Proceedings of the World Congress on Engineering 2012 Vol I
WCE 2012, July 4 - 6, 2012, London, U.K.

Power-Law Adjusted Failure-Time Models

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***Abstract*—A simple adjustment to parametric failure-time distributions, which allows for much greater flexibility in the shape of the hazard-rate function, is considered. Analytical expressions for the distributions of the power-law adjusted Weibull, gamma, log-gamma, generalized gamma, lognormal and Pareto distributions are given. Most of these allow for bathtub shaped and other multi-modal forms of the hazard rate. The new distributions are fitted to real failure-time data which exhibit a multi-modal hazard-rate function and the fits are compared.**

***Index Terms*—survival analysis; bathtub hazard; accelerated failure time (AFT) regression; power-law distribution.**

I. Introduction

Parametric distributions play an important role in the analysis of lifetime data especially in accelerated failure time (AFT) regression models. Generally speaking analysis based on a parametric model will be more precise than that based on a nonparametric or semi-parametric model, because it will have fewer unknown parameters. However this is contingent on it being possible to find a suitable parametric model to fit the data. Unfortunately for most of the common distributions employed there is very little flexibility in the shape of the hazard rate function. In particular none of the two-parameter distributions customarily employed can be used to model a bathtub-shaped hazard.

There are a number of three-parameter distributions which allow a bathtub-shaped hazard including the *exponentiated Weibull* [3], the *generalized Weibull* [4] and the *generalized gamma* (see *e.g.* [1]) distributions. An addition to these was proposed in a recent article by Reed [5]. This distribution, which is a special case of a *double Pareto-lognormal* distribution [6], can be characterised as the product of independent random variables, one with a lognormal distribution and the other with a power-law distribution on $[0, 1]$. For this reason the new distribution was called the *lognormal-power function* distribution. It can be thought of as an extension of the lognormal distribution.

In this article it is shown how any simple parametric failure-time distribution can be extended in a similar way to allow for much greater flexibility in its form, including in most cases the possibility of bathtub shaped hazard-rate functions. Precisely, the failure time T is modelled as the product $T \stackrel{d}{=} T_0U$, where T_0 follows the "simple" failure-time distribution and U follows the power-law distribution with density $\lambda u^{\lambda-1}$ on $[0, 1]$. Alternatively this can be expressed as $T \stackrel{d}{=} T_0/V$ where V has a Pareto distribution, with density $\lambda/v^{\lambda+1}$ on $[1, \infty)$.

As might be expected, it is not possible for every parametrically specified distribution (of T_0) to obtain an analytical expression for the resulting power-law modified density. However it turns out to be possible to do so for a number of the more common failure-time distributions including the lognormal (Reed, 2011), exponential, Weibull, gamma, log-gamma, Pareto and generalized gamma distributions. These distributions are considered in this article. In all cases, except the lognormal and Pareto, the resulting power-function modified densities can be expressed in terms of an incomplete gamma function.

In Sec.2 the distribution theory associated with the power-law modification is presented, and in Sec.3 maximum likelihood estimation discussed. In Sec.4 the results of fitting the various power-law modified failure-time distributions to data with a multi-modal shaped hazard rate, are presented.

Manuscript received March 9, 2012; revised March, 2012. This work was supported in part by NSERC Grant OGP 7252.

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II. Theory

Let T_0 be a random variable with a known continuous failure-time distribution. The power-law modified form of this distribution can be represented by a random variable T with

$$T \stackrel{d}{=} T_0U$$

where U, independent of T_0, follows the power-law distribution with density $\lambda u^{\lambda-1}$ $(\lambda > 0)$ on the interval $[0, 1]$. Taking logarithms leads to

$$X = \log(T) \stackrel{d}{=} Z_0 - \frac{1}{\lambda}E$$

where $Z_0 = \log T_0$ (with survivor function and density $S_0(z)$ and $f_0(z)$, say) and E is a standard (unit mean) exponential random variable. The survivor function for X can be found as a convolution as follows:

