

Stochastic Volatility Analysis using the Generalised Kolmogorov-Feller Equation

Jonathan Blackledge, Marc Lamphiere, Kieran Murphy, Shaun Overton and Afshin Panahi

Abstract—We consider an approach to analysing the Stochastic Volatility of a financial time series using the Generalised Kolmogorov-Feller Equation (GKFE). After reviewing the computation of the Stochastic Volatility using a phase only condition, a Green’s function solution to the GKFE equation is derived which depends upon the ‘memory function’ used to construct the GKFE. Using the Mittag-Leffler memory function, we derive an expression for the Impulse Response Function associated with a short time window of data which is then used to derive an algorithm for computing a new index using a standard moving window process. It is shown that application of this index to both a financial time series and its corresponding Stochastic Volatility provides a correlation between the start, direction and end of a trend depending on the sampling rate of the time series and the look-back window that is used.

Index Terms—stochastic volatility, generalised Kolmogorov-Feller equation, memory function, trend analysis.

I. INTRODUCTION

PRICE models involve the derivation and solution of a variety of stochastic differential and partial differential equations. A standard model for the price of a stock as a function of time $s(t)$ is [1]

$$\frac{d}{dt}s(t) = \mu s(t) + \sigma s(t)u(t) \quad (1)$$

where μ is the ‘Drift’, σ is the ‘Volatility’ and $u(t)$ is a stochastic function. This model is based on the idea that prices appear to be the previous price plus some random change and that these price changes are independent, i.e. asset price changes appear to be random and independent, prices being taken to follow some random walk-type behaviour. This is the basis for including a stochastic function $u(t)$. However the size of price movements also depends on the size of the price itself. The model is therefore revised to include this effect, the stochastic term $u(t)$ being replaced by $u(t)s(t)$ where σ determines the degree of randomness taken to influence a price change. In general, μ and σ vary with time, and, in the context of equation (1), $\sigma(t)$ is referred to as the ‘Stochastic Volatility’, e.g. [2], [3] and [4]. The drift function $\mu(t)$ tends to vary over longer periods of time reflecting the long term trends associated with a price index.

In principle, $u(t)$ could be any stochastic function with statistical behaviour conforming to a range of Probability Density Functions. A conventional model is to assume that the log price changes are Gaussian distributed so that $u(t)$ is taken to be a zero-mean Gaussian distributed function. If this function is taken to have a fixed standard deviation

of 1, then the volatility becomes a measure of the standard deviation, at least, for a (zero-mean) Gaussian model. The stock price model given by equation (1) then provides a method for estimating the volatility σ in terms of a lower bound as shown in Section II.

In this paper, we consider a solution to the Generalised Kolmogorov-Feller Equation to model the stochastic behaviour of a financial time series and its corresponding Stochastic Volatility. By defining an Impulse Response Function which is based on a parameter associated with the Mittag-Leffler memory function used to construct the KFE, we consider an algorithm for analysing the trends of the time series.

II. EVALUATION OF THE STOCHASTIC VOLATILITY

Let

$$f(t) = \mu + \sigma u(t)$$

where

$$f(t) = \frac{1}{s(t)} \frac{d}{dt}s(t) = \frac{d}{dt} \ln s(t)$$

and μ and σ are taken to be constant. We first obtain an estimate of the Drift by noting that, if the mean of $u(t)$ is approximately zero over $t \in [0, T]$, then

$$\int_0^T f(t)dt = \int_0^T \mu dt + \sigma \int_0^T u(t)dt \sim \mu T$$

so that

$$\mu \sim \frac{1}{T} \int_0^T f(t)dt \quad (2)$$

To obtain an estimate for the volatility, we now consider the case when the stochastic function $u(t)$ is a phase only function, i.e. given that

$$\tilde{u}(\omega) = \int_{-\infty}^{\infty} u(t) \exp(-i\omega t) dt$$

where ω is the (angular) frequency, we consider

$$\tilde{u}(\omega) = A \exp[i\theta(\omega)] \quad (3)$$

where the amplitude spectrum A is taken to be a constant for all values of ω . We also consider $u(t)$ to be a band-limited function $\omega \in [-\Omega/2, \Omega/2]$ with bandwidth Ω and a function of compact support $t \in [-T/2, T/2]$. Using Minkowski’s identity for Euclidean norms,

