

Predicting Wave Velocity and Stress of Impact of Elastic slug and Viscoelastic Rod Using Viscoelastic Discontinuity Analysis (Standard Linear Solid Model)

A.B. Musa, *Member, IAENG*

ABSTRACT

The study is about impact of a short elastic rod (or slug) on a stationary semi-infinite viscoelastic rod. The viscoelastic materials are modeled as standard linear solid which involve three material parameters and the motion is treated as one-dimensional. We first establish the governing equations pertaining to the impact of viscoelastic materials subject to certain boundary conditions for the case when an elastic slug moving at a speed V impacts a semi-infinite stationary viscoelastic rod.

The objective is to predict stresses and velocities at the interface following wave transmissions and reflections in the slug after the impact using viscoelastic discontinuity. If the stress at the interface becomes tensile and the velocity changes its sign, then the slug and the rod part company. If the stress at the interface is compressive after the impact, the slug and the rod remain in contact. After modelling the impact and solve the governing system of partial differential equations in the Laplace transform domain. We invert the Laplace transformed solution numerically to obtain the stresses and velocities at the interface for several viscosity time constants and ratios of acoustic impedances. In inverting the Laplace transformed equations, we used the complex inversion formula because there is a branch cut and infinitely many poles within the Bromwich contour. In the discontinuity analysis, we look at the moving discontinuities in stress and velocity using the impulse-momentum relation and kinematical condition of compatibility. Finally, we discussed the relationship of the stresses and velocities using numeric and the predictable stresses and velocities using viscoelastic discontinuity analysis.

Keyword: Interface stress, Interface velocity, ratios of acoustic impedances, viscoelastic, Viscosity time constants.

I. INTRODUCTION

There are materials for which a suddenly applied and maintained state of uniform stress induces an instantaneous deformation followed by a flow process which may or may not be limited in magnitude as time grows [1]. These materials are said to exhibit both an instantaneous elasticity effect and a creep characteristic. This behavior clearly cannot be described by either elasticity or viscosity theories alone as it combines features of each and is called viscoelastic. Viscoelasticity is a generalization of elasticity and viscosity. The ideal linear elastic element is the spring whilst the ideal linear viscous element is the dashpot. Energy is stored in springs as elastic strain energy and energy is dissipated in a dashpot as heat [2].

Manuscript received Mac 29, 2012; revised April 25, 2012. A.B. Musa is with the college of Foundation and General Studies, Uniten 43009 Malaysia. 603-89287237; fax: 603-89212108 e-mail: bakar@uniten.edu.my
In general there are four types of analysis for low speed collisions, associated with particle impact, rigid body

impact, transverse impact on flexible bodies (i.e. transverse wave propagation or vibrations) and axial impact on flexible bodies (i.e. longitudinal wave propagation) [3]. We are interested in the latter impact where it generates longitudinal waves which affect the dynamic analysis of the bodies. The unifying characteristic of waves is propagation of disturbances through the medium. The properties of the medium that affect the waves and determine the speeds of propagation are density ρ and Young's modulus E , of deformability.

In this paper, we predict the stress and velocity at the interface after a moving slug impacts a stationary semi-infinite rod using viscoelastic discontinuity analysis. If the stress at the interface becomes tensile and the velocity changes its sign, then the slug and the rod part company. If the stress at the interface is compressive after the impact, then the slug and the rod remain in contact. In the elastic impact considered by R.P. Menday [4], the stress becomes tensile if the ratio of acoustic impedances $z < 1$ and the stress becomes compressive if the ratio of acoustic impedances $z > 1$ when the wave set up in the slug by the impact has returned to the slug/rod interface. In this viscoelastic impact we investigate how the viscosity time constants in the slug and in the rod give rise to different interface stresses and interface velocities following wave transmission in the slug.

II. MATHEMATICAL MODEL OF VISCOELASTIC IMPACT

We model the impact by having a finite length slug, moving with speed v , impacting a stationary semi-infinite rod and we solve the problem in the Laplace transform domain for the general case of viscoelastic slug and rod. In deriving the numerical solution, we firstly consider the slug is elastic and the rod is viscoelastic.

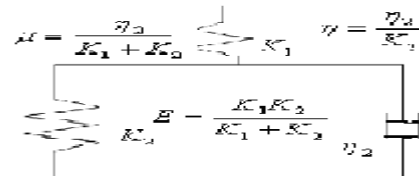


Fig. 1. Standard Linear Solid Model

We model the viscoelastic material as a standard linear solid (Fig. 1). Secondly, we consider the slug is viscoelastic and the rod is elastic and lastly, we consider both materials are viscoelastic. We then numerically compute the interface stress and interface velocity using the complex inversion formula [5].

III. GOVERNING EQUATIONS

Let $\tilde{u}$ and $\tilde{\bar{u}}$ be additional displacements in the slug and in the rod following the impact respectively, $\tilde{\sigma}$ and $\tilde{\bar{\sigma}}$ be the stress in the slug and in the rod respectively, E and $\bar{E}$ be the young modulus in the slug and in the rod respectively, ρ and $\bar{\rho}$ be the density in the slug and in the rod respectively. The quantities $\tilde{\mu}$, $\tilde{\bar{\mu}}$, $\tilde{\eta}$ and $\tilde{\bar{\eta}}$ are material constants with dimension of time where the $\sim$ notation indicates dimensional variables. We choose the origin of coordinates at the center of the interface and axis $0\bar{x}$ along the axis of the rod and we assume the impact takes place at time $\tilde{t} = 0$. When a slug moving at a speed V , impacts the rod at time $\tilde{t} = 0$ and at $\bar{x} = 0$, we write the position at time $\tilde{t}$ of the cross-section of the slug which was at location $\bar{x}$ at time $\tilde{t} = 0$ as

$$\bar{x}(\bar{x}, \tilde{t}) = \bar{x} + V\tilde{t} + \tilde{u}(\bar{x}, \tilde{t}) \quad \text{for } -h_s \leq \bar{x} \leq 0 \quad (1)$$

And the cross-section of the rod which was at location $\bar{x}$ at time $\tilde{t} = 0$ as

$$\bar{x}(\bar{x}, \tilde{t}) = \bar{x} + \tilde{\bar{u}}(\bar{x}, \tilde{t}) \quad \text{for } \bar{x} \geq 0 \quad (2)$$

