Common Fixed Point Theorems in TVS-Valued Cone Metric Spaces

Akbar Azam and B.E. Rhoades

Abstract— **We obtain common fixed points of a pair of mappings satisfying a generalized contractive type condition in TVS valued cone metric spaces. Our results generalize some well-known recent results in the literature.**

Index Terms— **contractive type mapping, cone metric space fixed point ,non-normal cones**

I. Introduction

Huang and Zhang [8] generalized the notion of metric space by replacing the set of real numbers by ordered Banach space and defined cone metric space and extended Banach type fixed point theorems for contractive type mappings. Subsequently, some other authors [1,4,5,7,10,12,13,14,15,17] studied properties of cone metric spaces and fixed points results of mappings satisfying contractive type condition in cone metric spaces. Recently Beg, Azam and Arshad [6], introduced and studied topological vector space(TVS) valued cone metric spaces which is bigger than that of introduced by Huang and Zhang [8]. TVS valued cone metric spaces were further considered by some other authors in [3,9,11,16,18]. In this paper we obtain common fixed points of a pair of mappings satisfying a generalized contractive type condition without the assumption of normality in TVS-valued cone metric spaces. Our results improve and generalize some significant recent results.

II. Preliminaries

We need the following definitions and results, consistent with [3,6] .

Let (E,τ) be always a topological vector space (TVS) and P a subset of E . Then, P is called a cone whenever :

(i) P is closed, non-empty and $P \neq \{0\}$,

(ii) $ax+by \in P$ for all $x, y \in P$ and non-negative real numbers a, b ,

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(iii) If x belongs to both P and $-P$, then $x=0$

We shall always assume that the cone P has a nonempty interior int P (such cones are called solid). Each cone P induces a partial ordering $\leq$ on E by $x \leq y$ if and only if $y - x \in P$, $x < y$ will stand for $x \leq y$ and $x \neq y$, while $x << y$ will stand for $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P .

Definition 1 Let X be a non-empty set. Suppose the mapping $d : X \times X \to E$ satisfies:

$(\mathbf{d}_1)$ $0 \leq d(x,y)$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y,$

$(\mathbf{d}_2)$ $d(x,y) = d(y,x)$ for all $x, y \in X,$

$(\mathbf{d}_3)$ $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X.$

Then d is called a TVS-valued cone metric on X and (X,d) is called a TVS-valued cone metric space. If E is a real Banach space then (X,d) is called cone metric space [8].

Definition 2 Let (X,d) be a TVS-valued cone metric space, $x \in X$ and $\{x_n\}_{n\geq 1}$ a sequence in X . Then

(i) $\{x_n\}_{n\geq 1}$ converges to x whenever for every $c \in E$ with $0 << c$ there is a natural number N such that $d(x_n, x) << c$ for all $n \geq N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x.$

(ii) $\{x_n\}_{n\geq 1}$ is a Cauchy sequence whenever for every $c \in E$ with $0 << c$ there is a natural number N such that $d(x_n, x_m) << c$ for all $n, m \geq N$.

(iii) (X,d) is a complete cone metric space if every Cauchy sequence is convergent.

Note that the results concerning fixed points and other results, in the case of cone metric spaces with non-normal solid cones, cannot be proved by reducing to metric spaces, because in this case neither of the conditions from Lemma 2.6 [9] holds. Further, the vector valued function cone metric is not continuous in the general case, i.e., from

$x_n \to x, y_n \to y$ it need not follow that $d(x_n, y_n) \to d(x, y)$ (see [9]).

III. Main Results

The following theorem improves/generalizes the results in [2, 8, and 14]

Theorem 3 *Let* (X, d) *be a complete TVS-valued cone metric space,* P *be a solid cone and* $0 < h < 1$ *. If the mappings* $S, T : X \to X$ *satisfy:*

$$d(Sx, Ty) \le hu(x, y) \qquad (1)$$

for all $x, y \in X$ *, where*

$$u(x, y) \in \left\{ d(x, y), d(x, Sx), d(x, Ty), \frac{d(y, Tx) + d(x, Sy)}{2} \right\},$$

then S *and* T *have a unique common fixed point.*

Proof We shall first show that fixed point of one map is a fixed point of the other. Suppose that $p = Tp$ and that $p \ne Sp.$ Then from (1)

$$d(Sp, p) = d(Sp, Tp) \le hu(p, p).$$

Case I.