$$\|f(t)\|_2 \leq \|\mu\|_2 + \|\sigma u(t)\|_2$$

where

$$\|x(t)\|_2 := \left(\int |x(t)|^2 dx \right)^{\frac{1}{2}}$$

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so that we can write

$$\sigma \|u(t)\|_2 \geq \|f(t)\|_2 - \mu\sqrt{T}$$

where μ is given by equation (2). Using Parseval's Theorem (Rayleigh's Energy Theorem), the condition expressed by equation (3) allows us to write

$$\int_{-T/2}^{T/2} |u(t)|^2 dt = \frac{1}{2\pi} \int_{-\Omega/2}^{\Omega/2} |\tilde{u}(\omega)|^2 d\omega = \frac{\Omega A^2}{2\pi}$$

We can therefore consider the equation

$$\sigma_{\min} = \frac{1}{A} \sqrt{\frac{2\pi}{\Omega}} (\|f(t)\|_2 - \mu\sqrt{T}) \quad (4)$$

which yields an expression for the lower bound of the volatility.

III. NUMERICAL COMPUTATION OF THE STOCHASTIC VOLATILITY

Consider a discrete signal denoted by the array f_n , $n = 1, 2, 3, \dots, N$ where a uniform sampling interval of Δt is assumed. In this case, the discrete version of equation (4) becomes

$$\sigma_{\min} = \frac{1}{A} \sqrt{\frac{2\pi}{\Omega}} (\sqrt{\Delta t} \|f_n\|_2 - \mu\sqrt{T})$$

where we invoke the usual definition for a vector (Euclidean) norm, i.e.

$$\|f_n\|_2 := \left(\sum_{n=1}^N |f_n|^2 \right)^{\frac{1}{2}}, \quad \mu = \frac{\Delta t}{T} \sum_{n=1}^N f_n$$

The sampling interval Δt of f_n is related to the sampling interval $\Delta\omega$ of the Discrete Fourier Transform of f_n by the equation

$$\Delta t \Delta\omega = \frac{2\pi}{N}$$

and since the bandwidth of the discrete spectrum of f_n is $N\Delta\omega$ it is clear that $\Delta t = 2\pi/\Omega$. Thus, given that the support of the signal is $T = N\Delta t$, we note that

$$T = \frac{2\pi N}{\Omega}$$

and therefore obtain

$$\sigma_{\min} = \frac{2\pi}{A\Omega} (\|f_n\|_2 - \sqrt{N}\mu), \quad \mu = \frac{1}{N} \sum_{n=1}^N f_n$$

The scaling constant $2\pi/(A\Omega)$ can then be used to define a re-scaled Stochastic Volatility given by

$$\hat{\sigma} := \sigma_{\min} \frac{A\Omega}{2\pi}$$

thereby yielding the expression

$$\hat{\sigma} = \|f_n\|_2 - \sqrt{N}\mu$$

Writing this result explicitly in terms of the price value s_n we obtain the equation

$$\hat{\sigma} = \left(\sum_{n=1}^{N-1} \left| \ln \left(\frac{s_{n+1}}{s_n} \right) \right|^2 \right)^{\frac{1}{2}} - \frac{1}{\sqrt{N-1}} \sum_{n=1}^{N-1} \ln \left(\frac{s_{n+1}}{s_n} \right) \quad (5)$$

To compute the 'Stochastic Volatility' σ_m , N is taken to determine the size of the data sampling window or 'look-back' window which is moved along the time series one element at a time so that we can write

$$\hat{\sigma}_m = \left(\sum_{n=1}^{N-1} \left| \ln \left(\frac{s_{m+n+1}}{s_{m+n}} \right) \right|^2 \right)^{\frac{1}{2}} - \frac{1}{\sqrt{N-1}} \sum_{n=1}^{N-1} \ln \left(\frac{s_{m+n+1}}{s_{m+n}} \right) \quad (6)$$

