A Theory of Piezoelectric Thickness Vibrator Transducer Based on the Rayleigh and Bishop Model

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***Abstract*—The theory of the thickness vibrator piezoelectric transducer is developed on the basis of the one dimensional Rayleigh-Love and Bishop models of longitudinal vibrations of rods. In the frames of this theory the lateral displacements are proportional to the product of the longitudinal strain, Poisson ratio, and the distance from the neutral line of the symmetric cross-section of the waveguide. The Hamilton variational principle is used for derivation of the equation of motion and for obtaining of the mechanical and electric boundary conditions. The electric impedance of the piezoelectric transducer is calculated.**

***Index Terms*—piezoelectric, Rayleigh and Bishop equation, vibration of thick rod.**

I. Introduction

The piezoelectric transducers generate and detect ultrasonic waves in continuous media such as fluids, solids, etc [1]. They have been developed for many industrial applications. In South Africa the transducers are manufactured by different companies for example, for navigation functions, and railroad inspection (CSIR), and ultrasonic level instruments (K-TEK), etc. The conventional theory of the thickness vibrator transducer is based on an assumption that its lateral vibrations are negligible. In this case the transducers' dynamics could be described in terms of the one-dimensional wave equation and a set of mechanical and electric boundary conditions. However, the main assumption of the model is only valid in the case of long and relatively thin rods. As a rule, the linear dimensions of thickness vibrators are comparable with the characteristic dimensions of their cross-sections and hence, it is necessary to take into consideration the lateral displacements of these transducers. In the present paper the theory of the thickness vibrator transducer is developed on the basis of the Rayleigh-Love [2], [3] and Bishop [4], [5] models of longitudinal vibrations of rods. In the frame of this theory it is supposed that the lateral displacements are proportional to the product of the longitudinal strain, Poisson ratio, and the distance from the neutral line of the transducers' cross-section. For proper application of this model an equivalent Poisson ratio, of the transversely isotropic piezoelectric material, is calculated from the principle of minimum strain energy. The Hamilton variational principle is used for derivation of the equation of motion and for obtaining the mechanical and electric boundary conditions. On the basis of the obtained Rayleigh-Love and Bishop models the electric impedance of the thickness vibrator is calculated. Possible generalizations of the proposed approach are considered and conclusions are formulated. The main theoretical results of the proposed paper are: formulation of a convenient method of equivalent Poisson ratio calculation by means of the minimum of the strain energy and use of the Hamilton variational principle to obtain the equations of motion of the Rayleigh-Love and Bishop models of a thickness vibrator transducer, and its mechanical and electric boundary conditions. The formalism explained in the present article could be used for analysis of N-stepped structures [6] with piezoelectric members.

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II. Theory of Thickness Vibrator Based on Rayleigh-Love and Bishop Model

Suppose that $Oz\left(03\right)$ - axis of polarization of a piezoelectric rod. According to the Rayleigh-Love and Bishop theories displacements in $Ox\left(01\right), Oy\left(02\right)$, Oz - directions are correspondingly $u, v,$ and w:

$$\left.\begin{aligned} u &= u\left(x,z,t\right) = -\eta x w' \\ v &= v\left(y,z,t\right) = -\eta y w' \\ w &= w\left(z,t\right) \end{aligned}\right\} \tag{1}$$

where prime means partial differentiation with regards to z, η - Poisson ratio and x, y - displacements from the neutral axis of any symmetric cross0section of the bar. For isotropic materials η are normally known. For piezoelectric materials, operating in the regime of thickness vibrator, it could be found from the principle of the strain energy minimum that

$\eta = \dfrac{c_{13}^D}{c_{11}^D + c_{12}^D}$ - equivalent Poisson ratio, where c_{ij}^D - elastic constants at constant electric charge density [5].

