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Fluid Dynamics of Nonaqueous Phase Contaminants in Groundwater: Analytical Solutions and Analogy with Zhukovsky's Trochoid

Obnosov Yu.V., Kacimov A.R.,

***Abstract*—An exact solution to a free-boundary, potential, 2-D flow of a Darcian fluid (mathematically equivalent to flow of a heavy irrotational ideal fluid) past a barrier is obtained by the theory of holomorphic functions. A volume of liquid contaminant contrasting in density with the ambient flowing groundwater makes a lens attached to the stoss or lee side of the barrier. The shape of the interface morphs in response to a pressure-velocity field in the dynamic and static liquid phases. The flow net and interface are plotted from explicit expressions found for the complex potential and complex velocity. As a particular case, we obtain a famous Zhukovsky's gas-bubble contour belonging to the class of trochoids. Serious caveats for remediation projects and artificial recharge of groundwater are inferred: more intensive descending seepage of ponded surface water through a heterogeneous aquifer may worsen the groundwater quality, contrary to what would occur in homogeneous porous media.**

***Index Terms*— environmental engineering, Darcian flow, groundwater contamination and clean-up, free surface, holomorphic functions, conformal mappings, hodograph transform.**

I. INTRODUCTION

Light and dense nonaqueous phase liquids (so-called LNAPLs-DNAPLs, e.g. crude oil, diesel, trichloroethylene, bitumen, [1]) contaminate groundwater/aquifers/vadose zone soil and require costly and protracted environmental engineering techniques in remediation, monitoring and controls (e.g., [2]). Often, NAPLs in a porous/fractured subsurface make macrovolumes (lenses, hydrocarbon traps, ganglia, blobs, streaks, etc. [3]-[6]) separated from groundwater by sharp interfaces. The locus and shape of the interface between two liquid-saturated domains of contrasting density depends on the ambient groundwater flow and the geological boundaries (e.g., bed-,cap-rock unconformities, troughs, bulges, anticline flanks, etc.). Detection of "alien entities" obstructing flow in such aquifers due to their contrast in texture with the main porous medium is problematic due to imprecision of geophysical tools and limited core samples in study areas. Simulation based on analytical and numerical models of multiphase flow and transport in the subsurface [7] is a vital step in many surveys, containment projects and engineering measures aimed at decontamination by pump-and-treat, air-sparging, permeable reactive barriers, LNAPL skimming, DNAPL scraping, etc.

In some NAPL-treatment approaches an intensive groundwater gradient is artificially generated by injection of fresh water through wells or infiltration ponds. In these hydrodynamically agitated clean-up schemes, NAPL can be flushed away or the interface can take a non-trivial (even counter-intuitive) shape. In this paper we study analytically the case of a NAPL attached as a static macrovolume to a solid barrier (an impermeable "alien" inclusion, e.g. a salt dome, constructed silo or tunnel) with a strong influence of the conjugated dynamic groundwater. We utilize the theory of holomorphic functions, in particular, conformal mappings and the hodograph method to analytically find the shape of NAPL interfaces, whose very existence is puzzling.

II. MATHEMATICAL MODEL

We consider a homogeneous porous matrix of hydraulic conductivity k with an impermeable barrier baffling a descending fully saturated Darcian seepage with velocity V_0 far away from the obstacle (Fig.1a presents a vertical cross-section).

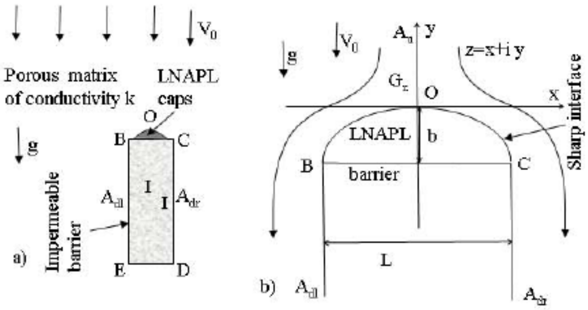

Fig.1 Vertical cross-section of a 2-D barrier with a cap of liquid contaminant

A curve-shaped cap BOC of LNAPL (Fig.1a) sits on the flat horizontal roof BC of the baffle as a static lens, morphed by the ambient seepage flow. The flowing water pushes the LNAPL volume to the stoss side of the barrier. If

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Obnosov Yurii is with Kazan Federal University, Russia. Tel: Tel: 007 (8432) 315 278 (e-mail: yobnosov@ksu.ru).