Equation (5) may be compared with other estimates for the Stochastic Volatility such as the Maximum Likelihood (ML) estimate given by [5]

$$\hat{\sigma}_{ML}^2 = \frac{1}{N-1} \sum_{n=1}^{N-1} \left[\ln \left(\frac{s_{n+1}}{s_n} \right) \right]^2 - \frac{1}{(N-1)^2} \left[\ln \left(\frac{s_N}{s_1} \right) \right]^2$$

The phase only condition used to derive equations (5) and (6) is equivalent to modelling the stochastic function $u(t)$ in terms of a random walk in the (complex) Fourier domain where the amplitude of each step is the same.

IV. DERIVATION OF THE GENERALISED KOLMOGOROV-FELLER EQUATION

For an arbitrary Characteristic Function $P(k)$ with Probability Density Function (PDF) $p(x)$, Einstein's evolution equation is [6]

$$u(x, t + \tau) = u(x, t) \otimes_x p(x)$$

where $u(x, t)$ is a 'density function' representing the concentration of a canonical ensemble of particles undergoing elastic collisions. Consider a Taylor series for the function $u(x, t + \tau)$, i.e.

$$u(x, t + \tau) = u(x, t) + \tau \frac{\partial}{\partial t} u(x, t) + \frac{\tau^2}{2!} \frac{\partial^2}{\partial t^2} u(x, t) + \dots$$

For $\tau \ll 1$

$$u(x, t + \tau) = u(x, t) + \tau \frac{\partial}{\partial t} u(x, t)$$

and we obtain the 'Classical KFE' [7], [8]

$$\tau \frac{\partial}{\partial t} u(x, t) = -u(x, t) + u(x, t) \otimes_x p(x) \quad (7)$$

Equation (7) is based on a critical assumption which is that the time evolution of the field $u(x, t)$ is influenced only by short term events and that longer term (historical) events have no influence on the behaviour of the field, i.e. the 'system' described by equation (7) has no 'memory'. This statement is the physical basis upon which we introduce the condition $\tau \ll 1$ thereby allowing the Taylor series expansion of the $u(x, t + \tau)$ to be made to first order. The question then arises as to how longer term temporal influences can be modelled, other than by taking an increasingly larger number of terms in the Taylor expansion of $u(x, t + \tau)$ which is not of practical analytical value. For arbitrary values of τ ,

$$\tau \frac{\partial}{\partial t} u(x, t) + \frac{\tau^2}{2!} \frac{\partial^2}{\partial t^2} u(x, t) + \dots = -u(x, t) + u(x, t) \otimes_x p(x)$$

We can model the effect on a solution for $u(x, t)$ of the series on the left hand side of this equation in terms of a 'memory function' $m(t)$ and write

$$\tau m(t) \otimes_t \frac{\partial}{\partial t} u(x, t) = -u(x, t) + u(x, t) \otimes_x p(x) \quad (8)$$

where \otimes_t is taken to denote the causal convolution integral over t . This is the Generalised KFE (GKFE) which reduces to the Classical KFE when

$$m(t) = \delta(t)$$

Note that for any memory function for which there exists a function or class of functions of the type $n(t)$, say, such that

$$n(t) \otimes_t m(t) = \delta(t)$$

then we can write equation (8) in the form

$$\tau \frac{\partial}{\partial t} u(x, t) = -n(t) \otimes_t u(x, t) + \hat{n}(t) \otimes_t u(x, t) \otimes_x p(x) \quad (9)$$

where the Classical KFE is recovered when $n(t) = \delta(t)$.