Then the equation of motion in the slug is

$$\frac{\partial \tilde{\sigma}}{\partial \bar{x}} = \rho \frac{\partial^2 \tilde{u}}{\partial \tilde{t}^2} \quad (3)$$

And the equation of motion in the rod is

$$\frac{\partial \tilde{\bar{\sigma}}}{\partial \bar{x}} = \bar{\rho} \frac{\partial^2 \tilde{\bar{u}}}{\partial \tilde{t}^2} \quad (4)$$

We model the slug and rod as a standard linear solid so that the equation of viscoelastic stress related to $\tilde{u}$ in the slug is

$$\tilde{\sigma} + \tilde{\mu} \frac{\partial \tilde{\sigma}}{\partial \tilde{t}} = E \left(\frac{\partial \tilde{u}}{\partial \bar{x}} + \tilde{\eta} \frac{\partial^2 \tilde{u}}{\partial \tilde{t} \partial \bar{x}} \right) \quad (5)$$

And the equation of viscoelastic stress related to $\tilde{\bar{u}}$ in the rod is

$$\tilde{\bar{\sigma}} + \tilde{\bar{\mu}} \frac{\partial \tilde{\bar{\sigma}}}{\partial \tilde{t}} = \bar{E} \left(\frac{\partial \tilde{\bar{u}}}{\partial \bar{x}} + \tilde{\bar{\eta}} \frac{\partial^2 \tilde{\bar{u}}}{\partial \tilde{t} \partial \bar{x}} \right) \quad (6)$$

We now define the non-dimensional quantities $x, \bar{x}, X, t, u, \bar{u}, \sigma, \bar{\sigma}, \mu, \eta, \bar{\mu}, \bar{\eta}$ by the non-dimensionalising scheme below

$$\begin{aligned} \bar{x} &= h_s X, \quad \bar{x} = h_s x, \quad \tilde{t} = \frac{h_s}{c} t, \quad \tilde{u} = \frac{V}{c} h_s u, \quad \tilde{\bar{u}} = \frac{V}{c} h_s \bar{u}, \\ \tilde{w} &= \frac{V}{c} h_s w, \quad \tilde{\sigma} = E \sigma, \quad \tilde{\bar{\sigma}} = E \bar{\sigma}, \quad \tilde{x} = h_s \bar{x}, \quad \tilde{\eta} = \frac{h_s}{c} \eta, \\ \tilde{\mu} &= \frac{h_s}{c} \mu, \quad \tilde{\eta} = \frac{h_s}{c} \eta, \quad \tilde{\mu} = \frac{h_s}{c} \mu, \quad \tilde{\eta} > \tilde{\mu}, \quad \tilde{\eta} > \tilde{\mu} \end{aligned} \quad (7)$$

Where $c^2 = \frac{E}{\rho}$, $\bar{c}^2 = \frac{\bar{E}}{\bar{\rho}}$, $z = \frac{\rho c}{\bar{\rho} \bar{c}}$ and $\alpha = \frac{c}{\bar{c}}$.

If we now use (7) to non-dimensionalize equations (1) and (2) for $\bar{x}$ and $\bar{x}$, the non-dimensional displacements $\bar{x}$ and $\bar{x}$ are given by

$$x = X + \frac{V}{c} (t + u(X, t)) \quad (8)$$

$$\bar{x} = X + \frac{V}{c} \alpha \bar{u}(X, t) \quad (9)$$

We then non-dimensionalize (3) – (6) to obtain the non-dimensional equation of motion and stress-strain relations

$$\frac{\partial \sigma}{\partial X} = \frac{V}{c} \frac{\partial^2 u}{\partial t^2} \quad (10)$$

$$\frac{\partial \bar{\sigma}}{\partial X} = \alpha^2 \frac{V}{\bar{c}} \frac{\partial^2 \bar{u}}{\partial t^2} \quad (11)$$

$$\sigma + \mu \frac{\partial \sigma}{\partial t} = \frac{V}{c} \left(\frac{\partial u}{\partial X} + \eta \frac{\partial^2 u}{\partial t \partial X} \right) \quad (12)$$

$$\bar{\sigma} + \alpha \bar{\mu} \frac{\partial \bar{\sigma}}{\partial t} = \frac{V}{\bar{c}} \left(\frac{\partial \bar{u}}{\partial X} + \alpha \bar{\eta} \frac{\partial^2 \bar{u}}{\partial t \partial X} \right) \quad (13)$$

IV. GENERAL SOLUTIONS

In order to solve for the additional displacements u and $\bar{u}$ of the waves propagating in the slug and the rod, we take Laplace transforms of the equations (10) – (13) with respect to t and solve the differential equations for the transformed displacement $\hat{u}$ and $\hat{\bar{u}}$ in the s domain. Taking the Laplace transform of the non-dimensionalized equations (10) and (11), after differentiating (12) with respect to x , gives

$$\frac{\partial \hat{\sigma}}{\partial X} = \frac{V}{c} \hat{u} s^2 \quad (14)$$

$$\frac{\partial \hat{\bar{\sigma}}}{\partial X} = \frac{V}{c} \frac{\partial^2 \hat{\bar{u}}}{\partial X^2} \beta^2(s) \quad (15)$$

Where

$$\beta^2(s) = \frac{1 + \eta s}{1 + \mu s} \quad \text{and} \quad \bar{\beta}^2(s) = \frac{1 + \alpha \bar{\eta} s}{1 + \alpha \bar{\mu} s}$$

Equating (14) and (15) yields the differential equation below for $\hat{u}(x, s)$

$$\beta^2(s) \frac{\partial^2 \hat{u}}{\partial X^2} - \hat{u} s^2 = 0 \quad (16)$$

Solving (16), we obtain the general solution for the transform of the additional displacement $\hat{u}(x, s)$ in the slug,

$$\hat{u}(X, s) = a(s) e^{\frac{sX}{\beta(s)}} + b(s) e^{-\frac{sX}{\beta(s)}} \quad (17)$$

Repeating the same process for the equations (11) and (13) gives the general solution $\hat{\bar{u}}(x, s)$ for the transform of the additional displacement in the rod,

$$\hat{\bar{u}}(X, s) = d(s) e^{\frac{\alpha s X}{\bar{\beta}(s)}} + f(s) e^{-\frac{\alpha s X}{\bar{\beta}(s)}} \quad (18)$$

V. BOUNDARY CONDITIONS

In order to find $a(s)$, $b(s)$, $d(s)$ and $f(s)$ in (17) and (18), we apply the boundary conditions described below.