$$\begin{aligned} S_X(x) &= \mathrm{P}(Z_0 - E/\lambda \geq x) \\ &= \mathrm{P}(E \leq \lambda(Z_0 - x)) \\ &= \mathcal{E}\{\mathrm{P}(E \leq \lambda(Z_0 - x))|Z_0\} \\ &= \mathcal{E}\{[1 - e^{-\lambda(Z_0-x)}]\ I[Z_0 - x > 0]\} \\ &= \int_x^\infty [1 - e^{-\lambda(z-x)}]f_0(z)dz \\ &= S_0(x) - e^{\lambda x}\int_x^\infty e^{-\lambda z}f_0(z)dz \qquad (1) \end{aligned}$$

where the expectation $\mathcal{E}$ is with respect to Z_0 and I is a Bernoulli indicator random variable. Upon integrating by parts one obtains

$$S_X(x) = \lambda e^{\lambda x}\int_x^\infty e^{-\lambda z}S_0(z)dz. \qquad (2)$$

From this, by differentiation and using (1), one obtains the corresponding formula for the density of X

$$f_X(x) = \lambda e^{\lambda x}\int_x^\infty e^{-\lambda z}f_0(z)dz. \qquad (3)$$

From (2) and (3) the survivor function and density of T in terms of those of T_0 ($S_{T_0}(t)$ and $f_{T_0}(t)$) can be easily obtained:

$$S_T(t) = \lambda t^{\lambda} \int_t^{\infty} u^{-\lambda-1} S_{T_0}(u) du. \tag{4}$$

$$f_T(t) = \lambda t^{\lambda-1} \int_t^{\infty} u^{-\lambda} f_{T_0}(u) du. \tag{5}$$

We now consider power-law modified forms of some specific failure-time distributions.

Weibull and exponential model. If T_0 has a Weibull distribution with hazard rate function $h_{T_0}(t) = \alpha\beta t^{\beta-1}$, its survivor function and density are $S_{T_0}(t) = \exp(-\alpha t^{\beta})$ and $f_{T_0}(t) = \alpha\beta t^{\beta-1}\exp(-\alpha t^{\beta})$. The hazard rate is monotone increasing for $\beta > 1$ and monotone decreasing for $\beta < 1$. In the case $\beta = 1$ it is constant and the Weibull distribution reduces to an exponential distribution. The survivor function and density for $Z_0 = \log T_0$ are

$$S_0(z) = \exp(-\alpha e^{\beta z}) \qquad \text{and} \qquad f_0(z) = \alpha\beta\exp(\beta z - \alpha e^{\beta z}).$$

From (2) and (3), the survivor function and density of $X = \log T$, where T follows the power-law adjusted Weibull distribution, are

$$S_X(x) = \frac{\lambda\alpha^{\lambda/\beta}}{\beta}\; e^{\lambda x}\; I(\alpha e^{\beta x}, -\lambda/\beta)$$

$$f_X(x) = \lambda\alpha^{\lambda/\beta}\; e^{\lambda x}\; I(\alpha e^{\beta x}, 1-\lambda/\beta)$$

where I is the incomplete gamma function

$$I(y,\theta) = \int_y^{\infty} u^{\theta-1} e^{-u} du. \tag{6}$$

Note that although the ordinary gamma function can be expressed as the integral $\Gamma(\theta) = \int_0^{\infty} u^{\theta-1}e^{-u}du$ only for $\theta > 0$, the incomplete gamma function $I(y,\theta)$ evaluated at $y > 0$ converges for all real θ. Thus $S_X(x)$ and $f_X(x)$ above are well-defined since $\alpha e^{\beta x} > 0$.

The survivor function, density and hazard-rate function for T are easily computed from the above as

$$S_T(t) = S_X(\log t); \qquad f_T(t) = \frac{1}{t} f_X(\log t); \qquad h_T(t) = \frac{f_T(t)}{S_T(t)}$$

Fig.1 (top row) illustrates three shapes that the hazard rate function of the power-law adjusted Weibull distribution can assume.