Any solution obtained to the GKFE will be dependent upon the choice of memory function $m(t)$ used. There are a number of choices that can be considered, each or which is taken to be a 'best characteristic' of the stochastic system in terms of the influence of its time history. However, it may be expected that the time history of physically significant random systems is relatively localised in time. This includes memory functions such as the Mittag-Leffler function [9]

$$m(t) = \frac{1}{\Gamma(1-\beta)t^\beta}, \quad 0 < \beta < 1$$

where

$$n(t) = \frac{1}{\Gamma(\beta-1)t^{2-\beta}}$$

given that

$$\int_0^\infty \frac{\exp(-st)}{\Gamma(\beta)t^{1-\beta}} dt = \frac{1}{s^\beta} \quad \text{and} \quad \int_0^\infty \delta(t) \exp(-st) dt = 1$$

V. SOLUTION TO THE GKFE USING THE GREEN'S FUNCTION METHOD

Consider equation (9) which can be written in the form

$$\tau \frac{\partial}{\partial t} u(x, t) + u(x, t) = u(x, t) - n(t) \otimes_t u(x, t) + n(t) \otimes_t u(x, t) \otimes_x p(x)$$

so that the Green's function solution is given by

$$u(x, t) = g(t) \otimes_t u(x, t) - g(t) \otimes_t n(t) \otimes_t u(x, t) + g(t) \otimes_t n(t) \otimes_t u(x, t) \otimes_x p(x) \quad (10)$$

where the Green's function is given by

$$g(t) = \frac{1}{\tau} \exp(-t/\tau), \quad t > 0$$

which is the solution to

$$\tau \frac{\partial}{\partial t} g(t-t_0) + g(t-t_0) = \delta(t-t_0)$$

and we assume the initial conditions $u(x, t=0) = 0$ and $g(t=t_0) = 0$. We can now analyse this solution in Fourier-Laplace space by taking the Fourier transform and the Laplace transform of equation (10) and using the

convolution theorems for the Fourier and Laplace transform, respectively, to obtain

$$\tilde{u}(k, s) = \bar{g}(s) \tilde{u}(k, s) - \bar{g}(s) \bar{n}(s) \tilde{u}(k, s) + \bar{g}(s) \bar{n}(s) \tilde{u}(k, s) \tilde{p}(k) \quad (11)$$

where

$$\tilde{u}(k, s) = \int_0^\infty \int_{-\infty}^\infty u(x, t) \exp(-ikx) dx \exp(-st) dt$$

$$\bar{g}(s) = \int_0^\infty g(t) \exp(-st) dt, \quad \bar{n}(s) = \int_0^\infty n(t) \exp(-st) dt$$

and

$$\tilde{p}(k) = \int_{-\infty}^\infty p(x) \exp(-ikx) dx$$

From equation (11) it is clear that we can write

$$\begin{aligned} \tilde{u}(k, s) &= -\frac{\bar{g}(s)}{1-\bar{g}(s)} \bar{n}(s) \tilde{u}(k, s) + \frac{\bar{g}(s)}{1-\bar{g}(s)} \bar{n}(s) \tilde{u}(k, s) \tilde{p}(k) \\ &= -\frac{\bar{n}(s)}{\tau s} \tilde{u}(k, s) + \frac{\bar{n}(s)}{\tau s} \tilde{u}(k, s) \tilde{p}(k) \end{aligned}$$

given that $\bar{g}(s) = (1+\tau s)^{-1}$ and thus we obtain the equation

$$\tilde{u}(k, s) = \bar{h}(s) \tilde{u}(k, s) \tilde{p}(k) \quad (12)$$

where

$$\bar{h}(s) = \frac{\bar{n}(s)}{\tau s + \bar{n}(s)}$$

or, upon inverse transformations

$$u(x, t) = h(t) \otimes_t u(x, t) \otimes_x p(x) \quad (13)$$

with

$$h(t) \leftrightarrow \frac{\bar{n}(s)}{\tau s + \bar{n}(s)}$$

where \leftrightarrow denotes the Laplace transformation, i.e. mutual transformation from t -space to s -space.

