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An Analytical Approach to EWMA Control Chart for Trend Stationary Exponential AR(1) Processes

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Abstract— **The main assumptions for the statistical process control models is the independent and identically distributed (i.i.d.) observations, but in practice the observed data from industrial processes or finance are serially correlated or have trending. The Exponentially Weighted Moving Average (EWMA) control chart can detect small shifts in the process mean more quickly than the Shewhart control chart. In this paper, we derive an explicit formula for the characteristic of EWMA control chart for trend stationary exponential AR(1) processes. We compare the results for Average Run Length (ARL) obtained from the explicit formula with values obtained from the integral equation. The new results are simple, easy to programming, which make it attractive to be used in practice by performers. Our results show that the explicit expressions reduce considerably the computational time used to evaluate the ARL, comparable with the integral equation approach.**

Index Terms— **analytical solution, EWMA control chart, ARL, exponential white noise**

I. Introduction

Traditionally, independent and identically distributed data is the main assumption when analyze the characteristics of statistical process control, but in practice data from industrial processes or finance has serial correlations or trending. The effect of autocorrelation on the statistical performance of the control charts has been studied by [1], [2], [3], [4], [5]. Usually, the performance of the control chart is measured by the average run length (ARL). The ARL_0 is defined as the expectation of false alarm time (τ) before an in-control process is taken to signal to be out of control. For practical purposes, a sufficient large in-control ARL_0 is desired. When the process is out-of-control, the performance of a control chart is usually used as ARL_1. The ARL_1 is defined as the expected number of observations taken from an out-of-control process until the control chart signals that the process is out-of-control. Ideally, ARL_1 should be small.

A control chart based on the Exponentially Weighted Moving Average (EWMA) model was first proposed by [6]. The methods for evaluating the performance of EWMA control charts for serially correlated have been studied by [7]. They used simulation method based on the presence of autocorrelation for EWMA control chart. The ARL, and steady state ARL of EWMA were estimated numerically by [8] using an integral equation approach and a Markov chain approach to investigating EWMA and CUSUM procedures for the process mean when data was described by an AR(1) process with additional random error. The EWMA control charts based on the observations which follow an AR(1) process, plus a random error, and to detect changes in the process mean, or in the process variance, the authors using a simulation approach is discussed by [5]. In [9] presented the ARL of the EWMA control chart for monitoring the mean of an AR(1) process, plus a random error by using an integral equation method. In [10] compared the ARL for the EWMAST chart, the CUSUM residual chart, the EWMA residual chart, the X residual chart, and X chart using simulation. In [11] calculated the ARL of $\bar{X}$ and EWMA charts using analytical and simulation techniques. In [12] studied the EWMA chart with residual-based approaches for detecting process shifts by using simulation. In [13] studied EWMA chart for an AR model and calculated ARL by Markov chain approach. In [14] evaluated the ARL of EWMA charts with heavy tailed distribution for monitoring the mean of the stationary processes by simulation method. In [15] computed exactly *ARL* with the Markov chain approach for a Poisson INAR(1) model of EWMA chart. In [16] designed the ARL performance of autocorrelated process control chart using a Monte Carla simulation. In [17] used finite Markov chain imbedding technique to investigate the run length properties for control charts when the process observations were autocorrelated. Recently, [18] have derived explicit formula for ARL_0 for EWMA control charts for AR(1) process observations with exponential white noise.

Recently, several researchers show increasing interest in the formulation and analysis of non-Gaussian models for serially correlated data, e.g, [19] and [20]. Exponential white noise has been studied in the connection with pollution problem (see [21]), and some paper has studied with exponential white noise by [21], [22], [23], [24], [25].

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In our study, an explicit formula for the EWMA control chart for trend stationary exponential AR(1) processes is presented. In the next section, the EWMA control chart for trend stationary exponential AR(1) processes will be given. An explicit formula will be briefly discussed in Section 3. Explicit formula for EWMA control chart for trend stationary exponential AR(1) processes and numerical comparisons with the analytical results are presented in Section 4 and 5, respectively. Finally, concluding will be provided in the last section.