Gamma model. If T_0 follows a gamma distribution with scale parameter θ^{-1} and shape parameter κ, then the density and survivor function of $Z_0 = \log T_0$ are

$$S_0(z) = \frac{I(\theta e^z, \kappa)}{\Gamma(\kappa)} \qquad \text{and} \qquad f_0(z) = \frac{\theta^{\kappa}}{\Gamma(\kappa)}\exp(\kappa z - \theta e^z)$$

From (2) and (3), the survivor function and density of $X = \log T$, where T follows the power-law adjusted gamma distribution, are

$$S_X(x) = \frac{1}{\Gamma(\kappa)}\left[I(\theta e^x, \kappa) - \theta^{\lambda} e^{\lambda x} I(\theta e^x, \kappa-\lambda)\right]$$

$$f_X(x) = \frac{\lambda\theta^{\lambda}}{\Gamma(\kappa)}\; e^{\lambda x} I(\theta e^x, \kappa-\lambda)$$

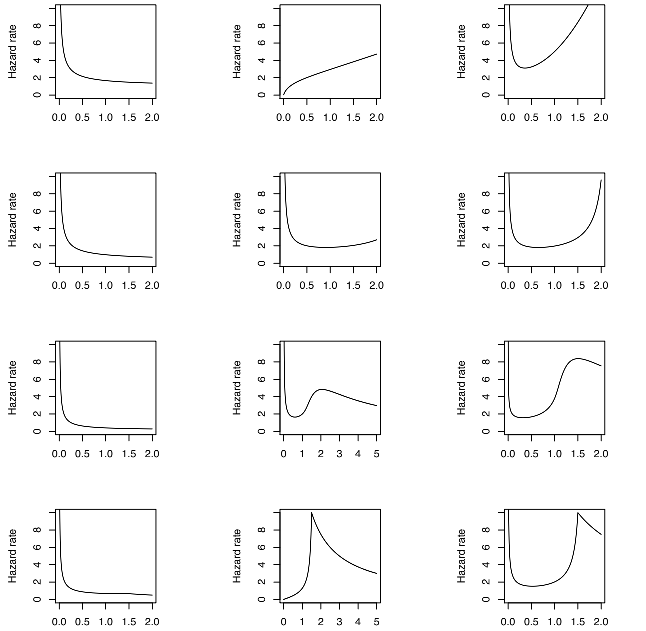

Fig. 1. Some shapes of the hazard rate function for for various power-law adjusted distributions. Top row: Weibull distribution with $\alpha = 1$: (l.hand) $\beta = 1$ (exponential distribution) and $\lambda = 0.02$; (centre) $\beta = 2$ and $\lambda = 2$; r.hand $\beta = 3$ and $\lambda = .02$. Second row: gamma distribution with $\theta = 0.25$: (l.hand) $\kappa = .01$ and $\lambda = 1$; (centre) $\kappa = .01$ and $\lambda = 2.5$; (r.hand) $\kappa = .1$ and $\lambda = 7$. Third row: log-gamma distribution with $\theta = 20$: (l.hand) $\kappa = 50$ and $\lambda = .01$; (centre) $\kappa = 10$ and $\lambda = .01$; (r.hand) $\kappa = 5$ and $\lambda = .5$. Bottom row: Pareto distribution with $\tau_0 = 1.5$: (l.hand) $\alpha = 1$ and $\lambda = 0.1$; (centre) $\alpha = 15$ and $\lambda = 2$; (r.hand): $\alpha = 15$ and $\lambda = 0.2$

Fig.1 (second row) illustrates some shapes that the hazard rate function of the power-law adjusted gamma distribution can assume.

Log-gamma model. If $Z_0 = \log T_0$ follows a gamma distribution, so that T_0 has density $f_{T_0}(t) = \frac{\theta^{\kappa}}{\Gamma(\kappa)} t^{-(\theta+1)}(\log t)^{\kappa-1}$ with support on $[1,\infty)$ then from (2) and (3), it is easy to show that the power-law adjusted random variable T has support on $(0,\infty)$ and that $X = \log T$ has survivor function and density

$$S_X(x) = \begin{cases} 1 - e^{\lambda x}\left(\frac{\theta}{\theta+\lambda}\right)^{\kappa} & \text{if } x \le 0 \\ \frac{1}{\Gamma(\kappa)}\left[I(\theta x,\kappa) - \left(\frac{\theta}{\theta+\lambda}\right)^{\kappa} e^{\lambda x} I([\theta+\lambda]x,\kappa)\right] & \text{if } x > 0 \end{cases}$$

and

$$f_X(x) = \begin{cases} \lambda e^{\lambda x}\left(\frac{\theta}{\theta+\lambda}\right)^{\kappa} & \text{if } x \le 0 \\ \lambda e^{\lambda x}\left(\frac{\theta}{\theta+\lambda}\right)^{\kappa} \frac{I([\theta+\lambda]x,\kappa)}{\Gamma(\kappa)} & \text{if } x > 0 \end{cases}$$

Fig.1 (third row) illustrates some shapes that the hazard rate function of the power-law adjusted log-gamma distribution can assume.