Consider the iteration of equation (13) defined by

$$u_{n+1}(x, t) = h(t) \otimes_t u_n(x, t) \otimes_x p(x) \quad (14)$$

for an initial solution $u_0(x, t)$ where $n = 1, 2, \dots, N$. The equivalent iteration in Fourier-Laplace space is, from equation (12)

$$\tilde{u}_{n+1}(k, s) = \bar{h}(s) \tilde{u}_n(k, s) \tilde{p}(k) \quad (15)$$

with initial solution $\tilde{u}_0(k, s)$. From equation (15) it is clear that, after N iterations, we can write

$$\tilde{u}_N(k, s) = [\bar{h}(s)]^N [\tilde{p}(k)]^N \tilde{u}_0(k, s)$$

so that upon inverse Fourier-Laplace transformation, equation (14) becomes

$$u_N(x, t) = \prod_{j=1}^N p(x) \prod_{k=1}^N h(t) \otimes_x \otimes_t u_0(x, t) \quad (16)$$

where

$$\prod_{j=1}^N f(t) \equiv f(t) \otimes_t f(t) \otimes_t f(t) \otimes_t \dots$$

denoting the N^{th} convolution of $f(t)$. The convergence criterion required for the iteration defined by equation (14) is given in the Appendix A.

VI. MITTAG-LEFFLER IMPULSE RESPONSE FUNCTION

Form equation (16), if the initial solution is an impulse (i.e. $u_0(x, t) = \delta(x)\delta(t)$) then the Impulse Response Function (IRF), denoted by $r(x, t)$, is given by

$$r(x, t) = \prod_{j=1}^N p(x) \prod_{k=1}^N h(t)$$

with ‘transfer function’

$$\tilde{r}(k, s) = [\bar{h}(s)\tilde{p}(k)]^N$$

For a memory function $m(t)$ modelled by the Mittag-Leffler function (for $0 < \beta < 1$)

$$m(t) \leftrightarrow \frac{1}{s^{1-\beta}}, \quad \bar{h}(s) = \frac{1}{1 + \tau s^\beta} \sim \frac{1}{\tau s^\beta}$$

so that

$$h(t) \sim \frac{1}{\tau \Gamma(\beta) t^{1-\beta}}$$

Similarly, suppose we consider a Mittag-Leffler PDF of the form

$$p(x) = \frac{1}{\Gamma(1 - \gamma) |x|^\gamma}, \quad 0 < \gamma < 1$$

so that the IRF becomes

$$r(x, t) \sim \prod_{j=1}^N \frac{1}{\Gamma(1 - \gamma) |x|^\gamma} \prod_{k=1}^N \frac{1}{\tau \Gamma(\beta) t^{1-\beta}}$$

Figure 1 shows the evolution of the function $u(x, t)$ for an initial solution $u_0(x, t)$ composed of a uniformly distributed stochastic field. The result is based on a discretisation of the equation

$$u_N(x, t) = r(x, t) \otimes_x \otimes_t u_0(x, t) \tag{17}$$

and shows grey-level images of the field $u_N(x, t)$ for $N = 1, 2, 3, 4$. The computation of the field is undertaken by multiple filtering the Fourier transform of $u_0(x, t)$ with the transfer function $|k|^{-\gamma} |s|^{-\beta}$ and shows how the field acquires structure from a uniformly distributed random space-time process.

Note that, from Appendix A, if $\|h(t)\| \times \|p(x)\| \ll 1$ then $r(x, t) \sim p(x)h(t)$, and, in the case of the Mittag-Leffler function used here, this will occur when $\tau \gg 1$. Also, note that $r(x, t) \rightarrow 0$ as $\gamma \rightarrow 1$ and as $\beta \rightarrow 0$.

VII. STOCHASTIC VOLATILITY ANALYSIS

On the basis of the results discussed in the previous section, we consider a short time series model given by (for an arbitrary PDF p)

$$\hat{u}(t) \equiv \int_{-\infty}^{\infty} p(x)h(t)dx = \frac{a}{t^{1-\beta}}, \quad \beta > 0$$

where a is a scaling constant. This model represents the IRF associated with a random scaling fractal signal $u(t)$ [1]. For the discrete case when $\hat{u}_n \equiv \hat{u}(t_n)$ (for $n = 1, 2, \dots, N$) is taken to represent a window of data taken from an input data stream,