II. Exponential Weighted Moving Average (EWMA) Control Chart for Trend Stationary Exponential AR(1) Processes

The EWMA can be applied to an AR(1) time series to detect shifts in the process mean. The EWMA statistic based on AR(1) process Q_t is given by:

$$R_t = (1-\lambda) R_{t-1} + Q_t \tag{1}$$

where λ is a smoothing constant $(0<\lambda<1)$, the sequence $\{Q_t, t=1,2,3,...\}$ consists of an AR(1) processes with trend, and the initial value of R_0 is usually selected to be the process target of $\{Q_t, t=1,2,3,...\}$ or the average of random data. An AR(1) with trend is assumed to be as follow

$$Q_t = a + bt + \varphi Q_{t-1} + y_t \quad , \quad t \ge 1 \tag{2}$$

where a is a constant, b is the trend slope in term of t, and φ is the autoregressive coefficient $(0<\varphi<1)$. Let y_t is the independent random error term at time t following $Exp(\alpha)$. The variance of R_t for the large t will be

$$\sigma_{R_t}^2 = \left(\frac{\lambda}{2-\lambda}\right)\frac{1+\varphi(1-\lambda)}{(1-\varphi^2)(1-\varphi(1-\lambda))}\sigma_y^2 \tag{3}$$

Therefore the upper and lower control limits for monitoring the process when plotting R_t versus the time t are

$$\begin{aligned} UCL &= \mu + L\sigma_y\sqrt{\left(\frac{\lambda}{2-\lambda}\right)\frac{1+\varphi(1-\lambda)}{(1-\varphi^2)(1-\varphi(1-\lambda))}} = U \\ LCL &= \mu - L\sigma_y\sqrt{\left(\frac{\lambda}{2-\lambda}\right)\frac{1+\varphi(1-\lambda)}{(1-\varphi^2)(1-\varphi(1-\lambda))}} = 0 \end{aligned} \tag{4}$$

where L is a constant to be chosen, and σ_y is the standard deviation of a known underlying probability distribution. The process will be declared to be in an out-of-control state when $R_t > UCL$. The alarm time for the EWMA in then given by

$$\tau = \inf\{t>0 : R_t > UCL\}. \tag{5}$$

Assume $E_\theta(.)$ denote the expectation at time θ, where $\theta \le \infty$. The ARLs of the EWMA control chart for the given process are that:

$$ARL_0 = E_\infty(\tau) = T, \tag{6}$$

where T is given (usually large) and

$$ARL_1 = E_1(\tau / \tau \ge 1). \tag{7}$$

III. Explicit Formula for EWMA Control Chart for Trend Stationary Exponential AR(1) Processes

The performance of a control chart is measured by the average run length (ARL). The ARL_0 is defined as the expected of false alarm time (τ) before an in-control process is taken to signal to be out of control. A sufficient large in-control ARL_0 is desired. When the process is out-of-control, the performance of a control chart is usually used as ARL_1. It is the expected number of observations taken from an out-of-control process until the control chart signals that the process is out-of-control. Ideally, ARL_1 should be small. The values of ARL_0 and ARL_1 for an EWMA control chart with a trend stationary exponential AR(1) processes with exponential white noise observations are derived by Suriyakat et al. (2010). The authors used an integral equation approach and derived a Fredholm integral equation of second type for the *ARL*. The explicit formulas obtained by solving the integral equations are:

$$ARL_0 = 1 - \frac{\lambda e^{\frac{(1-\lambda)u}{\lambda}}\left(e^{-\frac{B}{\lambda}}-1\right)}{\lambda e^{-(a+b+\varphi v)} + e^{-B} - 1} \tag{8}$$

and

$$ARL_1 = 1 - \frac{\lambda e^{\frac{(1-\lambda)u}{\lambda\alpha}}\left(e^{-\frac{B}{\lambda\alpha}}-1\right)}{\lambda e^{-\frac{a+b+\varphi v}{\alpha}} + e^{-\frac{B}{\alpha}} - 1} \tag{9}$$

where a is a constant, b is the trend slope in term of t, φ is the autoregressive coefficient $(0<\varphi<1)$, α is a parameter of the exponential distribution, λ is a smoothing parameter, u, v are initial values, and B is boundary value.

