0$ for any $\mu \in \partial B_M(\mathbb{R}^2)$. That is, condition (2) in Lemma 1 holds.

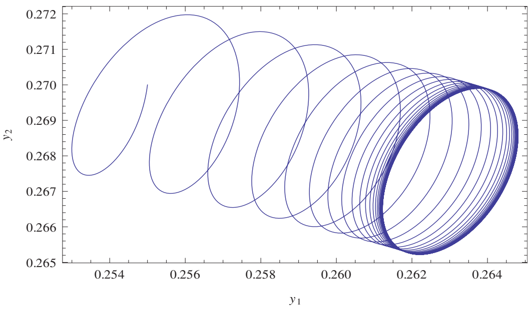

Fig. 1. Phase portrait of a solution of system (2) with 2π-periodic solution as its limit cycle.

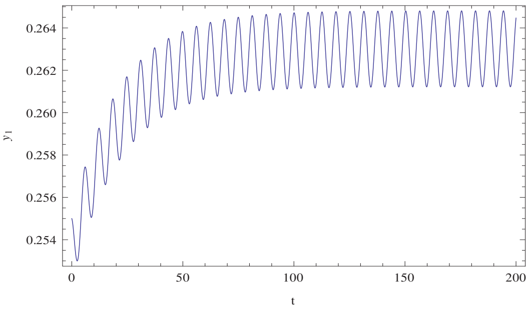

Fig. 2. Time-series of $y_1(t)$ evolved in system (2).

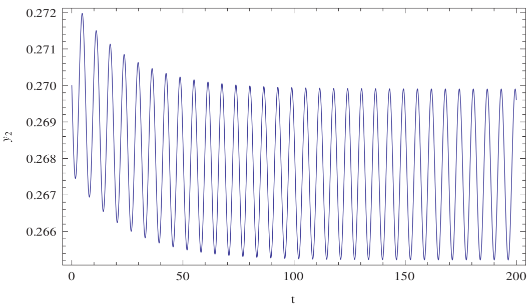

Fig. 3. Time-series of $y_2(t)$ evolved in system (2).

At last, we verify that condition (3) of Lemma 1 also holds. By assumption (1) of Theorem 1 and the formula for the Brouwer degree, a straightforward calculation shows that

$$\deg(g, B_M(\mathbb{R}^2)) = \sum_{\mu\in g^{-1}(0)\bigcap B_M(\mathbb{R}^2)} \operatorname{sign}\det Dg(\mu)$$

$$= \operatorname{sign}\left\{ \begin{vmatrix} \bar{a}_1 + \sum_{i=1}^{n} \bar{b}_{1i} & \sum_{j=1}^{m} \bar{c}_{1j} \\ \sum_{i=1}^{n} \bar{c}_{2i} & \bar{a}_2 + \sum_{j=1}^{m} \bar{b}_{2j} \end{vmatrix} e^{\mu_1+\mu_2} \right\} \neq 0.$$

By now all the assumptions required in Lemma 1 hold. It follows from Lemma 1 and Remark 1 that system (7) has an $\omega-$periodic solution. Returning to $y_i(t) = e^{x_i(t)}$, $i = 1, 2$, we conclude that (2) has at least one positive $\omega-$periodic solution. The proof of Theorem 1 is complete. □

IV. Some simulations

In this section, we shall discuss an example to illustrate our main results. For system (2), we take $r_1(t) = 0.05 + 0.01\sin t$, $r_2(t) = 0.05+0.01\cos t$, $a_1(t) = 0.08+0.05\sin t$, $a_2(t) = 0.08+0.05\cos t$, $b_{11}(t) = 0.05+0.01\sin t$, $b_{12}(t) = 0.05+0.01\cos t$, $b_{21}(t) = 0.05+0.01\cos t$, $b_{22}(t) = 0.05+0.01\cos t$, $c_{11}(t) = c_{12}(t) = 0.005 + 0.001\sin t$, $c_{21}(t) = 0.005 + 0.001\sin t$, $c_{22}(t) = 0.002 + 0.001\sin t$, $e_1(t) = 0.0002 + 0.00005\sin t$, $e_2(t) = 0.0002 + 0.00005\cos t$, $m = n = 2$, $\tau_1(t) = 0.08$, $\tau_2(t) = 0.05$, $\rho_1(t) = 0.13$, $\rho_2(t) = 0.09$, $\delta_1(t) = \delta_2(t) = 0.05$, $\eta_1(t) = 0.01$, $\eta_2(t) = 0.05$, $\sigma_1(t) = 0.07$, $\sigma_2(t) = 0.06$. When $t \leq 0$, we take $y_1(t) = 0.255+0.01\sin t$, $y_2(t) = 0.26+0.01\cos t$. It can be easily check that all conditions of Theorem 1 are satisfied. Then system (2) under the above conditions has at least one positive $\omega-$periodic solution (see Fig.1-Fig.3).

V. Conclusion

In this paper, a two-species competition system with general periodic neutral delay has been investigated. With the help of the continuation theorem for composite coincidence degree and some techniques, a set of sufficient conditions have been derived for the existence of at least one strictly positive periodic solution. Simulation examples have shown the effectiveness of the conditions presented in this paper.

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