$$\hat{u}_n = at_n^\alpha, \quad t_n > 0$$

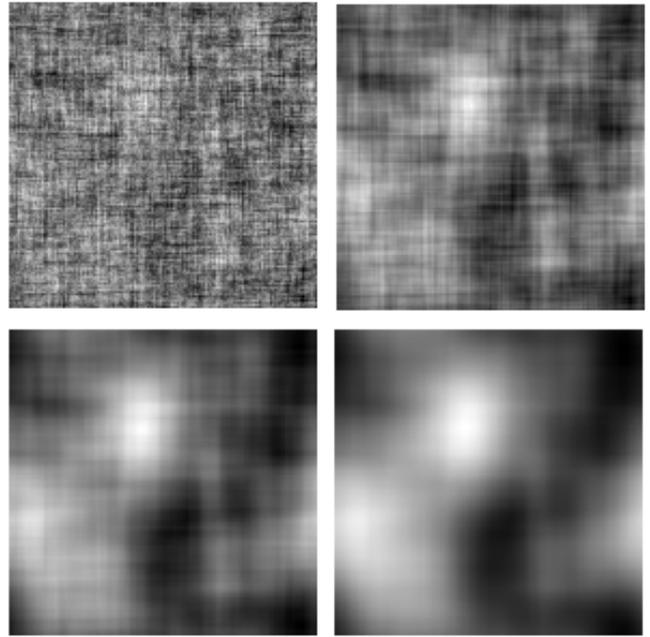


Fig. 1. Evolution of the field $u_N(x, t)$ for $N = 1, 2, 3, 4$ (top-left, top-right, bottom-left and bottom-right, respectively) based on application of equation (17) with $\gamma = 0.5$ and $\beta = 0.5$ computed using a 256×256 rectangular grid.

where $\alpha = \beta - 1$. Estimates of the parameters a and α are then chosen to minimise the error function

$$e(a, \alpha) = \|\ln \hat{u}_n - \ln u_n\|_2^2 \equiv \sum_{n=1}^N (\ln \hat{u}_n - \ln u_n)^2$$

where u_n is data which is taken to be normalised, i.e. $\|u_n\|_\infty = 1$. Differentiating with respect to $A = \ln a$ and α , it is trivial to show that

$$\alpha = \frac{\sum_{n=1}^N \ln u_n \sum_{n=1}^N \ln t_n - N \sum_{n=1}^N \ln u_n \ln t_n}{\left(\sum_{n=1}^N \ln t_n\right)^2 - N \sum_{n=1}^N (\ln t_n)^2} \tag{18}$$

and

$$a = \exp \left(\frac{\sum_{n=1}^N \ln u_n - \alpha \sum_{n=1}^N \ln t_n}{N} \right)$$

given that

$$\frac{\partial e}{\partial \alpha} = 0 \quad \text{and} \quad \frac{\partial e}{\partial A} = 0$$

Note that in general, $\alpha = \beta - 1$ may be greater than (for $\beta > 1$) or less than (for $0 < \beta < 1$) zero thereby providing a measure of any (long term) ascending or descending trends in the data u_n , respectively.

An example of computing the index α for a financial times series and for the Stochastic Volatility of the same time series is given in Figure 2. This figure shows the results of computing the Stochastic Volatility for FTSE (close-of-day) data (obtained from [11]) on a moving window basis using equation (6) with a look-back window of 100 elements. Figure 2 also shows the ‘ α -signatures’ for the same FTSE data (denoted by α_{Data}) and for the Stochastic Volatility