IV. Numerical Comparisons of Performance

We present an explicit formula for EWMA control chart for trend stationary exponential AR(1) processes. The numerical results for ARL_0 and ARL_1 for EWMA control chart for trend stationary exponential AR(1) processes were calculated from Eq. (8) and (9). To evaluate the characteristic of a control chart for monitoring trend stationary exponential AR(1) processes, we designed trend stationary AR(1) processes with levels of autocorrelation ranging from low to high, with numerical parameters $\varphi = 0.1, 0.3, 0.5, 0.7, 0.9, a = 0, b = 0.5$ and $Q_0 = v = 0.1$. The characteristics of the control charts measured in terms of

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ARL are examined for different values of shifts in the mean $\alpha = 1.1, 1.2, 1.3, 1.4, 1.5$ and $R_0 = u = 0.1$.

In Table 1-2, we compare the numerical results obtained by explicit formulas with the numerical results via integral equations method. Both methods gives ARL_0 for EWMA control chart for trend stationary exponential AR(1) processes with parameter value $\lambda = 0.3$ and 0.5. The explicit formulas give results which are very closed to the numerical integral equations. Notice that, calculations with explicit formula equation (8) and (9) are simple and considerable much faster from the point of view of computation times. For example, if we set φ=0.1, calculations time based on our technique takes less than 1 sec., while the CPU time required to obtain numerical solutions of integral equation for the EWMA run, show inside the brackets is 50-60 times larger.

TABLE 1 ARLS RESULTS OF EWMA CONTROL CHART FOR TREND STATIONARY EXPONENTIAL AR(1) PROCESSES, THE ENTRIES INSIDE THE PARENTHESES ARE THE CPU TIMES IN SECONDS

$\lambda = 0.3$		ARLs	
φ	B	Explicit Form	Integral Equation
0.1	0.178	10.919656	10.919655 (61.298)
	0.188	21.129169	21.129168 (70.648)
	0.198	353.993108	353. 993016 (71.647)
0.3	0.174	10.878219	10.878218 (68.992)
	0.184	21.391719	21.391718 (65.909)
	0.194	754.599882	754. 599488 (74.635)
0.5	0.170	10.784099	10.784099 (66.630)
	0.180	21.457852	21.457851 (56.465)
	0.190	4033.589724	4033. 579110 (53.810)
0.7	0.165	10.131421	10.131420 (55.790)
	0.175	19.369793	19.369792 (55.221)
	0.185	229.850853	229. 850818 (54.773)
0.9	0.161	9.944430	9.944430 (53.071)
	0.171	19.048355	19.048354 (52.705)
	0.181	234.960576	234. 960542 (58.294)

TABLE **2** ARLS RESULTS OF EWMA CONTROL CHART FOR TREND STATIONARY EXPONENTIAL AR(1) PROCESSES, THE ENTRIES INSIDE THE PARENTHESES ARE THE CPU TIMES IN SECONDS

$\lambda = 0.5$		ARLs	
φ	B	Explicit Form	Integral Equation
0.1	0.349	50.217201	50.217195 (69.967)
	0.353	100.052337	100. 052311 (68.424)
	0.357	13901.162634	13900. 677016 (65.623)
0.3	0.339	41.099995	41.099991 (80.982)
	0.343	70.609763	70.609751 (93.709)
	0.348	691.350539	691. 349398 (78.200)
0.5	0.332	45.969944	45.969939 (67.885)
	0.336	88.277837	88.277819 (72.331)
	0.340	1112.766716	1112. 763908 (76.580)
0.7	0.324	44.827491	44.827486 (70.607045)
	0.328	86.047477	86.047461 (74.649149)
	0.332	1073.303231	1073. 300749 (72.327185)
0.9	0.316	42.598448	42.598444 (76.191124)
	0.320	79.851980	79.851966 (80.355842)
	0.324	637.730463	637. 729630 (73.178844)