Kacimov Anvar is with the Sultan Qaboos University, Oman e-mail: anvar@squ.edu.om).

no ambient flow in Fig.1a, then LNAPL would rise to the regional water table, upwelled by the Archimedian force.

If V_0 is large enough the cap would be flushed downward, similarly to what is well-known for the so-called "hydrodynamic" hydrocarbon traps in anticlines [5]. Consequently, for both large and small incident velocities we have a standard problem of ideal fluid flow past a rectangular barrier in Fig.1a, tractable by conformal mappings [8]. The lens BOC in Fig.1 makes the problem more complicated.

We consider a near-field of the cap in Fig.1a i.e. assume that sufficiently close to the barrier edges, the flow becomes as uniform as upstream of the cap. LNAPL is separated from groundwater by a sharp interface, i.e. a capillary fringe is ignored. Our objective is to find the shape of BOC and flow characteristics, depending on the incident velocity, densities of groundwater, ρ_w, and LNAPL, $\rho_o, \rho_o < \rho_w$, barrier size, L, and k.

We select the origin O of a Cartesian coordinate system xOy coinciding with the apex O of the cap (Fig.1b). The height b of the interface BOC above BC is to be determined. The hydraulic head $h(x, y)$ is counted from point O i.e. $h(0,0) = 0$. The head is related to groundwater pore pressure p_w via the equation $p_w = \rho_w g(h - y + C_p)$ where C_p is a pressure head at point O (this constant vanishes from the analysis). A complex potential is defined as $w = \varphi + i\psi$, where $\varphi = -kh$ is the velocity potential and ψ is a complex-conjugated stream function. Both characteristic functions are harmonic in the flow domain G_z.

LNAPL pressure inside the cap follows Pascal's law. There is no pressure jump across BOC and from elementary analysis [4,9] the water potential on the interface satisfies the condition:

$$\varphi + C_d y = 0, \quad C_d = k\frac{\rho_w - \rho_o}{\rho_w} = \text{const} \ , \tag{1}$$

where for LNAPL $C_d > 0$ and for DNAPLs $C_d < 0$.

Point O is a stagnation point and BOC is symmetric with respect to Oy. Therefore, the streamline A_uO splits into two branches OCA_{dr} and OBA_{lr} and we select $\psi = 0$ along this streamline.

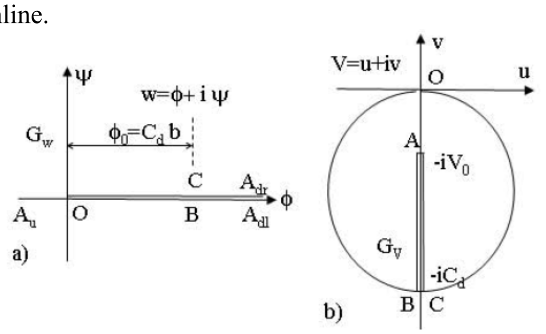

Fig.2. Complex potential domain (a) and hodograph (b)

The complex potential domain G_w is shown in Fig.2a where the potential at points B and C, located on two opposite sides of the cut, is $\varphi_0 = C_d b$, in congruency with (1).