$$d(Sp, p) \le hd(p, p) = 0, \text{ and } p = Sp.$$

Case II.

$$d(Sp, p) \le hd(p, Sp),$$

which, since we have assumed that

$$p \ne Sp \text{ yields } p = Sp,$$

Case III.

$$d(Sp, p) \le hd(p, Tp) = 0, \text{ and } p = Sp$$

Case IV.

$$d(Sp, p) \le h\left[\frac{d(p, Tp) + d(p, Sp)}{2}\right],$$

which implies that $d(Sp, p) \le \frac{h}{2} d(p, Sp),$ and hence $p = Sp$. In a similar manner it can be shown that any fixed point of S is also the fixed point of T . Let $x_0 \in X.$ Define

$$x_{2n+1} = Tx_{2n}$$
$$x_{2n+2} = Sx_{2n+1}, \quad n \ge 0.$$

We shall assume that $x_n \ne x_{n+1}$ for each $n.$ For, suppose there exists an n such that $x_{2n} = x_{2n+1}.$ Then $x_{2n} = Tx_{2n}$ and x_{2n} is a fixed point of T , hence a fixed point of $S.$ Similarly, if $x_{2n+1} = x_{2n+2}$ for some n , then x_{2n+1} is common fixed point of S and $T.$
From (1)

$$d(x_{2n}, x_{2n+1}) \le hud(x_{2n-1}, x_{2n}).$$

Case I.

$$d(x_{2n}, x_{2n+1}) \le hd(x_{2n-1}, x_{2n}).$$

Case II.

$$d(x_{2n}, x_{2n+1}) \le hd(x_{2n-1}, Sx_{2n-1}) \le hd(x_{2n-1}, x_{2n}).$$

Case III.

$$d(x_{2n}, x_{2n+1}) \le hd(x_{2n}, Tx_{2n}) = hd(x_{2n}, x_{2n+1}),$$

which implies that

$x_{2n} = x_{2n+1},$ a contradiction to our assumption.

Case IV.

$$d(x_{2n}, x_{2n+1}) \le \frac{h}{2}\left[d(x_{2n-1}, Tx_{2n}) + d(x_{2n}, Sx_{2n-1})\right]$$
$$= \frac{h}{2} d(x_{2n-1}, x_{2n}) + \frac{1}{2} d(x_{2n}, x_{2n+1}),$$

which implies that

$$d(x_{2n}, x_{2n+1}) \le hd(x_{2n-1}, x_{2n}).$$

Thus in all cases, we have $d(x_{2n}, x_{2n+1}) \le hd(x_{2n-1}, x_{2n}).$ Similarly, one can show that $d(x_{2n+1}, x_{2n+2}) \le hd(x_{2n}, x_{2n+1}).$ Thus, for all n ,

$$d(x_n, x_{n+1}) \le hd(x_{n-1}, x_n) \le h^2 d(x_{n-2}, x_{n-1}) ... \le h^n d(x_0, x_1)$$

Now for any $m > n,$

$$d(x_m, x_n) \le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m-1}, x_m)$$
$$\le \left[\lambda^n + \lambda^{n+1} + ... + \lambda^{m-1}\right] d(x_0, x_1)$$
$$\le \left[\frac{\lambda^n}{1-\lambda}\right] d(x_0, x_1).$$

Let $\mathbf{0} << c$ be given, choose a symmetric neighborhood V of 0 such that $c + V \subseteq \mathrm{int}P$. Also, choose a natural number N_1 such that $\left[\frac{\lambda^n}{1-\lambda}\right] d(x_0, x_1) \in V$, for all $n \ge N_1$. Then, $\frac{\lambda^n}{1-\lambda} d(x_1, x_0) << c$, for all $n \ge N_1$.