Linear stress-strain relations are:

$$\left.\begin{aligned} T_1 &= c_{11}^D S_1 + c_{12}^D S_2 + c_{13}^D S_3 - h_{31} D_3 \\ T_2 &= c_{12}^D S_1 + c_{11}^D S_2 + c_{13}^D S_3 - h_{31} D_3 \\ T_3 &= c_{13}^D \left(S_1 + S_2 \right) + c_{33}^D S_3 - h_{33} D_3 \\ T_4 &= c_{44}^D S_4 \\ T_5 &= c_{44}^D S_5 \\ T_6 &= 0 \end{aligned}\right\} \qquad (2)$$

Where strains: $S_1 = S_2 = -\eta w'$, $S_3 = w'$, $S_4 = -\eta y w''$, $S_5 = -\eta x w''$, $S_6 = 0$, are calculated.from (1) electric displacements $D_1 = D_2 = 0$, $D_3 = D_{30} e^{i\omega t}$, where $D_{30} = const$, and $\omega = 2\pi f$ is the frequency of excitation

Expression for kinetic energy:

$$K = \frac{\rho}{2} \int_0^l \left[A \dot{w}^2 + I_p \dot{w}'^2 \right] dz \qquad (3)$$

is used in both Rayleigh-Love and Bishop theories, where l - thickness of the transducer

Strain energy is as follows:

$$\begin{aligned} P &= \frac{1}{2} \int_0^l \int_{(A)} \left(T_1 S_1 + T_2 S_2 + T_3 S_3 + T_4 S_4 + T_5 S_5 + T_6 S_6 \right) dA dz \\ &= \int_0^l \int_{(A)} \left\{ \left(c_{11}^D + c_{12}^D \right) S_1^2 + 2 c_{13}^D S_1 S_3 + \frac{1}{2} c_{33}^D S_3^2 \right\} dA dz + \\ &+ \int_0^l \int_{(A)} \left\{ c_{44}^D \left(S_4^2 + S_5^2 \right) - h_{31} S_1 D_3 - \frac{1}{2} h_{33} S_3 D_3 \right\} dA dz \end{aligned} \qquad (4)$$

where A - cross-section area of the transducer. In the Rayleigh-Love theory shear strains and stresses are neglected ($S_4 = S_5 = 0$ and hence, $T_4 = T_5 = 0$), but in the Bishop theory it is supposed that $S_4 \neq 0, \quad S_5 \neq 0$.

Electric energy is:

$$\begin{aligned} W &= \frac{1}{2} \int_0^l \int_{(A)} \left(E_1 D_1 + E_2 D_2 + E_3 D_3 \right) dA dz \\ &= \int_0^l \int_{(A)} \left\{ -h_{31} S_1 D_3 - \frac{1}{2} h_{33} S_3 D_3 + \frac{1}{2} \beta_3^S D_3^2 \right\} dA dz \end{aligned} \qquad (5)$$

It is also necessary to keep in mind the following electric boundary condition:

$$\int_0^l E_3 \, dz = V = V(t) \qquad (6)$$

where $V(t) = V_0 e^{i\omega t}$ $\left(V_0 = const \right)$- excitation voltage applied to the thickness vibrator.

Lagrangian of the system:

$$\begin{aligned} L &= K - P + W + \lambda \cdot \left[\int_0^l E_3 \, dz - V(t) \right] \\ &= \frac{1}{2} \int_0^l \left\{ \rho A \dot{w}^2 - E A w'^2 + \rho I_p \dot{w}'^2 - c_{44}^D \eta^2 I_p w''^2 \right\} dz \\ &+ \frac{1}{2} \beta_3^S A l D_3^2 + \lambda \cdot \left\{ -H_{33} \left[w(l) - w(0) \right] + \beta_3^S l D_3 - V \right\} \\ &= L\left[\dot{w}, \dot{w}', w', w'', w(0), w(l), D_3, \lambda \right] \\ &= \int_0^l \Lambda\left[\dot{w}, \dot{w}', w', w'' \right] dz + \Delta L\left[w(0), w(l), D_3, \lambda \right] \end{aligned} \qquad (7)$$