1. The interface conditions state that the particle velocity in the slug and in the rod has to be the same at $\bar{x} = 0$ that is

$$V + \frac{\partial \tilde{u}}{\partial \tilde{t}}(0, \tilde{t}) = \frac{\partial \tilde{\bar{u}}}{\partial \tilde{t}}(0, \tilde{t}) \quad (19)$$

In non-dimensional form, the above equation becomes

$$1 + \frac{\partial u}{\partial t} = \alpha \frac{\partial \bar{u}}{\partial t} \quad (20)$$

Then we Laplace transform equation (20) and substitute from (17) and (18), we obtain

$$\frac{1}{s} + s[a(s) + b(s)] = \alpha s[d(s) + f(s)] \quad (21)$$

2. At the interface $\bar{x} = 0$, the stress in the slug and in the rod must be the same so we have

$$\tilde{\sigma} = \tilde{\bar{\sigma}} \quad (22)$$

If we non-dimensionalize and take Laplace transform of the above equation we have

$$E\left(\frac{V}{c} \frac{\partial \hat{u}}{\partial X} \beta^2(s)\right) = \bar{E}\left(\frac{V}{c} \frac{\partial \hat{u}}{\partial X} \bar{\beta}^2(s)\right) \quad (23)$$

Since

$$\hat{\sigma} = \frac{V}{c} \frac{\partial \hat{u}}{\partial X} \beta^2(s) \quad (24) \quad \text{and} \quad \hat{\sigma} = \frac{V}{c} \frac{\partial \hat{u}}{\partial X} \bar{\beta}^2(s) \quad (25)$$

Substituting the derivatives of (17) and (18) at $\bar{X} = 0$ into the equation (23), gives

$$z\beta(s)[a(s) - b(s)] = \alpha\bar{\beta}(s)[d(s) - f(s)] \quad (26)$$

3. When the wave in the slug reaches the boundary $X = -1$, the stress in the slug at $\bar{X} = -h_s$

is zero that is $\bar{\sigma} = 0$

Non-dimensionalize and take Laplace transform of the above equation to obtain

$$E\left(\frac{V}{c} \frac{\partial \hat{u}}{\partial X} \beta^2(s)\right) = 0$$

Substituting (17) at $X = -1$ into the above equation, we obtain

$$s\rho c V \beta(s) \left(a(s) e^{-\frac{s}{\beta(s)}} - b(s) e^{\frac{s}{\beta(s)}} \right) = 0 \quad (27)$$

4. The stress in the rod at $X = ah_s$ is zero, that is $\bar{\sigma} = 0$

Non-dimensionalize and take Laplace transform of the above equation to obtain

$$\bar{E}\left(\frac{V}{c} \frac{\partial \hat{u}}{\partial X} \bar{\beta}^2(s)\right) = 0$$

Substituting (18) at $X = a$ into the above equation, we obtain

$$s\alpha\bar{\rho}cV\bar{\beta}(s) \left(d(s) e^{\frac{\alpha as}{\bar{\beta}(s)}} - f(s) e^{-\frac{\alpha as}{\bar{\beta}(s)}} \right) = 0 \quad (28)$$

Equations (21), (26), (27) and (28) give four equations in the four unknowns $a(s)$, $b(s)$, $d(s)$ and $f(s)$. Solving for these unknowns and substituting into (17) and (18) gives the additional displacements $\hat{u}$ and $\hat{\bar{u}}$ in the Laplace transform domain,

$$\hat{u} = - \frac{\sinh\left(\frac{\alpha as}{\beta(s)}\right) \cosh\left(\frac{s}{\beta(s)}(1+X)\right)}{s^2 \left(z \frac{\beta(s)}{\bar{\beta}(s)} \sinh\left(\frac{s}{\beta(s)}\right) \cosh\left(\frac{\alpha as}{\bar{\beta}(s)}\right) + \sinh\left(\frac{\alpha as}{\beta(s)}\right) \cosh\left(\frac{s}{\beta(s)}\right) \right)} \quad (29)$$

$$\hat{\bar{u}} = - \frac{z\beta(s) \sinh\left(\frac{s}{\beta(s)}\right) \cosh\left(\frac{\alpha s}{\bar{\beta}(s)}(X-a)\right)}{s^2 \left(z\beta(s) \sinh\left(\frac{s}{\beta(s)}\right) \cosh\left(\frac{\alpha as}{\bar{\beta}(s)}\right) + \bar{\beta}(s) \sinh\left(\frac{\alpha as}{\beta(s)}\right) \cosh\left(\frac{s}{\beta(s)}\right) \right)} \quad (30)$$

Having found the general solution (29) and (30) for the displacements in the slug and the rod in the s-domain, we then can derive the equations for stress and the velocity in the slug and the rod. To do this we use the complex inversion formula to invert the transforms. As the general solution in the Laplace transform domain is particularly complicated, we consider its inversion in certain special cases. In solving for stress and velocity, we are considering a slug traveling at speed v which impacts semi-infinite rod. These solutions apply provided the interface stress remains compressive. If the stress at the interface becomes tensile, then the solution no longer valid since there can be no

longer tensile stress at the interface. It is then necessary to modify the solution by introducing waves traveling away from the interface stress at zero. If the stress is compressive, then the slug and the rod remain in contact until such time as the stress drops to zero.

Firstly, we consider the general case when both the slug and the rod are viscoelastic. Then we apply the Bromwich contour to lay-out the calculation of the complex integrals along the contour and determine the poles and branch points. Secondly, we compute the residue of the simple pole and numerically compute the rest of the residues and the complex integrals. We first consider an elastic slug impacting a viscoelastic rod where $\eta = 0$, $\mu = 0$, $\bar{\eta} \neq 0$, $\bar{\mu} \neq 0$, then we consider the slug is viscoelastic and rod is elastic where $\eta \neq 0$, $\mu \neq 0$, $\bar{\eta} = 0$, $\bar{\mu} = 0$ and lastly we consider that both materials are viscoelastic where $\eta \neq 0$, $\mu \neq 0$, $\bar{\eta} \neq 0$, $\bar{\mu} \neq 0$.