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Thus,

$$d(x_m,\ x_n) \le \left[\frac{\lambda^n}{1-\lambda}\right] d(x_0, x_1) << c,$$

for all $m > n$. Therefore, $\{x_n\}_{n\ge1}$ is a Cauchy sequence. Since X is complete, there exists $u \in X$ such that $x_n \to u.$ In case I, choose a natural number N_2 such that

$$d(x_n, u) << \frac{c}{2} \text{ for all } n \ge N_2.$$

Then for all $n \ge N_2$

$$\begin{aligned} d(u,Tu) &\le d(u, x_{2n+2}) + d(x_{2n+2}, Tu) \\ &\le d(u, x_{2n+2}) + d(Sx_{2n+1}, Tu) \\ &\le d(u, x_{2n+2}) + hd(x_{2n+1},\ u) \\ &\le d(u,\ x_{2n+2}) + d(x_{2n+1}, u) << \frac{c}{2} + \frac{c}{2} = c. \end{aligned}$$

For case II choose a natural number N_3 such that for all $n \ge N_3$

$$d(x_{n+1}, x_n) << \frac{c}{2} \text{ and } d(x_{n+1}, u) << \frac{c}{2}.$$

Then for all $n \ge N_3$

$$\begin{aligned} d(u,Tu) &\le d(u, x_{2n+2}) + d(x_{2n+2}, Tu) \\ &\le d(u, x_{2n+2}) + d(Sx_{2n+1}, Tu) \\ &\le d(u, x_{2n+2}) + hd(x_{2n+1},\ x_{2n+2}) \\ &\le d(u, x_{2n+2}) + d(x_{2n+1}, x_{2n+2}) << \frac{c}{2} + \frac{c}{2} = c. \end{aligned}$$

In case III choose a natural number N_4 such that

$$d(x_{n+1}, u) << c(1-h) \text{ for all } n \ge N_4.$$

Hence, for all $n \ge N_4$

$$\begin{aligned} d(u,Tu) &\le d(u, x_{2n+2}) + d(x_{2n+2}, Tu) \\ &\le d(u, x_{2n+2}) + d(Sx_{2n+1}, Tu) \\ &\le d(u, x_{2n+2}) + hd(u, Tu) \\ &\le \frac{1}{1-h} d(u, x_{2n+2}) << c. \end{aligned}$$

In case IV choose a natural number N_5 such that

$$d(x_{n+1}, u) << \frac{c(1-h)}{3} \text{ for all } n \ge N_5.$$

Then for all $n \ge N_5$

$$\begin{aligned} d(u,Tu) &\le d(u, x_{2n+2}) + d(x_{2n+2}, Tu) \\ &\le d(u, x_{2n+1}) + d(Sx_{2n+1}, Tu) \\ &\le d(u, x_{2n+1}) + h[d(u, Sx_{2n+1}) + d(x_{2n+1}, Tu)] \\ &\le d(u, x_{2n+1}) + h[d(x_{2n+2}, u) + d(x_{2n+1}, u) + d(u, Tu)] \\ &\le \frac{1}{1-h}[d(u, x_{2n+1}) + h(d(x_{2n+2}, u) + d(x_{2n+1}, u))] \\ &<< \frac{c}{3} + \frac{c}{3} + \frac{c}{3} = c. \end{aligned}$$

Thus

$$d(u,Tu) << \frac{c}{m}, \text{for all } m \ge 1,$$

so $\frac{c}{m} - d(u,Tu) \in P,$ for all $m \ge 1.$ Since $\frac{c}{m} \to \mathbf{0}\ (as\ m \to \infty)$ and P is closed $,\ -d(u,Su) \in P.$ But $d(u,Tu) \in P.$ Therefore, $d(u,Tu) = \mathbf{0},$ and $u = Tu.$ Similarly, by using

$$d(u,Tu) \le d(u, x_{2n+1}) + d(x_{2n+1}, Tu),$$

we can show that $u = Tu,$ which implies that u is a common point of S and T.