III. Analytical Solution by Hodograph Transformation

A complex Darcian velocity $V = u + iv$ is defined, where $u(x, y)$ and $v(x, y)$ are its horizontal and vertical components. It is well-known [10] that the image of BOC in the hodograph domain, G_V, shown in Fig.2b, is a circle of diameter C_d and centre at point $(0, -iC_d/2)$. The equation of the circle is $u^2 + v^2 + C_d v = 0$. In G_V the image of the barrier walls is a cut such that points at infinity A_u, A_{dr} and A_{lr} (different in G_z) merge into a single point. Obviously, from Fig.2b $0 < V_0 < C_d$ that determines the above discussed hydrodynamic constraint on the regime of Fig.1b. $V(x, y)$ is an antiholomoprhic function and $u - iv = dw/dz$ is holomorphic in G_z. We use the method of inversion [10] and invert the disk with a cut in Fig.2b into a half-plane G_ς with a cut in the plane dz/dw (Fig.3a). In other words, G_ω is a tetragon. Next, we map G_ω onto G_w via an auxiliary plane $\zeta = \xi + i\eta$ (Fig. 3b).

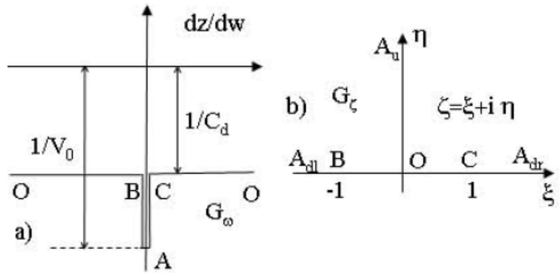

Fig.3. Inverted hodograph domain (a) and auxiliary half-plane (b)

By the Schwarz-Christoffel formula G_w is mapped on the half plane G_ς ($\eta > 0$) by the function:

$$w = C_d b \varsigma^2 \tag{2}$$

with the correspondence of points $B \to -1, \quad A \to 1, \quad O \to 0, \ A \to \infty$.

The domain G_ω is mapped on G_ς by the function:

$$\frac{dz}{dw} = A_1 \int_1^{\varsigma} \frac{d\tau}{\tau^2\sqrt{1-\tau^2}} - \frac{i}{C_d}, \tag{3}$$

where the branch of the radical is fixed in the upper half-plane and it is selected to be positive at $-1 < \tau < 1$. The mapping constant $A_1 > 0$ is evaluated from the locus of point A ($= -i/V_0$) in G_ω. Upon integration in (3) we obtain

$$\frac{dz}{dw} = f(\zeta) = \left(\frac{1}{C_d} - \frac{1}{V_0}\right)\frac{\sqrt{1-\zeta^2}}{\zeta} - \frac{i}{C_d}. \tag{4}$$

From (2) and (4) follows

$$z(\zeta) = b\left(\frac{C_d}{V_0} - 1\right)\left(\zeta\sqrt{1-\zeta^2} + \arcsin\zeta\right) - ib\zeta^2. \tag{5}$$

It should be $z(1) = L/2 - ib$ (see Fig.1) and therefore we find from (5)

$$b = \frac{LV_0}{\pi(C_d - V_0)}. \tag{6}$$

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For the sake of brevity we skip over calculating the index of our problem. We introduce dimensionless variables as (x^*, y^*, C_d^*, V_0^*, u^*, v^*, φ^*, ψ^*)= (x/L, y/L, V_0/k, C_d/k, u/k, v/k, $\varphi/(kL)$, $\psi/(kL)$) and drop the *-superscript.

We separate the real and imaginary parts of the function (5) and taking into account (6) obtain the parametric equations of the right half of the LNAPL sharp interface, OC:

$$x = \frac{1}{\pi}(\arcsin\xi + \xi\sqrt{1-\xi^2}),$$
$$y = -\frac{V_0}{\pi(C_d - V_0)}\xi^2, \quad 0 < \xi < 1, \qquad (7)$$

where $0 < C_d < 1, \quad V_0 < C_d$.