Considering the first case where the slug is elastic and the rod is viscoelastic, the Laplace transform of the stress in the slug is given by equation (24) and putting $\frac{V}{c} = 1$ and

considering the general solution of the slug displacement equation (29), the solution (24) in the case when $a \rightarrow \infty$ gives

$$\hat{\sigma}(X, s) = - \frac{\beta(s) \sinh\left(\frac{s}{\beta(s)}(1+X)\right)}{s \left(z \frac{\beta(s)}{\bar{\beta}(s)} \sinh\left(\frac{s}{\beta(s)}\right) + \cosh\left(\frac{s}{\beta(s)}\right) \right)} \quad (31)$$

In order to find the stress as a function of time, we have to invert the solution (31) and we employ the complex inversion formula [4].

VI. VISCOELASTIC DISCONTINUITY

Assume there are discontinuities in $\bar{v}$, $\bar{\epsilon}$, and $\bar{\sigma}$ across the surface $\frac{dx}{dt} = U$. Let v^- , ϵ^- , and σ^- denote velocity, strain and stress, respectively behind the moving surface U while v^+ , ϵ^+ , and σ^+ denote velocity, strain and stress, respectively ahead of the moving surface. As the surface moves, we consider the change of momentum between the times t and $t + \delta t$ where the velocity of the mass $\rho A U \delta t$ changes from v^+ to v^- to give the change as $\rho A U (\bar{v}^- - \bar{v}^+) \delta t$. The change of the momentum, must equal the impulse of the net force which gives

$$(\bar{\sigma}^+ - \bar{\sigma}^-) A \delta t = \rho A U (\bar{v}^- - \bar{v}^+) \delta t \quad (32)$$

We non-dimensionalise equation (32) and obtain

$$(\sigma^+ - \sigma^-) = \frac{V}{c} U (v^- - v^+) \quad (33)$$

$$-\frac{[\sigma]}{U} = \frac{V}{c} [v] \quad (34)$$

Considering equation (3.11) in Musa [6] and replacing $\bar{u}$ by u , we obtain

$$U \left[\frac{df}{dx} \right] + \left[\frac{df}{dt} \right] = 0 \quad (35)$$

Substituting f by u in equation (35) gives

$$-\frac{[v]}{U} = [\epsilon] \quad (36)$$

Considering equation (3.16) in Musa [6] for the non-dimensional stress-strain relation of viscoelastic material and replacing $\bar{\eta}$, $\bar{\mu}$ and $\bar{c}$ by η , μ and c respectively, we obtain

$$\mu[\sigma] = \frac{V}{c} \eta[\varepsilon] \quad (37)$$

Eliminating $[\sigma]$ and $[\varepsilon]$ in equations (34), (36) and (37) gives

$$\left(\mu U - \frac{\eta}{U} \right) [v] = 0 \quad (38)$$

Which gives $U = \pm \sqrt{\frac{\eta}{\mu}}$.

Considering from equation (3.19) to (3.23) in Musa [6], we obtain

$$-U[v] + \mu U^2 \left[\frac{\partial v}{\partial X} \right] - 2\mu U \frac{\delta}{\delta t} [v] = -\frac{[v]}{U} + \eta \left[\frac{\partial v}{\partial X} \right] \quad (39)$$

Further simplification, reduces equation (39) to

$$\frac{\delta}{\delta t} [v] = \frac{1}{2\mu} \left(\frac{1}{U^2} - 1 \right) [v] = \frac{1}{2} \left(\frac{1}{\eta} - \frac{1}{\mu} \right) [v] \quad (40)$$

Equation (40) integrates to give the variations of jump in $[v]$ as we move with the front, in the form

$$[v] = [v]_0 e^{\frac{\mu - \eta}{2\eta\mu} t} \quad (41)$$

Where $[v]_0$ is the value of the jump at time $t = 0$. It follows from equations (34) and (36) that

$$[\varepsilon] = [\varepsilon]_0 e^{\frac{\mu - \eta}{2\eta\mu} t} \quad (42) \text{ and } [\sigma] = [\sigma]_0 e^{\frac{\mu - \eta}{2\eta\mu} t} \quad (43)$$

Which agrees with result of Morrison [8]?

Applying the result to the impact problem where we solve the viscoelastic equations in the rod and slug subject to the boundary conditions

$$1 + v = \alpha \bar{v} \quad (44) \text{ and } \alpha z \sigma = \bar{\sigma} \quad (45)$$

At impact, we will have a discontinuity in velocity in both slug and rod and a discontinuity in stress. From (44) and (45), these are given by

$$1 + [v]_0 = \alpha [\bar{v}]_0 \quad (46) \text{ and } \alpha z [\sigma]_0 = [\bar{\sigma}]_0 \quad (47)$$

There will be a discontinuity in the slug along $\frac{dx}{dt} = -U$ and

in the rod $\frac{d\bar{x}}{dt} = \bar{U}$ and from the results in equation (34), we obtain

$$[\sigma]_0 = \frac{UV}{c} [v]_0 \quad (48) \text{ and } [\bar{\sigma}]_0 = -\alpha^2 \frac{\bar{U}V}{\bar{c}} [\bar{v}]_0 \quad (49)$$

Where $U = \sqrt{\frac{\eta}{\mu}}$ and $\bar{U} = \sqrt{\frac{\bar{\eta}}{\bar{\mu}}}$. Eliminating $[\bar{v}]_0$ gives

$$[v]_0 = -\frac{1}{1 + z^*} \quad (50) \text{ and } [\sigma]_0 = -\frac{V}{c} \sqrt{\frac{\eta}{\mu}} \quad (51)$$

and in the rod

$$[\bar{v}]_0 = \frac{z^*}{\alpha(1 + z^*)} \quad (52) \text{ and } [\bar{\sigma}]_0 = -\frac{V}{\bar{c}} z \sqrt{\frac{\bar{\eta}}{\bar{\mu}}} \quad (53)$$

