Investigations on Unit Distance Property of Clebsch Graph and Its Complement

Pratima Panigrahi and Uma kant Sahoo *‡

Abstract—**An n-dimensional unit distance graph is a simple graph which can be drawn on n-dimensional Euclidean space $\mathbb{R}^n$ so that its vertices are represented by distinct points in $\mathbb{R}^n$ and edges are represented by closed line segments of unit length. In this paper we show that the Clebsch graph is 3-dimensional unit distance graph, but its complement is not.**

Keywords: ***unit distance graph, strongly regular graphs, Clebsch graph, Petersen graph.***

In this article we consider only simple graphs, i.e. undirected, loop free and with no multiple edges. The study of dimension of graphs was initiated by Erdos et.al [3]. An *n-dimensional unit distance graph* is a simple graph which can be drawn on n-dimensional space $\mathbb{R}^n$ so that its vertices are represented by distinct points in $\mathbb{R}^n$ and edges are represented by closed line segments of unit length. Unit distance property of graphs have been studied by many authors in [4],[6],[7], [8], [9], and [10]. However most of them have discussed on 2-dimensional unit distance graphs only. In [1], Bagchi et. al. have given all possibilities for 2-dimensional unit distance strongly regular graphs. Recall that a *strongly regular graph* with parameters (v, k, λ, μ) is a regular graph of degree k on v vertices such that any two adjacent vertices have exactly λ common neighbours, while any two (distinct but) non-adjacent vertices have exactly μ common neighbours. Strongly regular graphs are of great importance in finite geometry. From the results of Bagchi et. al. [1] one finds that Clebsch graph and its complement are not 2-dimensional unit distance graphs. Clebsch graph and its complement are strongly regular graphs on parameters (16,5,0,2) and (16,10,6,6) respectively. These graphs are known to be unique in the respective parameters [2]. In this paper we give unit distance representation of Clebsch graph in the 3-dimensional Euclidean space $\mathbb{R}^3$. Also we show that the complement of Clebsch graph is not a 3-dimensional unit distance graph.

The Petersen graph is the strongly regular graph on parameters (10,3,0,1). This graph is also unique in its parameter set. It is known that Petersen graph is 2-dimensional unit distance graph (see [1],[3],[10]). It is also known that Petersen graph is a subgraph of Clebsch graph, see [[5], section 10.6].

Theorem 1: *The Clebsch graph has a 3-dimensional unit distance structure.*

Proof: To investigate the three dimensional unit distance structure of the Clebsch graph we need to realize it as a super graph of the Petersen graph, see Figure1 and Figure2, which consists of a regular pentagon and a concentric regular pentagram, with the vertices of the pentagon joined to the corresponding vertices of the pentagram by five bridges.

If both the pentagon and the pentagram are scaled to have unit sides, and then the plane of the pentagram is lifted up perpendicular to the plane of the pentagon by $h = \sqrt{1-(R-r)^2}$, where R and r are the distances of the vertices of the pentagon and the pentagram respectively from their corresponding centers, then a three dimensional unit distance representation of Petersen graph results.

Imagine that the pentagon of the Petersen graph i.e. 1-2-3-4-5 is on the XY plane with its center as the origin $O(0,0,0)$. Point 1 lies on the Y axis. The plane of the pentagram lies h units above the XY plane and is perpendicular to the Z axis. The center of this pentagram is $(0,0,h)$. Point 6 lies on YZ plane. All other points of

*Manuscript received March 23, 2012; revised April 03, 2012.
†Pratima Panigrahi is with the Department of Mathematics, Indian Institute of Technology, Kharagpur, WB, 721302, India E-mail: pratima@maths.iitkgp.ernet.in
‡Uma kant Sahoo is with School of Computer Application, KIIT University, Bhubaneswar-751021, India Email:umakant.iitkgp@gmail.com

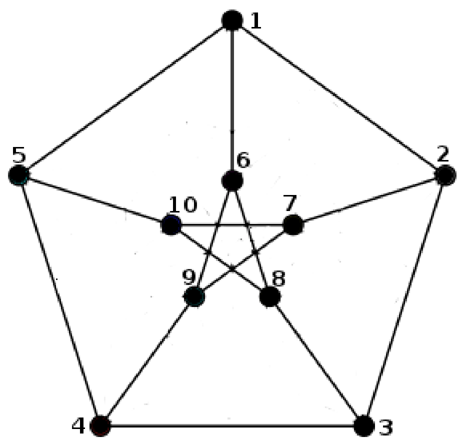

Figure 1: Petersen graph

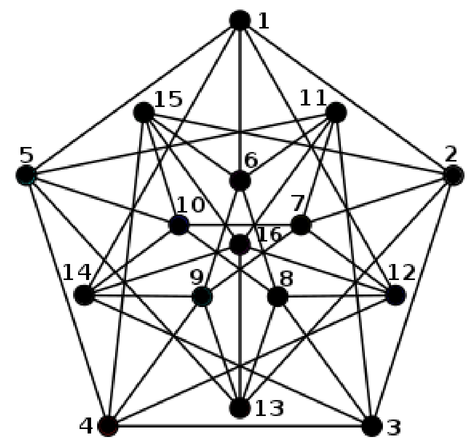

Figure 2: Clebsch graph

the Petersen graph are drawn accordingly. Table I shows the coordinates of the Petersen graph embedded in the Clebsch graph.