Pareto model. If T_0 follows a Pareto distribution with support on (τ_0,∞) and pdf $f_{T_0}(t) = \frac{\alpha}{\tau_0}\left(\frac{t}{\tau_0}\right)^{-(\alpha+1)}$ thereon, one can show that the power-law adjusted form has support on $(0,\infty)$ and (using (4)) that the survivor function of the power-law adjusted form is

$$S_T(t) = \begin{cases} 1 - \frac{\alpha}{\alpha+\lambda}\left(\frac{t}{\tau_0}\right)^{\lambda} & \text{if } t \le \tau_0 \\ \frac{\lambda}{\alpha+\lambda}\left(\frac{t}{\tau_0}\right)^{-\alpha} & \text{if } t > \tau_0 \end{cases}$$

ISBN: 978-988-19251-3-8
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

and using (5) that the corresponding pdf is

$$f_T(t) = \begin{cases} \frac{\alpha\lambda}{\alpha+\lambda}\ \frac{1}{\tau_0}\left(\frac{t}{\tau_0}\right)^{\lambda-1} & \text{if } t \le \tau_0 \\ \frac{\alpha\lambda}{\alpha+\lambda}\ \frac{1}{\tau_0}\left(\frac{t}{\tau_0}\right)^{-\alpha-1} & \text{if } t > \tau_0 \end{cases}$$

Fig.1 (bottom row) illustrates some shapes that the hazard rate function of the power-law adjusted Pareto distribution can assume.

Lognormal model. Consider the case where $Z_0 = \log T_0$ follows a normal distribution with mean μ and variance σ^2. Reed (2011) The power-law adjusted version of this distribution (the *lognormal-power function* or lNpf distribution) was considered in [5] where it is shown that the survivor function and density of $X = \log T$, where T follows the lNpf distribution, are

$$S_X(x) = \phi\left(\frac{x-\mu}{\sigma}\right)\left[R\left(\frac{x-\mu}{\sigma}\right) - R\left(\lambda\sigma + \frac{x-\mu}{\sigma}\right)\right]$$

and

$$f_X(x) = \lambda\phi\left(\frac{x-\mu}{\sigma}\right)R\left(\lambda\sigma + \frac{x-\mu}{\sigma}\right)$$

where R is *Mills' ratio* of the complementary cumulative distribution function (cdf) to the pdf of a standard normal distribution:

$$R(z) = \frac{\Phi^c(z)}{\phi(z)}.$$

Generalized gamma model. The three-parameter generalized gamma distribution includes the Weibull, gamma and lognormal models as special or limiting cases. It has density

$$f_{T_0}(t) = \alpha\theta^\kappa t^{\alpha\kappa-1}\exp(-\theta t^\alpha)/\Gamma(\kappa)$$

With some work using (2) and (3), the survivor function and density of $X = \log T$, where T follows the power-law adjusted gamma distribution, can be shown to be

$$S_X(x) = \frac{1}{\Gamma(\kappa)}\left[I(\theta e^{\alpha x},\kappa) - \theta^{\lambda/\alpha}e^{\lambda x}I(\theta e^{\alpha x},\kappa-\lambda/\alpha)\right]$$

$$f_X(x) = \frac{\lambda\theta^{\lambda/\alpha}}{\Gamma(\kappa)}\ e^{\lambda x}I(\theta e^{\alpha x},\kappa-\lambda/\alpha)$$

It should be noted that while the (unadjusted) log-gamma and Pareto distributions have support bounded away from zero, their power law adjusted versions have support on $[0,\infty)$ as indeed occurs in all of the power law adjusted models discussed in this paper. Thus in these models there are no problems with the range of support depending on a parameter, as occurs for example with the generalized Weibull distribution.