Where $z^* = z \sqrt{\frac{\eta\bar{\mu}}{\bar{\eta}\mu}}$. These results agree with results obtained by Musa [6] using the limit theorem (initial value

theorem) for the Laplace Transform solutions. As the discontinuity moves in the slug and reaches the free end ($X=-1$) at time $t = \frac{1}{\sqrt{\frac{\eta}{\mu}}} = \sqrt{\frac{\mu}{\eta}}$ at which time the magnitudes

$$[v] = -\frac{1}{1 + z^*} e^{-\frac{\eta - \mu}{2\eta\sqrt{\eta\mu}}} \quad (54)$$

$$[\sigma] = -\frac{V}{c} \sqrt{\frac{\eta}{\mu}} e^{-\frac{\eta - \mu}{2\eta\sqrt{\eta\mu}}} \quad (55)$$

The stress-free condition at $X=-1$ requires a reflected pulse to travel back along $\frac{dx}{dt} = U$ with its stress discontinuity

being equal and opposite to that given by (55). Since the discontinuity in v is now related to that in σ by equation (34), $[v]$ is still given by equation (54). This reflected pulse will reach the interface at time $t = 2\sqrt{\frac{\mu}{\eta}}$ at which time the

jumps in v and σ are given by

$$[v] = -\frac{1}{1 + z^*} e^{-\frac{\eta - \mu}{\eta\sqrt{\eta\mu}}} \quad (56) \text{ and } [\sigma] = -\frac{V}{c} \sqrt{\frac{\eta}{\mu}} e^{-\frac{\eta - \mu}{\eta\sqrt{\eta\mu}}} \quad (57)$$

VII. THE STRESS IN THE SLUG

In this case, our main objective is to determine if and when the slug and the rod part company. In order to do that, we need to examine the stress at the interface. The Laplace transform of the stress in the slug is given by

$$\hat{\sigma} = \frac{EV}{c} \frac{d\hat{u}}{dX} \beta^2(s) \quad (58)$$

Putting the constant $\frac{EV}{c} = 1$ and considering the general

solution of the displacement equation (29), the solution (58) in the case $a \rightarrow \infty$ gives

$$\hat{\sigma}(X, s) = -\frac{\beta(s) \sinh\left(\frac{s}{\beta(s)}(1 + X)\right)}{s \left(\frac{z\beta(s)}{\bar{\beta}(s)} \sinh\left(\frac{s}{\beta(s)}\right) + \cosh\left(\frac{s}{\beta(s)}\right) \right)} \quad (59)$$

In order to find the stress as a function of time, we have to invert the solution (59) using complex inversion formula [5]. The numerical results of the interface stress in the slug for several ratios of acoustic impedances and viscosity time constants are shown in Fig. 3 – Fig. 8 [7].

VIII. THE VELOCITY IN THE SLUG

The velocity at the position x in the slug is

$$\frac{\partial x}{\partial t} = \frac{V}{c} \left(1 + \frac{\partial u(X, t)}{\partial t} \right) \quad (60)$$

The Laplace transform of the displacement u is

$$\hat{u}(X, s) = -\frac{\cosh\left(\frac{s}{\beta(s)}(1 + X)\right)}{s^2 \left(\frac{z\beta(s)}{\bar{\beta}(s)} \sinh\left(\frac{s}{\beta(s)}\right) + \cosh\left(\frac{s}{\beta(s)}\right) \right)} \quad (61)$$

In order to find the velocity as a function of time, we invert equation (61) using complex inversion formula using a suitable Bromwich contour curve [5] as shown in Fig. 2. Then

$$u(X, t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} \hat{u}(X, s) ds \quad (62)$$

$$u(X, t) = -\frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\cosh\left(\frac{s}{\beta(s)}(1+X)\right) e^{st}}{s^2 \left(z \frac{\beta(s)}{\bar{\mu}(s)} \sinh\left(\frac{s}{\beta(s)}\right) + \cosh\left(\frac{s}{\beta(s)}\right) \right)} ds \quad (63)$$

The numerical results of the interface velocity in the slug for several ratios of acoustic impedances and viscosity time constants are shown in Fig. 3 – Fig. 8 [7].

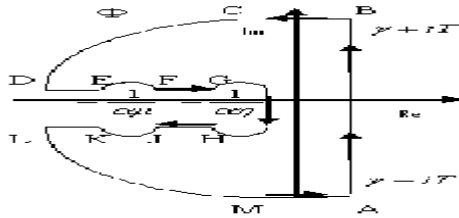


Fig. 2. Bromwich Contour Curve

IX. IMPACT OF ELASTIC SLUG AND VISCOELASTIC ROD

In order to make comparisons between the actual results and the discontinuity analysis, we consider equation (50) and we let $\beta = 1$ or $\eta = 0$ and $\mu = 0$ for elastic slug. We predict the initial velocity discontinuity at the interface for the case when the slug is elastic and the rod is viscoelastic to be

$$1 + [v]_0 = 1 - \frac{1}{1 + z \sqrt{\frac{\bar{\mu}}{\eta}}} = \frac{z^*}{1 + z^*} \quad (64)$$

And the initial interface stress discontinuity to be

$$[\sigma]_0 = -\frac{1}{1 + z \sqrt{\frac{\bar{\mu}}{\eta}}} = -\frac{1}{1 + z^*} \quad (65)$$

The predicted initial interface stress and initial interface velocities are shown in TABLE I for several ratios of acoustic impedances z^* . In TABLE I, we also display the initial interface stress and initial interface velocity based on the long time acoustic impedances. We calculate them by replacing z^* by z in (64) and (65).

Using the results of the elastic slug impacting an elastic rod in Appendix I in [6], we can predict the interface stress and interface velocity discontinuities at non-dimensional time $t = 2$ after the wave rebounds at $X = -1$ and reaches the interface. When the slug is elastic and the rod is viscoelastic, the predicted interface stress and interface velocity discontinuities are calculated by replacing z by z effective, $z^* = z \sqrt{\frac{\bar{\mu}}{\eta}}$ in (I 9) and (I 8) in [6], respectively

and we obtain

$$[\sigma]_1 = -\left(\frac{2}{z \sqrt{\frac{\bar{\mu}}{\eta}} + 1} \right) [\sigma]_0 = -\left(\frac{2}{z^* + 1} \right) [\sigma]_0$$

as the interface stress discontinuity at non-dimensional time $t = 2$ and

$$[v]_1 = -\left(\frac{2z \sqrt{\frac{\bar{\mu}}{\eta}}}{z \sqrt{\frac{\bar{\mu}}{\eta}} + 1} \right) [v]_0 = -\left(\frac{2z^*}{z^* + 1} \right) [v]_0$$

as the interface velocity discontinuity where $[\sigma]_0$ and $[v]_0$ are the incoming waves in the interface stress and interface velocity, respectively. The predicted interface stress and interface velocity after the wave rebounds at $X = -1$ are shown in TABLE II.