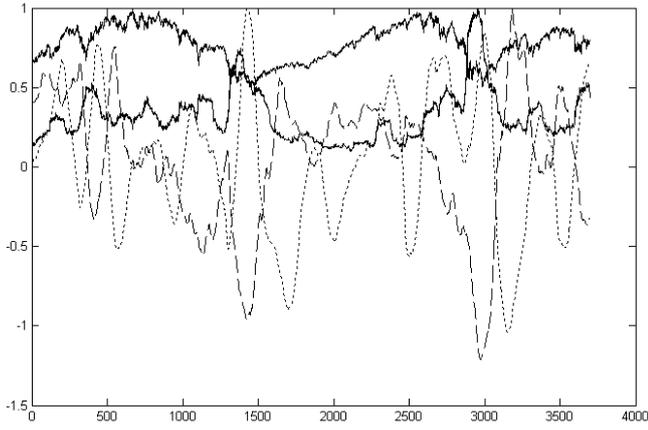


Fig. 2. Normalised FTSE index (close-of-day) from 13-03-1996 to 13-01-2012 (upper solid line), the (normalised) Stochastic Volatility (lower solid line) computed using equation (6) for a look-back window of 100, the (normalised) α_{Data} -index for the FTSE data (dashed line) and the α_{SV} -index for the Stochastic Volatility (dotted line) using equation (18) for a look-back window of 200.

(denoted by α_{SV}) using a look-back window consisting of 200 elements.

The results given in Figure 2 are informative in terms of the interpretation of the financial data, and, in particular, long term trend analysis. A closer inspection of Figure 2 shows a clear correlation between the upward and downward (long term) trends of the FTSE (close-of-day) data, the Stochastic Volatility and the polarity of the respective ‘ α -indices’, albeit with a lag determined by the size of the look-back window. For example, the upward trend observed between (approximately) days 1500 and 2500 in Figure 2 is characterised by $\alpha_{\text{Data}} > 0$ and values for α_{SV} that are predominantly < 0 . This result is repeated in the time series between data points > 3000 and 3500 (approximately). The behaviour is reversed in the downward trends that occur approximately between days 1000-1500 and 2500-3000. There is also a clear indication of the ‘turning points’ that occur at positions ~ 1400 and ~ 3000 in the time series when transitions occur between the downward and upward trends. These example results indicate the potential value of the approach in identifying the start, direction and end of a trend in a financial times series depending on the sampling rate of the data and the look-back window that is applied.

VIII. CONCLUSION

Compared to equations such as the Classical Diffusion and Fractional Diffusion Equations [10], the GKFE derived in Section IV and given by equation (8) represents a more accurate model for a density function describing random motion that conforms to Einstein’s evolution equation. We have considered the Green’s function solution of the GKFE given in Section V as a model for a financial time series (or a derived index). The time dependence of this solution depends upon the memory function used to model the higher order terms in the Taylor series expansion of the evolution equation, and, in this paper, we have used the Mittag-Leffler memory function. It has been shown that this choice provides a temporal solution that scales at t^α where $\alpha = \beta - 1$, $0 < \beta < 1$. For $\beta > 0$ the parameter α has been used to generate α -indices for both a financial time

series and its Stochastic Volatility using a standard moving window process. The sample results given in Section VII appear to have the potential to identify the start, direction and end of a trend of a signal.

It is noted that the behaviour of the α -indices for the data beyond 3500 days given in Figure 2 shows increasing volatility $\alpha_{\text{SV}} > 0$ associated with a continuing downward trend $\alpha_{\text{Data}} < 0$ and it should therefore be expected that, at the time of writing this paper (i.e. January 2012), the FTSE may be expected to continue on a downward trend thereby forecasting further recessive behaviour. The m-code used to compute the results given in Figure 2 is provided in Appendix B for interested readers to reproduce the results and to analyse other financial time series and/or derived indices.

APPENDIX A

CONDITION FOR CONVERGENCE OF EQUATION (14)

Consider the error function $\epsilon_n(x, t)$ at any iteration n so that $u_n(x, t) = u(x, t) + \epsilon_n(x, t)$ From equation (15) we can then write

$$\tilde{\epsilon}_{n+1}(k, s) = \bar{h}(s)\tilde{p}(k)\tilde{\epsilon}_n(k, s)$$

so that

$$\tilde{\epsilon}_n(k, s) = [\bar{h}(s)\tilde{p}(k)]^n \tilde{\epsilon}_0(k, s)$$

and it is clear that, since we require $\tilde{\epsilon}_n \rightarrow 0$ and $n \rightarrow \infty$, $[\bar{h}(s)\tilde{p}(k)] < 1 \forall (k, s)$. The condition for convergence therefore becomes