Example 4 *Let* $X = [0,1]$ *and* E *be the set of all real valued functions on* X *which also have continuous derivatives on* X. *Then* E *is a vector space over* R *under the following operations:*

$$(x+y)(t) = x(t) + y(t),\ \ (\alpha x)(t) = \alpha x(t),$$

for all $x, y \in E, \alpha \in \mathrm{R}.$ *Let* τ *be the strongest vector (locally convex) topology on* E. *Then* (X, τ) *is a topological vector space which is not normable and is not even metrizable. Define* $d\ :\ X \times X \to E$ *as follows:*

$$(d(x,y))(t) = |x-y|3^t,$$

$$P = \{(x \in E\ :\ x \ge 0\}.$$

Then (X,d) *is a TVS-valued cone metric space. Let* $0 < h < 1$ *and* $S\ ,T\ :\ X \to X$ *be such that*

$$|Sx - Ty| \le hu|x-y|$$

for all $x, y \in X$ *, where* $u|x-y| \in \left\{|x-y|, |x-Sx|, |x-Ty|, \frac{|y-Tx|+|x-Sy|}{2}\right\}.$ *Then* $S\ ,T$ *satisfies all conditions of the above theorem.*

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Theorem 5 *Let* (X,d) *be a complete TVS-valued cone metric space,* P *be a solid cone and* $0<h<1$ *. If a mapping* $T : X \to X$ *satisfies:*

$$d(Tx,Ty) \le hu(x,y)$$

for all $x,y \in X$ *, where*

$$u(x,y) \in \left\{ d(x,y), d(x,Tx), d(x,Ty), \frac{d(y,Tx)+d(x,Ty)}{2} \right\},$$

then T *has a unique fixed point.*

Corollary 6 *Let* (X,d) *be a complete cone metric space,* P *be a solid cone and* $0<h<1$ *. If a mapping* $S,T : X \to X$ *satisfies:*

$$d(Sx,Ty) \le hu(x,y)$$

for all $x,y \in X$ *, where*

$$u(x,y) \in \left\{ d(x,y), d(x,Sx), d(x,Ty), \frac{d(y,Tx)+d(x,Sy)}{2} \right\},$$

then S *and* T *have a unique common fixed point.*

The following example shows that the above corollary is an improvement and a real generalization of results [8, Theorems 1, 3, 4] and [5, Theorems 2.3, 2.6, 2.7].

Example 7 *Let* $X=\{1,2,3\}$, $E=R^2$ *and* $P=\{(x,y)\in E : x,y \ge 0\}$. *Define* $d : X \times X \to \mathrm{R}^2$ *as follows:*

$$d(x,y) = \begin{cases} (0,0) & \text{if } x=y \\ (\frac{4}{7},\frac{2}{7}) & \text{if } x \ne y \text{ and } x,y \in X-\{2\} \\ (1,\frac{1}{2}) & \text{if } x \ne y \text{ and } x,y \in X-\{3\} \\ (\frac{1}{2},\frac{1}{4}) & \text{if } x \ne y \text{ and } x,y \in X-\{1\}. \end{cases}$$

Define a mapping $T : X \to X$ *as follows:*

$$T(x) = \begin{cases} 3 & \text{if } x \ne 2 \\ 1 & \text{if } x = 2. \end{cases}$$

Note that

$$d(T(3),T(2)) = d(3,1) = (\frac{4}{7},\frac{2}{7})$$

(i). For $\alpha < 1,$ *we have*

$$\alpha d(3,2) < d(T(3),T(2))$$

(ii). For $\beta < \frac{1}{2}$

$$\beta[d(3,T(3)) + d(2,T(2))] < d(T(3),T(2))$$

(iv). For $\gamma < \frac{1}{2}$,

$$\gamma[d(3,T(2)) + d(2,T(3))] < d(T(3),T(2)).$$

Therefore the results in [8] and [14] are not applicable to obtain fixed point of T.

In order to apply the above corollary consider the mapping $Sx=3$ *for each* $x \in X$. *Then,*

$$d(Sx,Ty) = \begin{cases} (0,0) & \text{if } y \ne 2 \\ (\frac{4}{7},\frac{2}{7}) & \text{if } y = 2 \end{cases}$$

and for $h=\frac{4}{7}$

$$hd(y,Ty) = \left(\frac{4}{7},\frac{2}{7}\right) \text{ if } y=2.$$

It follows that S *and* T *satisfy all conditions of the above corollary and we obtain* $T(3)=3=S(3)$.

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