TABLE I

P_{ISD} = Predicted initial stress discontinuity and P_{IPV} = Predicted initial particle velocity P_{LTIS} and P_{LTV} are predicted initial stress discontinuity and initial velocity respectively.

z	P_{LTIS}	P_{LTV}	$\frac{z}{z^*}$	z^*	P_{ISD}	P_{IPV}
0.9	-0.526	0.474	2	0.636	-0.611	0.389
1	-0.5	0.5	2	0.707	-0.586	0.414
1.2	-0.455	0.545	2	0.849	-0.541	0.459
1.667	-0.375	0.625	2	1.179	-0.459	0.541
2	-0.333	0.667	2	1.414	-0.414	0.586
0.9	-0.526	0.474	5	0.402	-0.713	0.287
1	-0.5	0.5	5	0.447	-0.691	0.309
1.2	-0.455	0.545	5	0.537	-0.651	0.349
2.5	-0.286	0.714	5	1.118	-0.472	0.528
3	-0.25	0.75	5	1.342	-0.427	0.573

TABLE II

P_{SIJ} = Predicted stress jump and P_{VIJ} = Predicted velocity jump at the interface after the wave first rebound at $X = -1$ in the slug for several ratios of effective acoustic impedances z^*

z	$\frac{z}{z^*}$	z^*	P_{SIJ}	P_{VIJ}
0.9	2	0.636	0.747	-0.475
1	2	0.707	0.686	-0.485
1.2	2	0.849	0.585	-0.497
1.667	2	1.179	0.422	-0.497
2	2	1.414	0.343	-0.485
0.9	5	0.402	1.017	-0.409
1	5	0.447	0.955	-0.427
1.2	5	0.537	0.847	-0.455
2.5	5	1.118	0.445	-0.498
3	5	1.342	0.364	-0.489

X. DISCUSSION AND CONCLUSIONS

Fig. 3 shows that as t increases, the stress settle down and show that the initial interface stress discontinuity is between the long term initial interface stress, -0.526 and the short term initial interface stress, -0.611 when $z = 0.9$ or the effective ratios of acoustic impedances $z^* = 0.636$ as shown in TABLE I. When the viscosity time constants are $\bar{\eta} = 0.02$, $\bar{\mu} = 0.01$, the initial interface stress is about -0.55 which is close to the long term initial interface stress and when the viscosity time constants are $\bar{\eta} = 20$ and $\bar{\mu} = 10$, the initial interface stress is about -0.61 which close to the short term initial interface stress.

Fig. 6 shows that as t increases, the stress settle down and show that the initial interface stress discontinuity is between the long term initial interface stress, -0.455 and the short term initial interface stress, -0.651 when $z = 1.2$ or the effective ratios of acoustic impedances $z^* = 0.537$. When the viscosity time constants are

$\bar{\eta} = 0.05, \bar{\mu} = 0.01$, the initial interface stress is about -0.47 which is close to the long term initial interface stress and when the viscosity time constants are $\bar{\eta} = 50$ and $\bar{\mu} = 10$, the initial interface stress is about -0.65 which close to the short term initial interface stress. The results shown in Fig. 3 – Fig. 8 show that the actual initial interface stress is between the long term initial interface stress and the short term initial interface stress as shown in TABLE I. Moreover the results shown in Fig. 3 – Fig. 8 also show that the actual interface velocity is between the long time initial interface velocity and the short term initial interface velocity. As the viscosity time constants $\bar{\eta}$ and $\bar{\mu}$ increase, the predicted value in TABLE I approximates the initial interface stress initial interface velocity better. It is shown in Figures (A3), (A4), (A7) and (A8) in [6] that small values of $\bar{\eta}$ and $\bar{\mu}$, the material undergoes rapid creep and stress relaxation over time scale which is short compared with the travel time of a wave in the slug. This means that the material effectively behaves like an elastic material with the long time elastic constant E . For large values of $\bar{\eta}$ and $\bar{\mu}$, the strain and stress remain virtually constant over the time for the transit of the wave in the slug and the material behaves approximately like an elastic material with the short time(instantaneous) elastic modulus $E \sqrt{\frac{\bar{\eta}}{\bar{\mu}}}$.

Fig. 5 shows that the interface velocity jump at $t = 2$ when the viscosity time constants are $\bar{\eta} = 0.02, \bar{\mu} = 0.01$ and $z = 1.2$ is about -0.52 and when the viscosity time constants, $\bar{\eta} = 20$ and $\bar{\mu} = 10$ the interface velocity jump is about -0.5 . However the TABLE II shows that the interface velocity jump prediction at non-dimensional time $t = 2$ when $z = 1.2$ or $z^* = 0.849$ and ratio of viscosity time constants $\frac{\bar{\eta}}{\bar{\mu}} = 2$, is -0.479 . Moreover Fig. 4 shows that the interface velocity jump at $t = 2$ when the viscosity time constants are $\bar{\eta} = 0.05, \bar{\mu} = 0.01$ and $z = 0.9$ is about -0.46 and when the viscosity time constants, $\bar{\eta} = 50$ and $\bar{\mu} = 10$ the interface velocity jump is about -0.41 . TABLE II shows that the interface velocity jump prediction at non-dimensional time $t = 2$ when $z = 0.9$ or $z^* = 0.447$ and ratio of viscosity time constants $\frac{\bar{\eta}}{\bar{\mu}} = 5$, is -0.409 . This shows

that as we increase the viscosity time constants from $\bar{\eta} = 0.02, \bar{\eta} = 0.05$ and $\bar{\mu} = 0.01$ to $\bar{\eta} = 20, \bar{\eta} = 50$ and $\bar{\mu} = 10$, the prediction approximates the interface velocity jump at $t = 2$ better.