$$\|\bar{h}(s)\tilde{p}(k)\| \leq \|\bar{h}(s)\| \times \|\tilde{p}(k)\| < 1$$

or, for Euclidian norms, and, using Rayleigh’s theorem,

$$\|\bar{h}(s)\|_2 \times \|\tilde{p}(k)\|_2 < \frac{1}{\sqrt{2\pi}}$$

In (k, t) -space

$$\tilde{\epsilon}_n(k, t) = \prod_{k=1}^n h(t)[\tilde{p}(k)]^n \otimes \tilde{\epsilon}_0(k, t)$$

so that, using Hölder’s inequality

$$\begin{aligned} \|\tilde{\epsilon}_n(k, t)\| &\leq \left\| \prod_{k=1}^n h(t)[\tilde{p}(k)]^n \right\| \times \|\tilde{\epsilon}_0(k, t)\| \\ &\leq \|h(t)\|^n \times \|\tilde{p}(k)\|^n \times \|\tilde{\epsilon}_0(k, t)\| \end{aligned}$$

and the condition for convergence becomes

$$\|h(t)\|_2 \times \|p(x)\|_2 < \frac{1}{\sqrt{2\pi}}$$

APPENDIX B

M-CODE USED TO COMPUTE FIGURE 2

```
clear;%Clear memory.
%Read file from txt file into data array.
fid=fopen('C:\PATH\DATA.txt','r');
[data n]=fscanf(fid,'%g',[inf]);
fclose(fid);
%Set length of look-back window for
%computing the stochastic function to
w1=100; %and set length of look-back
>window for computing the index
```

```

w2=round(w1+100);
%Normalise the data
data=data./max(data);
%Begin moving window process
%required to compute the
%Stochastic Volatility using
%a working array length of
m=n-w1;
for i=1:m
%Window the data.
for j=1:w1
s(j)=data(i+j-1);
end
s=s./max(s);
%Compute the Stochastic Volatility.
sigma(i)=volatility(s,w1);
end
%Begin the moving window process
%required to compute the alpha
%index using a working array
%length of
n=m-w2;
for i=1:n
%Window the data.
for j=1:w2
vol(j)=sigma(i+j);
sig(j)=data(i+w1+j-1);
end
%Compute the alpha index for
%the volatility and the signal.
indexvol(i)=alpha(vol,w2);
indexsig(i)=alpha(sig,w2);
%End the moving window process.
end
%Prepare the original signal and
%the Stochastic Volatility for a
%a comparative plot.
i=1;
for j=1:n
signal(i)=data(j+w1+w2-1);
stochvol(i)=sigma(j+w2);
x(i)=i; i=i+1;%time element
end
%Normalise the data
signal=signal./max(signal);
indexvol=indexvol./max(indexvol);
indexsig=indexsig./max(indexsig);
stochvol=stochvol./max(stochvol);
%and plot the results.
figure(1);
plot(x,signal,'k-',x,stochvol,'r-',...
x,indexvol,'g-',x,indexsig,'b-');
grid on;
%
function sigma=volatility(s,n)
%Function to compute the volatility.
%Compute the log price differences.
for i=1:n-1
ds(i)=log(s(i+1)/s(i));
end
ds(n)=ds(n-1);%Set end point value

```

```

%Compute first and second terms.
term1=sqrt(sum(abs(ds.*ds)));
term2=sum(ds)/sqrt(n);
%Return the volatility.
sigma=term1-term2;
%
function index=alpha(data,N)
%Computation of the 'alpha-index'
%using the least squares algorithm.
%
%Compute the logarithm of the data.
for i=1:N
ydata(i)=log(data(i));
xdata(i)=log(i);
end
%Compute each term of the
%least squares formula.
term1=sum(ydata).*sum(xdata);
term2=sum(ydata.*xdata);
term3=sum(xdata.^2);
term4=sum(xdata.^2);
%Compute and return the alpha index
index=(term1-(N*term2))/(term3-(N*term4));

```

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