Where $E = c_{33}^D - 4\eta c_{13}^D + 2\eta^2 \left(c_{11}^D + c_{12}^D \right) = c_{33}^D - \dfrac{2\left(c_{13}^D \right)^2}{c_{11}^D + c_{12}^D}$ - equivalent modulus of elasticity $H_{33} = h_{33} - 2\eta h_{31} = h_{33} - \dfrac{2 c_{13}^D h_{31}}{c_{11}^D + c_{12}^D}$ - equivalent piezoelectric constant. Lagrangian (7) characterizes the Bishop model. In the Rayleigh-Love model it is necessary to neglect term $c_{44}^D \eta^2 I_p w''^2$ and hence, suppress w'' - dependence in the Lagrangian.

Variation of the Lagrangian for the Bishop model:

$$\begin{aligned} \delta L &= \int_0^l \delta\Lambda\left[\dot{w}, \dot{w}', w', w'' \right] dz + \delta\Delta L\left[w(0), w(l), D_3, \lambda \right] \\ &= \int_0^l \left\{ \frac{d}{dt} \left[\frac{\partial \Lambda}{\partial \dot{w}} \delta w + \frac{\partial \Lambda}{\partial \dot{w}'} \delta w' \right] \right\} dz + \\ &+ \int_0^l \left\{ \left[\frac{d^2}{dz^2} \left(\frac{\partial^2 \Lambda}{\partial w''} \right) + \frac{d^2}{dt dz} \left(\frac{\partial^2 \Lambda}{\partial \dot{w}'} \right) \right] \delta w \right\} dz - \\ &- \int_0^l \left\{ \left[\frac{d}{dt} \left(\frac{\partial \Lambda}{\partial \dot{w}} \right) + \frac{d}{dz} \left(\frac{\partial \Lambda}{\partial w'} \right) \right] \delta w \right\} dz + \\ &+ \left[\frac{\partial \Delta L}{\partial w(0)} - \frac{\partial \Lambda}{\partial w'} + \frac{d}{dt} \left(\frac{\partial \Lambda}{\partial \dot{w}'} \right) + \frac{d}{dz} \left(\frac{\partial \Lambda}{\partial w''} \right) \right]_{x=0} \delta w(0) + \\ &+ \left[\frac{\partial \Delta L}{\partial w(l)} + \frac{\partial \Lambda}{\partial w'} - \frac{d}{dt} \left(\frac{\partial \Lambda}{\partial \dot{w}'} \right) - \frac{d}{dz} \left(\frac{\partial \Lambda}{\partial w''} \right) \right]_{x=l} \delta w(l) - \\ &- \left[\frac{\partial \Lambda}{\partial w''} \delta w' \right]_{x=0} + \left[\frac{\partial \Lambda}{\partial w''} \delta w' \right]_{x=l} + \frac{\partial \Delta L}{\partial D_3} \delta D_3 + \frac{\partial \Delta L}{\partial \lambda} \delta \lambda \end{aligned} \qquad (8)$$

For the Rayleigh-Love model terms with $\dfrac{\partial \Lambda}{\partial w''}$ are absent.

Hence, the equation of motion in a general form for the Bishop model is:

$$\frac{d^2}{d z^2} \left(\frac{\partial^2 \Lambda}{\partial w''} \right) + \frac{d^2}{dt \, d z} \left(\frac{\partial^2 \Lambda}{\partial \dot{w}'} \right) - \frac{d}{dt} \left(\frac{\partial \Lambda}{\partial \dot{w}} \right) - \frac{d}{d z} \left(\frac{\partial \Lambda}{\partial w'} \right) = 0 \qquad (9)$$

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For the Rayleigh-Love model term $\frac{d^2}{dz^2}\left(\frac{\partial^2\Lambda}{\partial w''}\right)$ is excluded. In the explicit form Eq.(9) is as follows (see [4]):

$$\rho A\ddot{w} - EAw'' - \rho\eta^2 I_p \ddot{w}'' + G\eta^2 I_p w^{IV} = 0 \qquad (9a)$$