Fig. 5 also shows that the interface stress jump at $t = 2$ when the viscosity time constants are $\bar{\eta} = 0.02, \bar{\mu} = 0.01$ and $z = 1.2$ is about 0.40 and when the viscosity time constants, $\bar{\eta} = 20$ and $\bar{\mu} = 10$ the interface velocity jump is about 0.55 . The predicted interface stress as shown in TABLE II is 0.585 . Fig. 4 also shows that the interface stress jump at $t = 2$ when the viscosity time constants are $\bar{\eta} = 0.02, \bar{\eta} = 0.05, \bar{\mu} = 0.01$ and $z = 0.9$ is about 0.55 and when the viscosity time constants, $\bar{\eta} = 50$ and $\bar{\mu} = 10$ the interface velocity jump is about 1.0 . The

predicted interface stress as shown in TABLE II is 1.017 . This shows that as we increase the viscosity time constants from $\bar{\eta} = 0.02, \bar{\eta} = 0.02$ and $\bar{\mu} = 0.01$ to $\bar{\eta} = 20, \bar{\eta} = 50$ and $\bar{\mu} = 10$, the prediction approximates the interface stress jump at $t = 2$ better. Fig. 4 also shows that when the viscosity time constants are $\bar{\eta} = 0.05, \bar{\mu} = 0.01$ and $\bar{\eta} = 0.5, \bar{\mu} = 0.1$, the interface stress is decreasing rapidly after $t = 2$. However the stress is decreasing gently when the viscosity time constants $\bar{\eta} = 5, \bar{\mu} = 1$ and $\bar{\eta} = 50, \bar{\mu} = 10$. This trend of results can be explained by the relaxation test in Figure (A8) in Appendix A in [6]. The stress response for the relaxation test decreasing rapidly when the viscosity time constants are $\bar{\eta} = 0.05, \bar{\mu} = 0.01$ and the viscoelastic material behaves like the long elastic material whereas when the viscosity time constants are $\bar{\eta} = 50, \bar{\mu} = 10$, the interface stress remains constant and the viscoelastic material behaves like the short time elastic material. Fig. 3- Fig. 6 show that the stress becomes tensile at non-dimensional time $t = 2$ when the viscosity time constants $\bar{\eta} \geq 0.02, \bar{\mu} \geq 0.01$ for $z = 0.9 (z^* = 0.636)$ for $\frac{\bar{\eta}}{\bar{\mu}} = 2$ and $z^* = 0.402$ for $\frac{\bar{\eta}}{\bar{\mu}} = 5$) and $z = 1.2 z = 0.9 (z^* = 0.849$ for $\frac{\bar{\eta}}{\bar{\mu}} = 2$ and $z^* = 0.537$ for $\frac{\bar{\eta}}{\bar{\mu}} = 5$). However in the elastic impact by R.P. Menday [4],

the stress is compressive for $z > 1$ at time $t = 2$ (it takes 2 times unit for the wave to travel backward and forward in the elastic slug), This implies that the viscosity time constants in the rod play an important role in determining the stress at the interface as well as the effective ratio of acoustic impedances z^* . When the stress becomes tensile at the interface, the solution is no longer valid because the slug and the rod will part company. It then becomes necessary to initiate waves traveling away from the interface in both slug and rod, in such a way as to maintain zero stress at the interface.

ACKNOWLEDGMENT

The work in this paper was studied during my PhD assignment at Loughborough University, England under the supervision of Dr W.A. Green and Dr Martin Harrison. I would like to express my sincere gratitude to both of them for their patience, motivation, enthusiasm and immense knowledge in the area.

REFERENCES

- [1] Christensen, R.M.(1971) Theory of Viscoelasticity: An Introduction. First edition, Academic Press
- [2] Bland, D.R.(1960) International Series of Monographs on Pure and Appl. Math. The Theory Of Linear Viscoelasticity. Pergamon Press
- [3] Kolsky, H.(1963)Stress Waves in Solids. First edition, Dover Publication
- [4] Menday, R.P.(1999) The Forced Vibration of Partially Delaminated Beam. PhD Thesis, Loughborough University, England
- [5] Spiegel, M.R.(1965)Laplace Transforms. First edition, McGraw-Hill Book company
- [6] Musa, A.B.(2005) Wave Motion and Impact Effects in Viscoelastic Rods, PhD thesis, Loughborough University, England
- [7] Musa, A.B.(2006) Numerical Approach of Viscoelastic Impact, Regional Conference Paper, EduResearch UPSI 2006, Kuala Lumpur.
- [8] Morrison, J.A.(1956) Wave Propagation in Rods of Voigt Material and Visco-elastic Materials with Three-Parameter Models. Quart. Appl Math., Volume XIV, No.2., 153-169