III. Parameter estimation by maximum likelihood.

The parametric likelihood for much failure-time data is proportional to

$$\prod_{i=1}^{n}[f_{T_i}(t_i)]^{\delta_i}[S_{T_i}(t_i)]^{1-\delta_i}$$

where δ_i is an indicator variable with value 1 for an observed failure time, and value 0 for a right-censored observation. If there are no covariates and the failure times are considered to be identically distributed following a power-law adjusted distribution with pdf and survivor function f_T and S_T, then up to an additive constant the log-likelihood is

$$\sum_{i=1}^{n}\delta_i\log f_T(t_i) + \sum_{i=1}^{n}(1-\delta_i)\log S_T(t_i)$$

which is the same as

$$\sum_{i=1}^{n}\delta_i\log f_X(\log t_i) + \sum_{i=1}^{n}(1-\delta_i)\log S_X(\log t_i) - \sum_{i=1}^{n}\log t_i$$

Thus for each of the models discussed above an analytical expression for the log-likelihood can be obtained. This will need to be maximized numerically to obtain maximum likelihood estimates using an optimization routine such as optim in R. For starting values one can use the MLEs of the two parameters of the unadjusted distribution and an arbitrary value (say 1) for λ.

Covariates $\boldsymbol{Z}^{\boldsymbol{T}} = (Z_1, Z_2, \ldots, Z_p)$ can be incorporated in an accelerated failure time (AFT) regression model:

$$\log T = \beta_0 + \boldsymbol{\beta}^{\boldsymbol{T}}\boldsymbol{Z} + X \tag{7}$$

where X is a random variable with one of the power-law adjusted distributions of the previous section. Note that for all but the log-gamma these distributions can be reparameterized in terms of a location parameter and two other parameters. In these cases the intercept term β_0 in (7) is not needed (and indeed will result in a non-identifiable model if it is included).

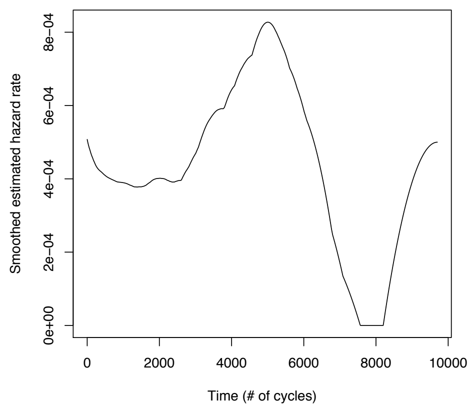

Fig. 2. Kernel smoothed non-parametric estimate of the hazard rate function for electrical appliances data. The Epanechnikov kernel with a bandwith of 1500 was used. Note that the right-hand part (> 6000) of the estimated hazard is unreliable, being based on only two observations.

IV. An example.

Electrical appliances. Lawless (p. 256) [2] presents data on the numbers of cycles to failure for 60 electrical appliances put on test. All of the sixty appliances eventually failed, the largest failure times being 6065 and 9701 cycles. Fig.2 shows a kernel-smoothed non-parametric estimate of the hazard rate for these data. There is clearly a suggestion of multi-modality. To assess and compare the various power-law adjusted models discussed in the previous section each was fitted to these data. Maximization of the log-likelihood

ISBN: 978-988-19251-3-8
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

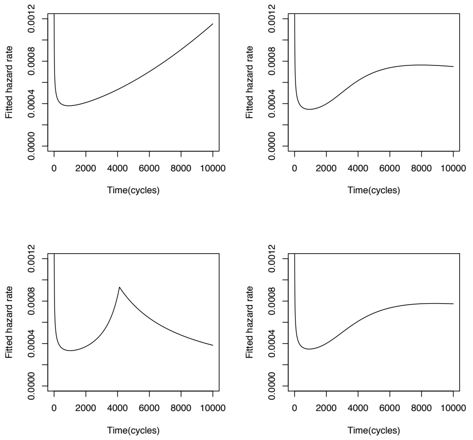

Fig. 3. Maximum likelihood estimates of various power-law adjusted distributions for the electrical appliance data. They are (clockwise from upper left) Weibull, log gamma, lognormal and Pareto.