In the Rayleigh-Love [7] model term $G\eta^2 I_p w^{IV}$ is absent

The boundary conditions in the general form for the Rayleigh-Bishop model are:

$[w]_{x=0,l} = 0$ - for fixed ends and

$$\left.\begin{aligned}&\left[\frac{\partial \Delta L}{\partial w(0)} - \frac{\partial\Lambda}{\partial w'} + \frac{d}{dt}\left(\frac{\partial\Lambda}{\partial \dot{w}'}\right)\right]_{x=0} = 0\\ &\left[\frac{\partial \Delta L}{\partial w(l)} - \frac{\partial\Lambda}{\partial w'} + \frac{d}{dt}\left(\frac{\partial\Lambda}{\partial \dot{w}'}\right)\right]_{x=l} = 0\end{aligned}\right\} \text{ for free ends.}$$

For the Bishop model the boundary conditions are:

$[w]_{x=0,l} = 0$ and $\left[\frac{\partial\Lambda}{\partial w''}\right]_{x=0,l} = 0$ for fixed ends and

$[w']_{x=0,l} = 0$;

$$\left.\begin{aligned}&\left[\frac{\partial \Delta L}{\partial w(0)} - \frac{\partial\Lambda}{\partial w'} + \frac{d}{dt}\left(\frac{\partial\Lambda}{\partial \dot{w}'}\right) + \frac{d}{dz}\left(\frac{\partial\Lambda}{\partial w''}\right)\right]_{x=0} = 0\\ &\left[\frac{\partial \Delta L}{\partial w(l)} - \frac{\partial\Lambda}{\partial w'} + \frac{d}{dt}\left(\frac{\partial\Lambda}{\partial \dot{w}'}\right) + \frac{d}{dz}\left(\frac{\partial\Lambda}{\partial w''}\right)\right]_{x=l} = 0\end{aligned}\right\}$$

for free ends.

It is also necessary to keep in mind the electric boundary condition: $\frac{\partial \Delta L}{\partial \lambda} = 0$.

Let us give an explicit form of the free ends boundary conditions for the Bishop case of the thickness vibrator:

$$\left.\begin{aligned}&\left[EAw' + \rho\eta^2 I_p \ddot{w}' - G\eta^2 I_p w'''\right]_{x=0,l} - H_{33}AD_3 = 0\\ &-H_{33}\left[w(l) - w(0)\right] + \beta_3^S l D_3 - V(t) = 0\\ &[w']_{x=0,l} = 0\end{aligned}\right\} \qquad (10)$$

Hence, we have six boundary conditions. For the Rayleigh-Love case corresponding boundary conditions are:

$$\left.\begin{aligned}&\left[EAw' + \rho\eta^2 I_p \ddot{w}'\right]_{x=0,l} - H_{33}AD_3 = 0\\ &-H_{33}\left[w(l) - w(0)\right] + \beta_3^S l D_3 - V(t) = 0\end{aligned}\right\} \qquad (11)$$

Hence, in this case we have three boundary conditions.

Solution of the Sturm-Liouville problem corresponding to (9a) equation in the Bishop case is

$$\begin{aligned}W(x,\omega) = {}& C_1\cos\left[k_1(\omega)x\right] + S_1\sin\left[k_1(\omega)x\right] +\\ &+ C_2\cosh\left[k_2(\omega)x\right] + S_2\sinh\left[k_2(\omega)x\right]\end{aligned} \qquad (12)$$

Where

$$k_{1,2}(\omega) = \sqrt{\sqrt{\left(\frac{EA - \omega^2\rho\eta^2 I_p}{2c_{44}^D\eta^2 I_p}\right)^2 + \frac{\omega^2\rho A}{c_{44}^D\eta^2 I_p}} \mp \frac{EA - \omega^2\rho\eta^2 I_p}{2c_{44}^D\eta^2 I_p}} \qquad (13)$$

In the Rayleigh-Love case with the term $G\eta^2 I_p w^{IV}$ is absent.