In Tables 3-4, we compare the numerical results obtained by explicit formulas with results by integral equations were both gives ARL_0 and ARL_1 for EWMA control chart for trend stationary exponential AR(1) processes with parameter value $\lambda = 0.3$ and 0.5. It is obvious that explicit formula gives numerical results which are very closed to the numerical integral equations method, and the calculations via Eq. (8) and (9) are simple with a faster computational times. As an illustration, if the boundary value is equal to 0.198 and $\varphi = 0.1$ for $\alpha = 1.5$, then evaluation time of $ARL_1 = 2.388221$, based on our technique takes less than 1 sec. CPU time, while the CPU time required for numerical integral equation for the EWMA run show in the brackets, is significant more larger.

TABLE 3 ARLS RESULTS OF EWMA CONTROL CHART FOR TREND STATIONARY EXPONENTIAL AR(1) PROCESSES, THE ENTRIES INSIDE THE PARENTHESES ARE THE CPU TIMES IN SECONDS

$\lambda = 0.3$		ARLs	
(φ, B)	α	Explicit Form	Integral Equation
$(0.1, 0.198)$	1.0	353.993107	353. 993016 (71.647)
	1.1	7.981882	7.981882 (65.589)
	1.2	4.501575	4.501575 (65.530)
	1.3	3.328658	3.328658 (74.410)
	1.4	2.740849	2.740849 (65.885)
	1.5	2.388221	2.388221 (68.786)
$(0.3, 0.194)$	1.0	754.599882	754. 599488 (74.635)
	1.1	7.982314	7.982314 (65.724)
	1.2	4.480692	4.480692 (65.964)
	1.3	3.309202	3.309202 (64.527)
	1.4	2.723826	2.723826 (82.486)
	1.5	2.373255	2.373255 (77.150)
$(0.5, 0.190)$	1.0	4033.589724	4033. 579110 (53.810)
	1.1	7.959805	7.959805 (71.019)
	1.2	4.453766	4.453766 (69.190)
	1.3	3.286985	3.286985 (44.998)
	1.4	2.705217	2.705217 (52.664)
	1.5	2.357254	2.357254 (49.904)

TABLE 4 ARLS RESULTS OF EWMA CONTROL CHART FOR TREND STATIONARY EXPONENTIAL AR(1) PROCESSES, THE ENTRIES INSIDE THE PARENTHESES ARE THE CPU TIMES IN SECONDS

$\lambda = 0.5$		ARLs	
(φ, B)	α	Explicit Form	Integral Equation
$(0.5, 0.340)$	1.0	1112.766716	1112. 763908 (76.580)
	1.1	7.754489	7.754489 (44.399)
	1.2	4.385658	4.385658 (54.231)
	1.3	3.258533	3.258533 (57.771)
	1.4	2.694310	2.694310 (52.070)
	1.5	2.355613	2.355613 (65.627)
$(0.7, 0.332)$	1.0	1073.303231	1073. 300749 (72.327)
	1.1	7.657627	7.657627 (50.935)
	1.2	4.335042	4.335042 (51.449)
	1.3	3.223493	3.223493 (50.188)
	1.4	2.667173	2.667173 (49.755)
	1.5	2.333297	2.333297 (52.589)
$(0.9, 0.324)$	1.0	637.730463	637. 729630 (73.178)
	1.1	7.534325	7.534325 (52.346)
	1.2	4.277649	4.277649 (52.219)
	1.3	3.185384	3.185384 (54.057)
	1.4	2.638281	2.638281 (52.111)
	1.5	2.309840	2.309840 (52.795)

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V. Conclusion

We derived simple explicit formulas for ARL_0 and ARL_1 for EWMA control chart for trend stationary exponential AR(1) processes. Our new results are easy to be implemented in any computer program. The performance of the analytical results for ARL_0 and ARL_1 compared with the numerical integral equation, shows a considerably reduction in the CPU time. This make it useful in practical applications to design optimal EWMA control charts.

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