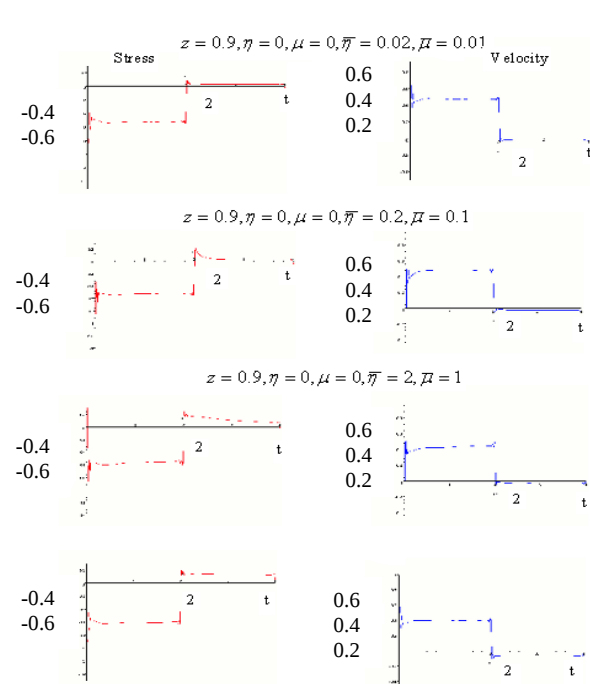


Fig. 3 For $z=0.9$ and several viscosity constants

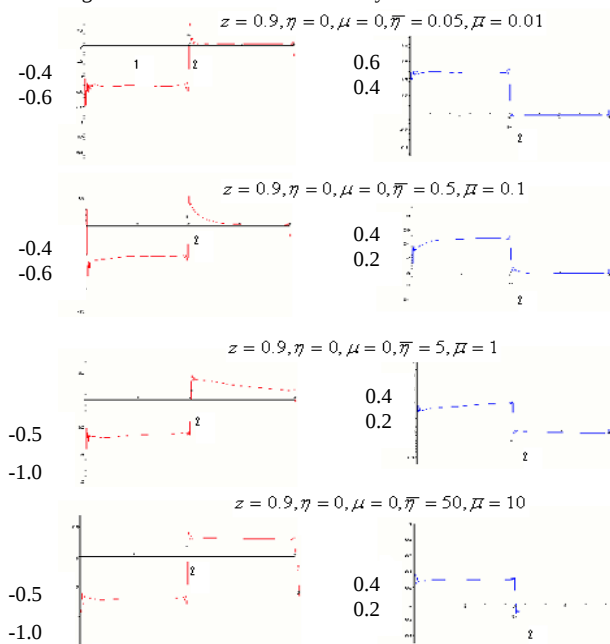


Fig. 4 For $z=0.9$ and several viscosity constants

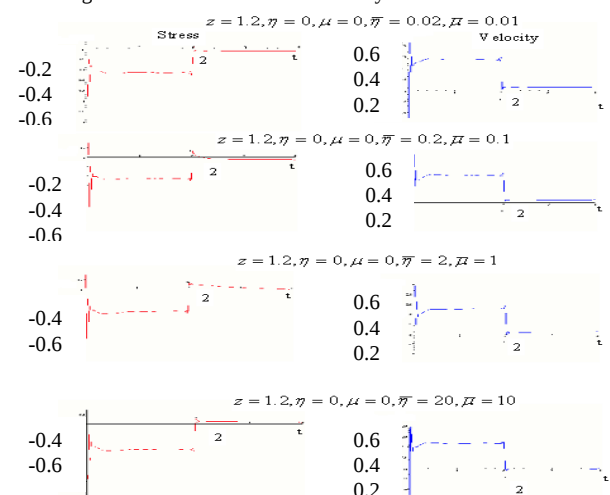


Fig. 5 For $z=1.2$ and several viscosity constants

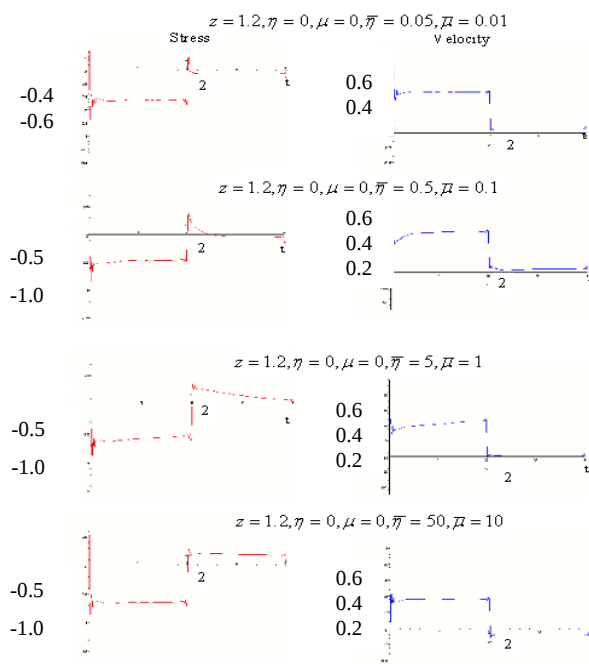


Fig. 6 For $z=1.2$ and several viscosity constants

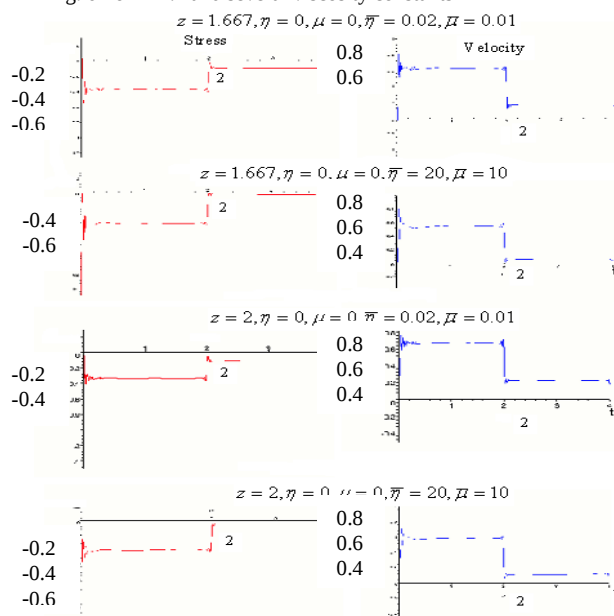


Fig. 7 For $z=1.667$ and $z=2$ and several viscosity constants

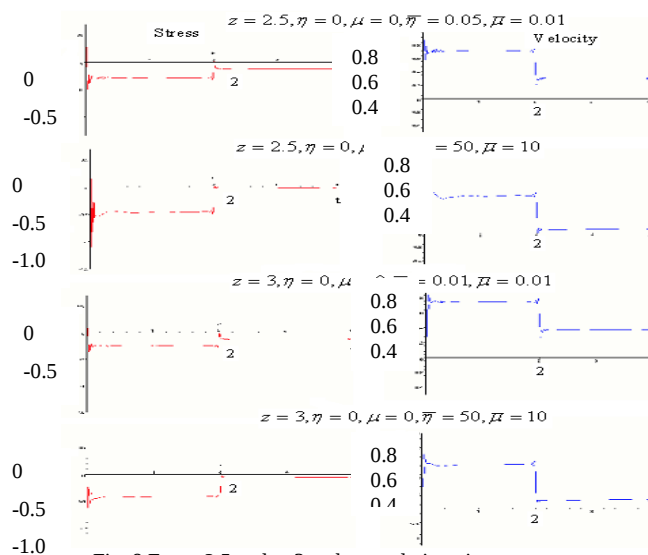


Fig. 8 For $z=2.5$ and $z=3$ and several viscosity constants