Next we add five more points i.e. $11, 12, 13, 14$ and 15 to the 3-dimensional unit distance Petersen graph, 2. Note that point 11 lies at a unit distance from points $3, 5, 6, 7$. It is easy to see that points $3, 5, 6, 7$ form a cyclic quadrilateral. In order to find coordinates of point 11, we first find the circum center of the cyclic quadrilateral 3-5-6-7 and then move it along the perpendicular to the plane containing 3-5-6-7 till it is at unit distance from points $3, 5, 6$ and 7. Now the circum center can be moved either above or below the plane. In this construction we have moved it above the plane. Similar calculations are done to find coordinates of points $12, 13, 14$ and 15. Due to symmetry, points $11, 12, 13, 14$ and 15 lie on the same plane which is parallel to XY plane. Also these points form a regular pentagon structure with center on Z axis. So from the z-coordinate of either of the points $11, 12, 13, 14$ or 15, we can find the coordinates of the center of regular pentagon 11-12-13-14-15. Now on moving this point along the Z axis till it is at distance 1 from points $11, 12, 13, 14$ and 15. Hence we get the location of point 16. Note that here also the center of 11-12-13-14-15 can be moved in two directions, upwards or downwards. In this construction, it is moved upwards. Hence we have a unit distance structure of Clebsch graph in three dimensions.

Table I: Coordinates of Petersen graph embedded in Clebsch graph

Point label	Co-ordinates
1	$(0, R, 0)$
2	$(Rcos\frac{\pi}{10}, Rsin\frac{\pi}{10}, 0)$
3	$(Rcos\frac{3\pi}{10}, -Rsin\frac{3\pi}{10}, 0)$
4	$(-Rcos\frac{3\pi}{10}, -Rsin\frac{3\pi}{10}, 0)$
5	$(-Rcos\frac{\pi}{10}, Rsin\frac{\pi}{10}, 0)$
6	$(0, r, h)$
7	$(rcos\frac{\pi}{10}, rsin\frac{\pi}{10}, h)$
8	$(rcos\frac{3\pi}{10}, -rsin\frac{3\pi}{10}, h)$
9	$(-rcos\frac{3\pi}{10}, -rsin\frac{3\pi}{10}, h)$
10	$(-rcos\frac{\pi}{10}, rsin\frac{\pi}{10}, h)$

Next to prove that the complement of Clebsch graph doesn't have a three dimensional unit distance structure we use the lemma below. In fact this lemma gives a general property of points in 3-dimensional space and hence this may be useful for the study of unit distance property of other graphs.

Lemma 1: *Given any three distinct points in $\mathbb{R}^3$ there are at most two points at unit distance from all of them.*

Proof: First let us take any two distinct points a_1 and a_2 in $\mathbb{R}^3$. Let S_1 and S_2 be unit spheres with

ISBN: 978-988-19251-3-8
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

centers a_1 and a_2 respectively. Points at a unit distance from a_1 and a_2, if they exist, lie on a circle C_1 in the perpendicular plane, bisecting the line between them. Now lets consider another distinct point a_3 in $\mathbb{R}^3$. Points unit distance away from a_3 lie on a unit sphere S_3 with it as the center. If a_3 has common neighbours with a_2 then S_3 intersects S_2 at the most in a circle, say C_2. Circles C_1 and C_2 intersects at the most two points and so the result of the Lemma .

Theorem 2: *The complement of Clebsch graph does not have a 3-dimensional unit distance structure.*

Proof: Let G be the complement of the Clebsch graph. So the vertex set of G is $\{1,2,\ldots, 16\}$ (same as the vertex set of Clebsch graph in Figure 2), and two of them are adjacent in G if and only if they are not adjacent in the Clebsch graph. Therefore, in G vertex 2 is not adjacent with 1 and 15 both, where as 1 and 15 are adjacent to each other. Next we use the fact that G is a strongly regular graph on parameters (16,10,6,6). Let A be the set of nine neighbours of 1 other than 15. Let B and C be the set of common neighbours of 1 and 15; and, 1 and 2 respectively. Then $|B| = |C| = 6$, $B \subset A$ and $C \subset A$. One checks that $|B \bigcap C| \geq 3$ i.e. we get at least three common neighbours of vertices 1, 2 and 15. If G would have been a 3- dimensional unit distance graph then we would get in $\mathbb{R}^3$ at least three vertices at unit distance from 1, 2 and 15 (where the vertices 1, 2 and 15 are also in $\mathbb{R}^3$), which contradicts to Lemma 1.

References

[1] Bagchi, B., Panigrahi, P., Sahoo, U. K.: On the strongly regular unit distance graphs, JCMCC, accepted.

[2] Colbourn, C. J., Dinitz, J. H., Hand book of combinatorial designs, CRC Press, (2007).

[3] Erdos, P., Harary, F., Tutte, W. T.: On the dimension of a graph, Mathematika, 12, 118-122 (1965).

[4] Gervacio, S. V., Lim, Y. F., Maehara, H.: Planar unit-distance graphs having planar unit-distance complement, Discrete Math. 308, 19731784, doi:10.1016/j.disc.2007.04.050, (2008).

[5] Godsil, C., Royle, G.: Algebraic Graph Theory, Springer Verlag, (2001).

[6] Horvat, B., Pisanski, T.: Products of unit distance graphs, Discrete Math., 310, 1783-1792 (2010).

[7] Maehara, H. Distances in a rigid unit-distance graph in the plane, Discrete Appl. Math. 31, 193170, doi:10.1016/0166-218X(91)90070-D, (1991).

[8] Maehara, H. Rdl, V.: On the dimension to represent a graph by a unit distance graph, Graphs Combin., 6, 365177, doi:10.1007/BF01787703, (1990).

[9] Spencer, J., Szemerdi, E., Trotter, W. T.: Unit distances in the Euclidean plane, Graph Theory and Combinatorics, Academic Press, 293178, (1984).

[10] Zitnik, A., Horvat, B., Pisanski, T.: All generalized Petersen graphs are unit-distance graphs, J. Korean Math. Soc., 49, 475-491, (2012).

ISBN: 978-988-19251-3-8
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)