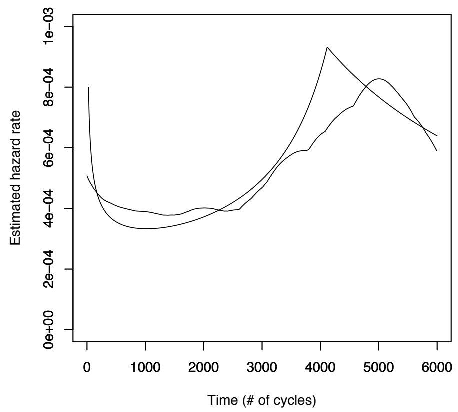

Fig. 4. Kernel smoothed non-parametric estimate of the hazard rate function for electrical appliances data and the MLE of the power-law adjusted Pareto hazard-rate.

was performed in R using the Nelder-Mead method in the routine optim and in all cases required only a minute or two of computation.

The values of the maximized log-likelihood and of the Akaike Information Criterion (AIC) for the power-law adjusted forms of the two-parameter models are given in Table 1. In all cases, the improvement in fit obtained by including the power-law adjustment was highly significant ($P << .001$) as one would expect since none of the two-parameter forms allows for a bathtub shape. From Table 1 it can be seen that the power-law adjusted Pareto distribution provides the best fit of these models.

Fig.3 shows the MLES of the hazard rate for (clockwise from upper left) the power-law adjusted Weibull, log-gamma, lognormal and Pareto distributions. While these plots may appear very different to the non-parametric estimate of the hazard function (Fig.2) at the upper end, it should be noted that the upper part of the non-parametric estimate is not very precise, since in the dataset there are only two observations greater than 6000 (with values 6065 and 9701). Fig.4 shows the fitted power-law adjusted Pareto hazard rate function superimposed on the non-parametric estimate on the range 0 to 6000 cycles. Also Fig.5 shows the Kaplan-Meier estimate of the survivor function and the fitted survivor function for the power-law adjusted Pareto distribution. Both plots suggest a good fit.

Attempts at fitting the four-parameter power-law adjusted generalized gamma distribution were not successful, with different maxima arising with different starting values. This suggests the possibility of identifiability problems with this model. Indeed the generalized gamma distribution without the power-law adjustment is capable of exhibiting a bathtub shaped hazard.

For comparison purposes the three 3-parameter distributions mentioned in the introduction which have been previously used to model data with a bathtub shaped hazard (exponentiated Weibull, generalized Weibull and generalized gamma) were fitted to the electrical appliances data. The results are shown in Table 2. From comparison with Table 1 it can be seen that of all eight models the best fitting is the power-law adjusted Pareto, followed by the generalized Weibull. Furthermore all of the power-law adjusted 2-parameter models, save the Weibull, have a better fit than the generalized gamma and the exponentiated Weibull distributions, suggesting that the consideration of power-law adjusted models may provide a useful addition to the toolkit of practitioners.

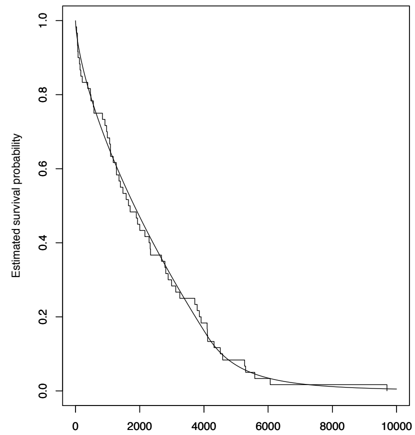

Fig. 5. Non-parametric Kaplan-Meier estimate (step function) of the survivor function for the electrical appliance data and the maximum likelihood estimate of the survivor function using the power-law adjusted Pareto distribution.

V. Conclusions.

This article shows how existing parametric failure-time distributions can be modified by a simple power-law adjustment, thereby rendering them more flexible, including in many cases having the possibility of a bathtub shaped hazard-rate function. The power-law adjustment involves the introduction of an extra parameter. While the article considers only distributions for which there are analytical expressions for the density and survivor function, the idea could still be applied to other common failure distributions (*e.g.* log-logistic, Gompertz, *etc.*) In such cases the density and survivor function would need to be computed numerically, using quadrature methods for evaluating the integrals (2) and (3). This would replace the computation involved

ISBN: 978-988-19251-3-8
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

in evaluating the incomplete gamma functions which occur in the distributions discussed in this paper and so the extra computation involved might not be too great.

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ISBN: 978-988-19251-3-8
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)