Longitudinal displacement of the waveguide is

$$W(x,\omega) = C\cos\left[k(\omega)x\right] + S\sin\left[k(\omega)x\right] \qquad (14)$$

where

$$k(\omega) = \omega\sqrt{\frac{\rho A}{EA - \omega^2\rho\eta^2 I_p}} \qquad (15)$$

Current through the thickness vibrator is calculated

$$I(\omega) = A\dot{D}_3 = i\omega A\left|D_3\right| \qquad (16)$$

Electric impedance is calculated as follows:

$$Z(\omega) = \frac{V_0}{I(\omega)} \qquad (17)$$

III. EXAMPLE

Let us consider a thickness vibrator made from PZT-4 *Vernitron*^{Co} piezoceramic. Its radius $a = 25\ mm$ and length $l = 25\ mm$, hence its area of cross-section is $A = \pi a^2 = 19.63\ mm^2$ and moment of inertia $I_p = \frac{\pi a^4}{2} = 61.36\ mm^4$.

Graphs of wave numbers versus frequency in the Bishop case is shown in Fig 1.

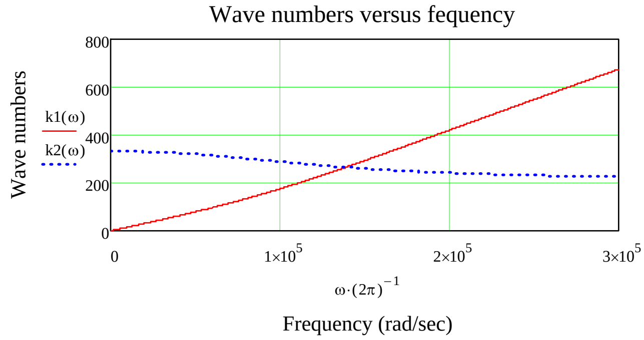

Fig. 1. Graphs of wave numbers versus frequency

Graph of phase velocity versus frequency is shown in Fig 2

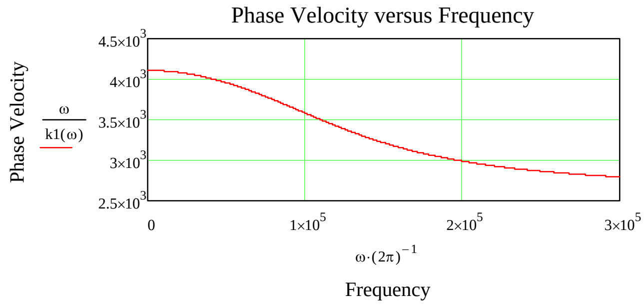

Fig. 2. Graphs of phase velocity versus frequency

Graph of electric impedance versus frequency is shown in Fig 3

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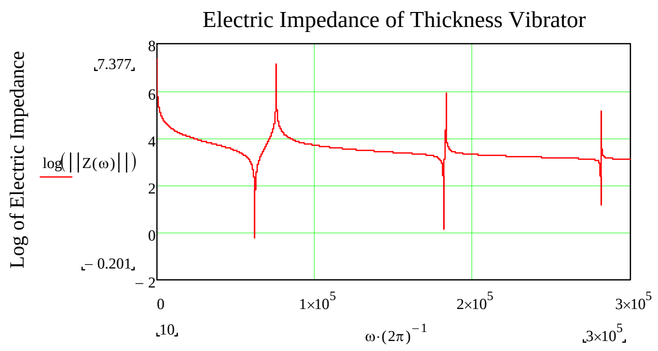

Fig.3. Graph of electric impedance (in logarithmic scale) versus frequency.

IV. Conclusion

Theories of thickness vibrator transducer are formulated on the basis of two main models of longitudinally vibrating bars, Rayleigh-Love and Bishop, which take into consideration the lateral effects.

For formulation of the theories the variational approach is systematically used and mechanical and electrical boundary conditions are obtained.

Electric impedance is calculated as function of excitation frequency.

Numerical example of practical calculation of the main properties of the piezoelectric transducer is considered.

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