Contours (7) are shown in Fig.4a for C_d =3/4, V_0=2/3, 1/2 and 1/9 (curves 1-3, correspondingly). As is evident from this Figure, the increase of the incident velocity causes a drastic increase of the size of the lens. The flow net is shown in Fig.4b for C_d =0.2, V_0=0.15. The solid curves in Fig.4b represent the streamlines (in particular, the interface and barrier walls) and dashed line are equipotentials.

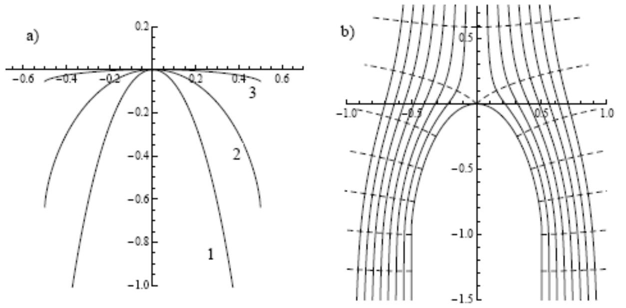

Fig.4. Sharp interfaces for C_d =3/4, V_0=2/3, 1/2 and 1/9 curves 1-3 (a), flow net for C_d =0.2, V_0=0.15.

Amazingly, curve (7) is a scaled trochoid. Moreover, if the fluids' densities and incident velocity satisfy the condition C_d/V_0=3/2 our solution coincides with the classical solution of Zhukovsky [11], [12] who studied a steady flow of a heavy ideal fluid baffled by a gas bubble injected through a straight-wall nozzle. Zhukovsky's solution served as a starting point for numerous studies in fluid mechanics (see [13] for reviews). Why are two seemingly different engineering problems – ours and Zhukovsky's– mathematically identical? This stems from the idea that the concepts and principles of interface fluid dynamics occur across the traditional science and engineering disciplines [14]. Indeed, in both conceptual models (ideal and Darcian flows) the velocity potential obeys the Laplace equation; our free interface condition (1) is equivalent to Zhukosvky's [11], [12] linear relation between the magnitude of velocity and sin of the angle between the velocity vector and the abscissa axis; far away from the baffle the flow velocity in both cases is uniform.

The flow pattern in Fig.1a-b can be inverted in the following manner: Instead of LNAPL we can consider DNAPL. Then this dense volume can be statically pending in the ambient groundwater flow if V_0 is oriented against gravity (ascending flow). In this case the interface BOC confines a DNAPL volume from below, i.e. BOC is attached to the bottom ED of the barrier in Fig.1a. This volume (and the whole mathematical solution above, upon rescaling the constant C_d in (1)) is symmetric to the LNAPL in Fig.1b, with respect to the horizon $A_{dr}A_{lr}$ in Fig.1a.

The very presence of attached LNAPL-DNAPL lenses as in Fig.1a-b is counterintuitive to many hydrogeologists. However, the existence of stable NAPL lenses is deleterious in managed aquifer recharge projects. Indeed, a large infiltration pond seeping for a long time into an allegedly clean subjacent porous medium may hydrodynamically entrap LNAPLs originally present in the vadose zone. Then a long-term degradation of quality of water stored/recovered in/from the aquifer due to dispersion-diffusion [1] will occur. Our rigorous analytical solution also illustrates that more intensive leaching from an infiltration pond located above the baffle-lens in Fig.4 will cause an opposite effect to what a remediation engineer may expect: instead of an improved clean-up, more hydrodynamic entrapment of LNAPL will occur. Such puzzling results can be also related to Toth's models of gravity-controlled groundwater flows in heterogeneous and aquifuge-baffled aquifers [15] where in one-phase flows (no NAPLs) an unexpectedly complex flow topology emerges. Another fascinating discovery of our work is the shape (7) which adds the trochoid to the arsenal of beautiful analytical curves (e.g., catenary, conchoid of Nicomedes, Taylor-Saffman's bubble, parabola, etc. [16]-[19]) emerging as exact solutions of various free-boundary problems in subsurface fluid dynamics.

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